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# CLOSED SUBSETS OF POWERS OF NATURAL NUMBERS

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CLOSED SUBSETS OF POWERS OF NATURAL NUMBERS

by

M. Hušek and J. van der Slot *)

It is known that for each non-measurable cardinal $\alpha$ the product $N^{2^\alpha}$ contains a closed discrete subspace of power $2^\alpha$ (see Juhasz [3]). It is clear that such a subspace cannot be C-embedded. Indeed, $N^{2^\alpha}$ contains a dense set of power $\alpha$ so there are only $2^\alpha$ continuous functions on $N^{2^\alpha}$.

It is natural to ask whether there exists a closed discrete non-C-embedded subspace of $N^{2^\alpha}$ which has cardinal $\alpha$. In this note we show that these subspaces certainly exist if $\alpha = \aleph_0$, i.e., $N^{2^{\aleph_0}}$ contains a closed non-C-embedded copy of $N$. We thus give a different approach than in Gillman and Jerison [1] page 97, who constructed a pseudocompact space which contains a closed non-C-embedded copy of $N$.

Recall that a subset D of a space X is called C-*embedded* provided that each continuous function on D can be extended continuously over X. Furthermore, not that a closed subspace of a normal space is C-embedded.

Denote by $R^*$ the real numbers supplied with the half open interval topology (i.e. the subsets $[a,b)$ for $a,b \in R$ form a base for the open sets) and let $S = R^* \times R^*$. We will first show that the space S contains a closed countable discrete subset which is not C-embedded. Let the discrete subspace $D \subset S$ be defined by $\{(x,y) \mid x + y = 1\}$ and $D = D_1 \cup D_2$ where $D_1$ and $D_2$ are dense on the line D (considered as subspace of the plane) and disjoint. The following proposition may be well-known (see also [4] pp. 134).

PROPOSITION 1. $D_1$ *and* $D_2$ *have no disjoint neighborhoods in* S.

PROOF. Suppose that U and V are open neighborhoods of $D_1$ and $D_2$

*) This work was done during the first author's stay at the Mathematical Center Amsterdam (February 1971).

respectively, and suppose $U = \cup\{U(p) \mid p \in D_1\}$; $V = \cup\{U(p) \mid p \in D_2\}$ where each $U(p)$ is a basic n.b.h. of $p$ which intersects $D$ only in $p$. For $n = 1,2,\ldots$ let $L_n$ be a line parallel to the line $D$ and on a distance $\frac{1}{n}$ from $D$. Then $A_n = \{p \in D_1 \mid U(p) \cap L_n \neq \emptyset\}$ and $B_n = \{p \in D_2 \mid U(p) \cap L_n \neq \emptyset\}$ are nowhere dense subsets of the line $D$ for sufficiently large $n$. Because $D$ is the union of the $A_n$'s and $B_n$'s we get a contradiction with Baire's category theorem.

PROPOSITION 2. $D_1$ *(and also* $D_2$*) is not* C*-embedded in* S.

PROOF. We may suppose that $D_1$ is countable. Let $D_1 = \{p_n \mid n=1,2,\ldots\}$ and define $f\colon D_1 \to R$ by $f(p_n) = n$. $f$ cannot be extended over S. Indeed, suppose that $\bar{f}$ is such an extension. For each $n = 1,2,\ldots$ let $U_n$ be a basic clopen neighborhood of $p_n$ in S such that $\bar{f}(U_n) \subset (n-\frac{1}{4}, n+\frac{1}{4})$ and $U_n \cap D = \{p_n\}$. Obviously $\{U_n \mid n=1,2,\ldots\}$ is a discrete collection of closed sets in S (because $\{\bar{f}^{-1}(n-\frac{1}{4}, n+\frac{1}{4}) \mid n=1,2,\ldots\}$ is discrete in S), so $G = \cup\{U_n \mid n=1,2,\ldots\}$ is closed. It follows that $G$ is a closed n.b.h. of $D_1$ which does not intersect $D_2$. This is impossible by Proposition 1.

Our main result is now proved if we can show that the space S is homeomorphic with a closed subspace of a product of continuously many copies of $N$. Indeed, $R^*$ and hence also S satisfies the following condition:

(*) Every maximal centered system of clopen sets with the countable intersection property has a non-empty intersection,

and it is well-known (see e.g. [2]) that such a (realcompact) space is homeomorphic with a closed subspace of $N^{C(X,N)}$ ($C(X,N)$ is the set of all continuous functions of X into $N$).

Hence, if $\underline{c}$ is the cardinal of the continuum,

THEOREM. $N^{\underline{c}}$ *contains a closed countable discrete subspace which is not* C*-embedded.*

REMARK. The set $D_1$ in Prop. 2 may have every cardinality between $\aleph_0$ and $\underline{c}$ (if the continuum hypothesis is not supposed). *Hence* $N^{\underline{c}}$ *contains for each* $\alpha$ *with* $\aleph_0 \leq \alpha \leq \underline{c}$ *a closed discrete subspace of cardinality* $\alpha$ *which is not* C-*embedded.*

The remark leads furthermore to the following two problems:

PROBLEMS. 1. Is it true that for each $\alpha$ $N^{2^\alpha}$ contains a closed discrete subspace of cardinal $\alpha$ which is not C-embedded? The above theorem says that this is valid for $\alpha = \aleph_0$.

2. Can in the theorem $\underline{c}$ be decreased to a smaller cardinal ($> \aleph_0$ because $N^{\aleph_0}$ is metrizable).

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