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C.G. Lekkerkerker

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The elements of the theory of modular forms, derived from properties of theta series

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§1. Introduction. In this lecture I wish to consider the modular function $J(\tau)$ and the modular forms $g_2 = g_2(\omega_1,\omega_2)$, $g_3 = g_3(\omega_1,\omega_2)$, $\Delta=\Delta(\omega_1,\omega_2)$ from a point of view different from the usual one. Here and in the sequel by ω_1, ω_2, τ are meant complex variables, subject to the relations

$$(1.1)\qquad \omega_1 \neq 0,\ \omega_2 \neq 0,\ \tau = \frac{\omega_1}{\omega_2},\ \text{Im}\,\tau > 0,$$

whereas g_2, g_3, Δ are understood to be defined by

$$(1.2)\qquad \begin{cases} g_2 = g_2(\omega_1,\omega_2) = 60\sum{}' \dfrac{1}{(m\omega_1+n\omega_2)^4} \\ g_3 = g_3(\omega_1,\omega_2) = 140\sum{}' \dfrac{1}{(m\omega_1+n\omega_2)^6} \\ \Delta = \Delta(\omega_1,\omega_2) = g_2^3-27g_3^2, \end{cases}$$

the summation being extended over all pairs of integers $(m,n) \neq (0,0)$. In the theory of elementary modular forms various relations and properties for g_2, g_3, Δ and other modular forms are proved. For instance the form Δ possesses the remarkable property, that it can be expanded as the following infinite product

$$(1.3)\qquad \Delta = \left(\frac{2\pi}{\omega_2}\right)^{12} r \prod_{n=1}^{\infty} (1-r^n)^{24}, \qquad r = e^{2\pi i\tau};$$

the infinite product converges absolutely, since by (1.1) we have $|r|<1$. In the classical presentation of the theory to a high degree use is made of elliptic functions.

In order to illustrate this mode of procedure by a (rather important) example, I sketch a possible classical proof of (1.3). The elliptic function

$$\wp(u) = \frac{1}{u^2} + \sum{}' \left\{ \frac{1}{(u-m\omega_1-n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2} \right\}$$

satisfies the differential equation

$$\{\wp'(u)\}^2 = 4\wp^3(u) - g_2\wp(u) - g_3.$$

If we put $\omega_3 = -\omega_1-\omega_2$, $\wp(\frac{1}{2}\omega_i) = e_i$ $(i = 1,2,3)$, then the zeros of $\wp'(u)$ in the fundamental period parallelogram of $\wp(u)$, $\wp'(u)$ are given by $u = \frac{1}{2}\omega_i$ $(i = 1,2,3)$ and the values e_1, e_2, e_3 of $\wp(u)$ are assumed only in the points $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, $\frac{1}{2}\omega_3$ respectively. So the numbers e_1, e_2, e_3 are different; furthermore they are zeros of the cubic $4z^3-g_2z-g_3$. The discriminant of this cubic form is equal to $g_2^3-27g_3^2 = \Delta$, yielding

$$\Delta = 16\{(e_1-e_2)(e_2-e_3)(e_3-e_1)\}^2.$$

(Hence $\Delta \neq 0$ for any admissable choice of ω_1, ω_2). A further step in the proof consists in considering the function

$$\sigma(u) = u \prod{}' \left\{ \left(1 - \frac{u}{m\omega_1+n\omega_2}\right) e^{\frac{u}{m\omega_1+n\omega_2} + \frac{u^2}{2(m\omega_1+n\omega_2)^2}} \right\},$$

related to $\wp(u)$ by the formula

$$\frac{d^2}{du^2} \log \sigma(u) = \wp(u);$$

furthermore, in view of the periodicity properties of $\wp(u)$, one finds

$$\sigma(u+\omega_i) = -e^{\eta_i(u+\frac{1}{2}\omega_i)} \sigma(u) \qquad (i = 1,2),$$

where η_1, η_2 are certain constants. Hence the function

$$e^{-\frac{\eta_2 u^2}{2\omega_2} + \pi i \frac{u}{\omega_2}} \sigma(u)$$

is left invariant under the transformation $u \rightarrow u+\omega_2$, so is a one-valued function $s(t)$ of $t = e^{2\pi i \frac{u}{\omega_2}}$. Reguarding also the behaviour under the transformation $u \rightarrow u+\omega_1$, i.e. $t \rightarrow e^{2\pi i \frac{\omega_1}{\omega_2}} t = rt$, one finds that the zeros of $s(t)$ are given by $t = r^n$ $(n = 0, \pm 1, \pm 2, \ldots)$ and also that $s(t)$ is given by the infinite product

$$s(t) = -\frac{\omega_2}{2\pi i}(1-t) \prod_{n=1}^{\infty} \frac{(1-tr^n)(1-t^{-1}r^n)}{(1-r^n)^2}.$$

Another property of the function $\sigma(u)$ is the fact, that

$$-\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)},$$

in view of the behaviour of $\sigma(u)$ under the above mentioned transformations, is an elliptic function in each of the variables u, v; in fact this function is equal to

$$\wp(u) - \wp(v).$$

Noting the relation $\wp(\frac{1}{2}\omega_i) - \wp(\frac{1}{2}\omega_j) = e_i - e_j$ we are now able to express Δ as a fraction, both terms of which are finite products of certain values of $\sigma(u)$, such as $\sigma(\frac{1}{2}\omega_1)$, $\sigma(\frac{1}{2}\omega_2)$, $\sigma(\frac{1}{2}\omega_2+\frac{1}{2}\omega_3)$, etc. Putting $e^{\pi i \tau} = q$, implying $q^2 = r$, these values correspond with $t = q, 1, q^{-1}$, etc. Carrying out carefully the calculations the result turns out to be astonishing simple; in fact we find the above formula (1.3).

It is desirable to have a more natural proof of (1.3) and more generally a more satisfactory approach to the theory of modular forms without long digressions about functions which, as to their nature and general properties, have little to do with modular forms. In the following I shall avoid completely the use of elliptic functions. Even the fact, that $\Delta = g_2^3 - 27g_3^2$ is a non-vanishing function, shall be proved in an entirely different way. Of course I need a starting-point; as such I chose the theory of theta series, these functions showing themselves modular properties.

In § 2 the transformation formulae for theta functions are deduced and the well known expansions in infinite products are given. Here the calculation of the factor $C = C(\tau)$, leading f.i. to a simple proof of the formula

$$\prod_{n=1}^{\infty} (1-x^n)^3 = \sum_{n=0}^{\infty} (-1)^n(2n+1)x^{\frac{1}{2}n(n+1)},$$

is performed in a direct way. Thereafter two functions $f_2(\tau)$, $h(\tau)$, closely related to g_2, Δ, are constructed by means of theta series.

In the next section the modular group is introduced in a geometrical way. The definition and some well known theorems on modular forms and functions are given, e.g. concerning the class of integral modular forms of a given negative dimension; the modular invariant $J(\tau)$ is defined in terms of $f_2(\tau)$, $h(\tau)$.

In § 4 I establish the connection between Eisenstein series, especially G_2 and G_3, and modular forms, by means of the theorems proved in § 3. A generating function for Eisenstein series is found in $\log \vartheta_{11}(z \mid \tau)$.

2. Theta series.

Let z and τ be complex variables with $\operatorname{Im} \tau > 0$. Then we define

$$\vartheta(z \mid \tau) = \sum_{n=-\infty}^{+\infty} e^{\pi i \tau n^2} e^{2\pi i z n}$$

and more generally for arbitrary integers g, h

$$\vartheta_{gh}(z \mid \tau) = \sum_{n=-\infty}^{+\infty} (-1)^{nh} e^{\pi i \tau (n+\frac{1}{2}g)^2} e^{2\pi i z(n+\frac{1}{2}g)}. \tag{2.1}$$

The series (2.1) is absolutely convergent on account of $\operatorname{Im} \tau > 0$.

Evidently we have the relation

$$\vartheta_{g+2k,h+2j}(z \mid \tau) = (-1)^{hk} \vartheta_{gh}(z \mid \tau) \qquad (k, j \text{ integers}), \tag{2.2}$$

showing that there are essentially only four theta functions, namely $\vartheta_{00}(z \mid \tau)$, $\vartheta_{01}(z \mid \tau)$, $\vartheta_{10}(z \mid \tau)$, $\vartheta_{11}(z \mid \tau)$. The different theta functions are related to each other by the following formula, which is an immediate consequence of the definition (2.1),

$$\vartheta_{gh}(z+\tfrac{1}{2}m\tau+\tfrac{1}{2}l \mid \tau) = e^{\frac{1}{2}\pi i g h} e^{-\frac{1}{4}\pi i \tau m^2} e^{-\pi i z m} \vartheta_{g+m,h+l}(z \mid \tau). \tag{2.3}$$

Of great importance for our purpose are the transformation formulae referring to the variable τ

$$\vartheta_{gh}(z \mid \tau+1) = e^{\frac{1}{4}\pi i g^2} \vartheta_{g,g+h+1}(z \mid \tau) \tag{2.4}$$

$$\vartheta_{gh}(\tfrac{z}{\tau} \mid -\tfrac{1}{\tau}) = \sqrt{-i\tau}\, e^{-\frac{1}{2}\pi i g h} e^{\frac{\pi i z^2}{\tau}} \vartheta_{hg}(z \mid \tau), \tag{2.5}$$

where $\operatorname{Re}(\sqrt{-i\tau})$ is taken positive.

The first formula is verified at once. As to the second, in virtue of Poisson's summation formula

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\pi i n x} f(x)\,dx,$$

we deduce

$$\vartheta_{gh}(\tfrac{z}{\tau}|-\tfrac{1}{\tau}) = \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{2\pi inx}\, e^{\pi ihx}\, e^{-\frac{\pi i}{\tau}(x+\frac12 g)^2}\, e^{2\pi i\frac{z}{\tau}(x+\frac12 g)}\,dx =$$

$$= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{\pi i}{\tau}\{(x+\frac12 g)^2-2z(x+\frac12 g)-2(n+\frac12 h)x\tau\}}\,dx =$$

$$= \sum_{n=-\infty}^{+\infty} e^{-\pi ig(n+\frac12 h)}\int_{-\infty}^{\infty} e^{-\frac{\pi i}{\tau}\{(x+\frac12 g)-z-(n+\frac12 h)\tau\}^2}\, e^{+\frac{\pi i}{\tau}\{z+(n+\frac12 h)\tau\}^2}\,dx=$$

$$= e^{-\frac12\pi igh}\, e^{\frac{\pi i}{\tau}z^2}\sum_{n=-\infty}^{+\infty}(-1)^{gn}\, e^{\pi i\tau(n+\frac12 h)^2}\, e^{2\pi iz(n+\frac12 h)}\int_{-\infty}^{\infty} e^{-\frac{\pi i}{\tau}x^2}\,dx =$$

$$= \sqrt{-i\tau}\; e^{-\frac12\pi igh}\, e^{\frac{\pi i}{\tau}z^2}\,\vartheta_{hg}(z|\tau).$$

A consequence of (2.3) is the fact that $\vartheta_{gh}(z|\tau)$ is a quasi-periodic function in z with periods $1,\tau$

$$\vartheta_{gh}(z+1|\tau) = (-1)^g\,\vartheta_{gh}(z|\tau),$$
$$\vartheta_{gh}(z+\tau|\tau) = (-1)^h\, e^{-\pi i\tau}\, e^{-2\pi iz}\,\vartheta_{gh}(z|\tau).$$

Hence, integrating in the positive sense along the sides of a parallelogram π with vertices $\zeta,\zeta+1,\zeta+1+\tau,\zeta+\tau$, where ζ is chosen in the z-plane in such a way that $\vartheta_{gh}(z|\tau)\neq 0$ on the boundary of π and denoting differentiation with respect to z by "'", we find

$$\frac{1}{2\pi i}\int_{\pi}\frac{\vartheta'_{gh}(z|\tau)}{\vartheta_{gh}(z|\tau)}dz = -\frac{1}{2\pi i}\int_{\zeta}^{\zeta+1}\left\{\frac{\vartheta'_{gh}(z+\tau|\tau)}{\vartheta_{gh}(z+\tau|\tau)}-\frac{\vartheta'_{gh}(z|\tau)}{\vartheta_{gh}(z|\tau)}\right\}dz +$$
$$+\frac{1}{2\pi i}\int_{\zeta}^{\zeta+\tau}\left\{\frac{\vartheta'_{gh}(z+1|\tau)}{\vartheta_{gh}(z+1|\tau)}-\frac{\vartheta'_{gh}(z|\tau)}{\vartheta_{gh}(z|\tau)}\right\}dz = 1.$$

Henceforth the function $\vartheta_{gh}(z|\tau)$ possesses exactly one zero in π.

Now we direct our attention especially towards the function $\vartheta_{01}(z|\tau)$. Clearly we have $\vartheta_{01}(\frac12\tau|\tau) = 0$. So the zeros of $\vartheta_{01}(z|\tau)$ are given by $m+(n+\frac12)\tau$ with $m,n = 0,\pm1,\pm2,\ldots$ This leads to the consideration of the (absolutely convergent) infinite product

$$\varphi(z,\tau) = \prod_{n=1}^{\infty}(1-e^{2\pi i\tau(n-\frac12)}e^{2\pi iz})(1-e^{2\pi i\tau(n-\frac12)}e^{-2\pi iz}),$$

this product representing a function with the same zeros. The quotient $\vartheta_{01}(z|\tau)\,\{\varphi(z,\tau)\}^{-1}$ is an analytic function. Moreover, as is easily verified, this quotient is left invariant under the transformations $z\rightarrow z+1$ and $z\rightarrow z+\tau$. Hence it only depends on τ. This means that there exists a function $C(\tau)$, such that

$$(2.6)\qquad \vartheta_{01}(z|\tau) = C(\tau)\prod_{n=1}^{\infty}(1-e^{2\pi i\tau(n-\frac12)}e^{2\pi iz})(1-e^{2\pi i\tau(n-\frac12)}e^{-2\pi iz});$$

or putting $e^{\pi i\tau} = q$,

$$(2.7)\qquad \vartheta_{01}(z|\tau) = C(\tau)\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2\pi z+q^{4n-2}).$$

We proceed to determine the factor $C(\tau)$. Each function $\vartheta_{gh}(z|\tau)$ satisfies the differential equation

$$\frac{\partial}{\partial\tau}\vartheta_{gh}(z|\tau) = \frac{1}{4\pi i}\frac{\partial^2}{\partial z^2}\vartheta_{gh}(z|\tau).$$

So in particular we find, in view of $\frac{\partial}{\partial z}(1-2q^{2n-1}\cos 2\pi z+q^{4n-2})_{z=0} = 0$,

$$\begin{aligned}\frac{\partial}{\partial\tau}\log\vartheta_{01}(0|\tau) &= \frac{1}{4\pi i}\left[\frac{1}{\vartheta_{01}(z|\tau)}\frac{\partial^2}{\partial z^2}\vartheta_{01}(z|\tau)\right]_{z=0} = \\ &= \frac{1}{4\pi i}\left[\sum_{n=1}^{\infty}\frac{+8\pi^2 q^{2n-1}\cos 2\pi z}{1-2q^{2n-1}\cos 2\pi z+q^{4n-2}}\right]_{z=0} = \\ &= -2\pi i\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2} = \\ &= -2\pi i\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} + 2\pi i\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}.\end{aligned}$$

By the last formula already $C(\tau)$ is determined. In order to find a neat expression we proceed as follows. On account of

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \qquad (|x|<1)$$

we deduce

$$\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} nx^{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n(x^m)^n = \sum_{m=1}^{\infty}\frac{x^m}{(1-x^m)^2}.$$

Hence we find

$$\begin{aligned}\frac{\partial}{\partial\tau}\log\vartheta_{01}(0|\tau) &= \pi i\left\{2\sum_{n=1}^{\infty}\frac{-nq^n}{1-q^n} - \sum_{n=1}^{\infty}\frac{-2nq^{2n}}{1-q^{2n}}\right\} = \\ &= \frac{\partial}{\partial\tau}\log\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{1-q^{2n}}.\end{aligned}$$

Since for $q\to 0$, i.e. $\tau\to i\infty$, by (2.1) the function $\vartheta_{01}(0|\tau)$ tends to 1, we may conclude

$$\vartheta_{01}(0|\tau) = C(\tau)\prod_{n=1}^{\infty}(1-q^{2n-1})^2 = \prod_{n=1}^{\infty}\frac{(1-q^n)^2}{1-q^{2n}},$$

hence

$$C(\tau) = \prod_{n=1}^{\infty}\left\{\frac{(1-q^n)^2}{1-q^{2n}}\frac{1}{(1-q^{2n-1})^2}\right\} = \prod_{n=1}^{\infty}\left\{\frac{(1-q^n)^2}{1-q^{2n}}\left(\frac{1-q^{2n}}{1-q^n}\right)^2\right\}.$$

So we have

(2.8) $$C(\tau) = \prod_{n=1}^{\infty}(1-q^{2n}).$$

(2.9) $$\vartheta_{01}(z|\tau) = \prod_{n=1}^{\infty}\left\{(1-q^{2n})(1-q^{2n-1}e^{2\pi iz})(1-q^{2n-1}e^{-2\pi iz})\right\}.$$

Applying (2.3) with $g = 0$, $h = 1$; $m = 1$, $n = 0$, we deduce

$$\vartheta_{01}(z+\tfrac{1}{2}\tau|\tau) = e^{-\frac{1}{4}\pi i\tau}e^{-\pi iz}\vartheta_{11}(z|\tau),$$

(2.10) $$\begin{cases}\vartheta_{11}(z|\tau) = q^{\frac{1}{4}}e^{\pi iz}\prod_{n=1}^{\infty}\left\{(1-q^{2n})(1-q^{2n}e^{2\pi iz})(1-q^{2n-2}e^{-2\pi iz})\right\} = \\ \qquad = q^{\frac{1}{4}}(e^{\pi iz}-e^{-\pi iz})\prod_{n=1}^{\infty}\left\{(1-q^{2n})(1-q^{2n}e^{2\pi iz})(1-q^{2n}e^{-2\pi iz})\right\}.\end{cases}$$

Applying (2.3) with $g = 0$, $h = 1$ and $m = 0$, $n = -1$ or $m = 1$, $n = -1$, we find in the same way

(2.11) $$\vartheta_{00}(z|\tau) = \prod_{n=1}^{\infty}\left\{(1-q^{2n})(1+q^{2n-1}e^{2\pi iz})(1+q^{2n-1}e^{-2\pi iz})\right\}$$

(2.12) $$\vartheta_{10}(z|\tau) = q^{\frac{1}{4}}(e^{\pi iz}+e^{-\pi iz})\prod_{n=1}^{\infty}\left\{(1-q^{2n})(1+q^{2n}e^{2\pi iz})(1+q^{2n}e^{-2\pi iz})\right\}.$$

Some well known formulae are an easy consequence of our considerations. First we have, by (2.10) and (2.1) respectively,

$$\vartheta'_{11}(0|\tau) = 2\pi iq^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{2n})^3$$

and

$$\vartheta'_{11}(0|\tau) = \frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty}(-1)^n\, q^{(n+\frac{1}{2})^2}\; 2\,\sin(2\pi z(n+\tfrac{1}{2}))\right]_{z=0} =$$

$$= 2\pi iq^{\frac{1}{4}}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)},$$

yielding

(2.13) $$\prod_{n=1}^{\infty}(1-q^{2n})^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)}. \qquad (|q|<1)$$

Secondly by (2.9), (2.11), (2.12) we have

$$\vartheta_{00}(0|\tau)\,\vartheta_{01}(0|\tau)\,\vartheta_{10}(0|\tau) =$$

$$= 2q^{\frac{1}{4}}\prod_{n=1}^{\infty}\left\{(1-q^{2n})(1-q^{2n-1})^2(1-q^{2n})(1+q^{2n-1})^2(1-q^{2n})(1+q^{2n})^2\right\} =$$

$$= 2q^{\frac{1}{4}}\prod_{n=1}^{\infty}\left\{(1-q^{2n})^3(1-q^{2n-1})^2(1+q^n)^2\right\} = 2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{2n})^3,$$

since

$$\prod_{n=1}^{\infty}\left\{(1-q^{2n-1})(1+q^n)\right\} = \prod_{n=1}^{\infty}\left\{\frac{1-q^n}{1-q^{2n}}\,\frac{1-q^{2n}}{1-q^n}\right\} = 1;$$

so we find

(2.14) $$\vartheta'_{11}(0|\tau) = \pi i\,\vartheta_{00}(0|\tau)\,\vartheta_{01}(0|\tau)\,\vartheta_{10}(0|\tau).$$

We are now in a position to construct two functions which are of the utmost importance for our further treatment. Since from now on we need theta functions only with $z = 0$, for brevity we put

(2.15) $$\vartheta_{gh}(0|\tau) = \vartheta_{gh}(\tau) \qquad (g,h = 0,0;\ 0,1;\ 1,0;\ 1,1).$$

We define

(2.16) $$24f_2(\tau) = \vartheta_{00}^8(\tau)+\vartheta_{01}^8(\tau)+\vartheta_{10}^8(\tau)$$

(2.17) $$256h(\tau) = \vartheta_{00}^8(\tau)\,\vartheta_{01}^8(\tau)\,\vartheta_{10}^8(\tau).$$

The behaviour of the functions $f_2(\tau)$, $h(\tau)$ under the transformations $\tau\rightarrow\tau+1$ and $\tau\rightarrow-\frac{1}{\tau}$ easily can be deduced from the formulae (2.4), (2.5), (2.2). For we have

$$\begin{cases}\vartheta_{00}(\tau+1) = \vartheta_{01}(\tau) & \vartheta_{00}(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\vartheta_{00}(\tau)\\ \vartheta_{01}(\tau+1) = \vartheta_{00}(\tau) & \vartheta_{01}(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\vartheta_{10}(\tau)\\ \vartheta_{10}(\tau+1) = e^{\frac{1}{4}\pi i}\vartheta_{10}(\tau) & \vartheta_{10}(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\vartheta_{01}(\tau).\end{cases}$$

Hence we find

(2.18) $$f_2(\tau+1) = f_2(\tau), \quad f_2(-\tfrac{1}{\tau}) = \tau^4 f_2(\tau),$$

$$(2.19) \qquad h(\tau+1) = h(\tau), \quad h(-\tfrac{1}{\tau}) = \tau^{12}h(\tau).$$

Finally, inspecting the proof of (2.14), we find for $h(\tau)$ the following infinite product

$$(2.20) \qquad h(\tau) = q^2 \prod_{n=1}^{\infty}(1-q^{2n})^{24} \qquad (q = e^{\pi i\tau}).$$

This shows that $h(\tau)$ is a regular function, different from zero for finite τ.

3. The modular group. Modular forms.

Let ω_1, ω_2 be two complex numbers with

$$(3.1) \qquad \omega_1 \neq 0, \quad \omega_2 \neq 0, \quad \operatorname{Im}\frac{\omega_1}{\omega_2} > 0.$$

Plotting down the numbers in the complex plane, the numbers $m\omega_1+n\omega_2$ (m,n integers) form a lattice, Λ say, for which ω_1, ω_2 constitute a basis.

Let ω_1', ω_2' be two complex numbers with $\omega_1' \neq 0$, $\omega_2' \neq 0$, $\operatorname{Im}\omega_1'/\omega_2' > 0$, which generate the same lattice Λ. Then, since ω_1', ω_2' themselves are lattice points, we can write

$$\omega_1' = \alpha\,\omega_1+\beta\omega_2, \qquad \omega_2' = \gamma\,\omega_1+\delta\omega_2,$$

where $\alpha, \beta, \gamma, \delta$ are integers. On the other hand, since ω_1', ω_2' generate the lattice Λ, ω_1 and ω_2 are expressible in terms of ω_1' and ω_2' in an analoguous manner. So we have $\alpha\delta-\beta\gamma = \pm 1$; the minus sign can be rejected on account of $\operatorname{Im}\omega_1/\omega_2$, $\operatorname{Im}\omega_1'/\omega_2' > 0$.

The transformations

$$(3.2) \qquad \omega_1' = \alpha\,\omega_1+\beta\omega_2, \qquad \omega_2' = \gamma\,\omega_1+\delta\omega_2,$$

where $\alpha, \beta, \gamma, \delta$ are integers with $\alpha\delta-\beta\gamma = 1$, evidently form a group, Γ say. Putting $\tau = \omega_1/\omega_2$, $\tau' = \omega_1'/\omega_2'$, this group Γ induces a group $\overline{\Gamma}$ of linear fractional substitutions $\tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$. The mapping $\Gamma \rightarrow \overline{\Gamma}$ is a homomorphism with a kernel of order 2: we have $\tau = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$ for all τ with $\operatorname{Im}\tau > 0$, if and only if $\beta = \gamma = 0$, $\alpha = \delta = +1$ or -1.

We call two numbers τ, τ' in the upper halfplane equivalent, if and only if there exists a relation

$$(3.3) \qquad \tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta} \qquad (\alpha, \beta, \gamma, \delta \text{ integers with } \alpha\delta-\beta\gamma = 1).$$

A fundamental region for the group $\overline{\Gamma}$ of transformations (3.3), operating in the upper half of the τ-plane, is obtained easily. Given a number τ chose ω_1, ω_2, satisfying (3.1) and the relation $\tau = \omega_1/\omega_2$ (e.g. $\omega_2 = 1$, $\omega_1 = \tau$). Determine a basis ω_1', ω_2' of the corresponding lattice Λ by the following procedure:

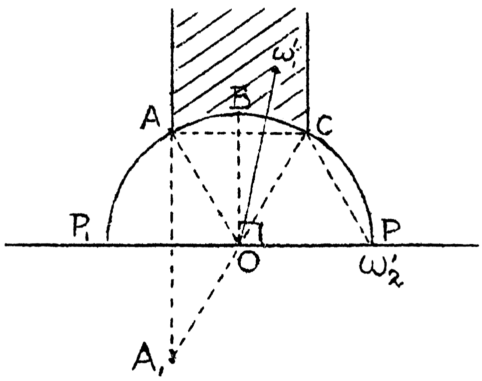
1. chose ω_2', such that $|\omega_2'|$ is minimal,
2. thereafter, out of the set of numbers ω_1'', generating together with ω_2' the lattice Λ, chose ω_1' so as to satisfy

$$(3.4) \qquad -\tfrac{1}{2} \leq \operatorname{Re}(\omega_1'/\omega_2') < \tfrac{1}{2}.$$

The definition of ω_1', ω_2' is unique, unless we have $|\omega_1'| = |\omega_2'|$. In that case, replacing ω_2', ω_1' by $\omega_1', -\omega_2'$ (this being in fact an integral unimodular transformation) if needed, we can fulfill the additional requirement

3. $\mathrm{Re}(\omega_1'/\omega_2') \leq 0$.

The choice of ω_1', ω_2' is still ambiguous if $\omega_1' = i\omega_2'$ or if $\omega_1' = \rho\,\omega_2'$ ($\rho = -\frac{1}{2}+\frac{1}{2}i\sqrt{3}$, the point of intersection of $\mathrm{Re}\,\tau = -\frac{1}{2}, |\tau| = 1$). In the first case we may replace ω_2', ω_1' by $\omega_1', -\omega_2' = i\omega_1'$ ($P,B \to B,P_1$); both choices lead to the same value $\tau = i$. In the second case we may replace ω_2', ω_1' by $\omega_1', -(\omega_1'+\omega_2')$ or by $-(\omega_1'+\omega_2'), \omega_2'$ ($P,A \to A,A_1$ or $P,A \to A_1,P$); all three choices lead to the same value $\tau = \rho$. We have the result: to each number τ there corresponds exactly one equivalent number in the region



(3.5) $G:\ -\frac{1}{2} \leq \mathrm{Re}\,\tau < \frac{1}{2},\ |\tau| \begin{cases} \geq 1 & \text{if } \mathrm{Re}\,\tau \leq 0 \\ > 1 & \text{if } \mathrm{Re}\,\tau > 0; \end{cases}$

notwithstanding for some $\tau \in G$ there exists a non-identical transformation L of the form (3.3) with $L(\tau) = \tau \in G$, namely for $\tau = i$ the transformation $\tau' = -\frac{1}{\tau}$ and for $\tau = \rho$ the transformations $\tau' = -\frac{\tau+1}{\tau}$ and $\tau' = -\frac{1}{\tau+1}$.

The region G is the required fundamental region; the transforms of G by all possible transformations (3.3) cover the upper halfplane completely and without overlappings (each image of i or ρ occurring two and three times respectively). If S and T are the transformations $\tau' = -\frac{1}{\tau}$ and $\tau' = \tau+1$ respectively, then evidently the transforms of G by S, T, T^{-1} are the only neighbouring regions of G. We shall prove now that S,T generate the group Γ.

First we remark that the set of images of the points i and ρ has no points of accumulation above the real axis (if τ_0 were such a point, then also the equivalent point in G). Hence we can join an arbitrary given point τ with $\mathrm{Im}\,\tau > 0$ withnthe equivalent point τ' in G by a path, which avoids the images of i and ρ, and which only passes through a finite number of transformed regions. If a certain region is obtained on applying to G a transformation $V \in \Gamma$, then the three neighbouring regions are found if we apply the transformations VS, VT, VT^{-1} on G. Hence there exists a transformation, built up from the transformations S, T, T^{-1}, which transforms τ' into τ, c.q. G into a region which contains τ.

Summarizing we have found that the group Γ, consisting of the transformations (3.3), has G, given by (3.5) for a fundamental region, and is generated by the transformations $\tau' = -\frac{1}{\tau}$, $\tau' = \tau \pm 1$. The group Γ is called the modular group; the transformations (3.3) are called modular transformations.

Definition. A modular function is a function $f(\tau)$, which is not identical zero and which possesses the following properties.

1. $f(\tau)$ is regular for $\mathrm{Im}\,\tau > 0$, except for poles,

2. $f(\tau)$ is invariant for the modular transformations; in particular $f(\tau)$ is a one-valued function $g(r)$ of $r = e^{2\pi i\tau} = q^2$,

3. the function $g(r)$ is regular or has a pole at $r = 0$, i.e. $\tau = i\infty$.

Definition. A modular form is a function $f(\tau)$, which is not identical zero and which possesses the following properties.

1'. $f(\tau)$ is regular for $\mathrm{Im}\,\tau > 0$, except for poles,

2'. there exists a real number $-k$, which is called the dimension of $f(\tau)$, such that for each modular transformation (3.3) we have

$$(3.6) \qquad f\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^k\, f(\tau),$$

implying that $f(\tau)$ is a one-valued function $g(r)$ of $r = e^{2\pi i\tau} = q^2$ [1]),

3'. the function $g(r)$ is regular or has a pole at $r = 0$, i.e. $\tau = i\infty$.

Definition. A modular form $f(\tau)$ is called an integral modular form, if $f(\tau)$ is regular for each τ with $\mathrm{Im}\,\tau > 0$, and if also $g(r)$ is regular for $r = 0$.

If $f(\tau)$ is a modular form with dimension $-k$, then consider the function $F(\omega_1,\omega_2) = \omega_2^{-k} f(\tau)$, where ω_1, ω_2 are non-vanishing complex numbers with $\tau = \omega_1/\omega_2$. Let $\alpha, \beta, \gamma, \delta$ be integers with $\alpha\delta - \beta\gamma = 1$. Then on account of (3,6) we have

$$F(\alpha\omega_1+\beta\omega_2, \gamma\omega_1+\delta\omega_2) = (\gamma\omega_1+\delta\omega_2)^{-k} f\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) =$$
$$= (\gamma\omega_1+\delta\omega_2)^{-k}(\gamma\tau+\delta)^k f(\tau) = \omega_2^{-k} f(\tau) = F(\omega_1,\omega_2).$$

Hence the property 2' is equivalent with

2". the function $F(\omega_1,\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2)$ is invariant for the transformations (3.2).

Definition. If $f(\tau)$ is a modular form of dimension $-k$, then $F(\omega_1,\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2)$ is also called a modular form of dimension $-k$.

Theorem 1. The dimension of a modular form is integral and even. If a function $f(\tau)$ satisfies 2' with even k for the transformations S and T, and if moreover 1' and 3' hold, then $f(\tau)$ is a modular form of dimension $-k$.

Proof. Applying (3.6) with $\beta = \gamma = 0$, $\alpha = \delta = -1$, we obtain $f(\tau) = (-1)^k f(\tau)$. Since $f(\tau)$ is not identical zero, this implies that k is even. This already proves the first part of the theorem.

Now let V be a modular transformation $\tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$. Then V is a finite product of factors S, T, T^{-1}. To each transformation V, S, T, T^{-1} there correspond two transformations, say V^*, S^*, T^*, $(T^{-1})^*$ respectively, of the type (3.2) for the pair ω_1, ω_2. Also V^* is a product of factors S^*, T^*, $(T^{-1})^*$. Now consider the function $F(\omega_1,\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2)$. This function is invariant for the transformations S^*, T^*, $(T^{-1})^*$, by our assumptions, hence also for the transformation V^*. So $F(\omega_1,\omega_2)$ possesses the property 2". This proves the last part of the theorem.

---

1) For we have $f(\tau+\beta) = f(\tau)$ for each integer β.

Theorem 2. The functions $f_2(\tau)$ and $h(\tau)$, defined in § 2 by (2.16), (2.17), are integral modular forms of dimensions -4, -12 respectively. The quotient

$$(3.7) \qquad J(\tau) = \{f_2(\tau)\}^3 \{h(\tau)\}^{-1}$$

is a modular function, which in the fundamental region G takes each value in exactly one point.

The first part of the theorem follows from the relations (2.18), (2.19) and theorem 1; the regularity finite τ is immediate and the regularity in $r \neq 0$ follows, since $f_2(\tau)$, $h(\tau)$ are regular for $2 \neq 0$ and continuous for $r \to 0$.

By the remark at the end of § 2 the function $J(\tau)$ is regular for finite τ. From (2.18), (2.19), (3.7) it follows that $J(\tau)$ is a modular function. Inspecting the formulae (2.1) or (2.9) - (2.12), we see, that for $\tau \to i\infty$, i.e. $r \to 0$, the functions $\vartheta_{00}(\tau)$, $\vartheta_{01}(\tau)$, $\vartheta_{10}(\tau)$ behave as follows:

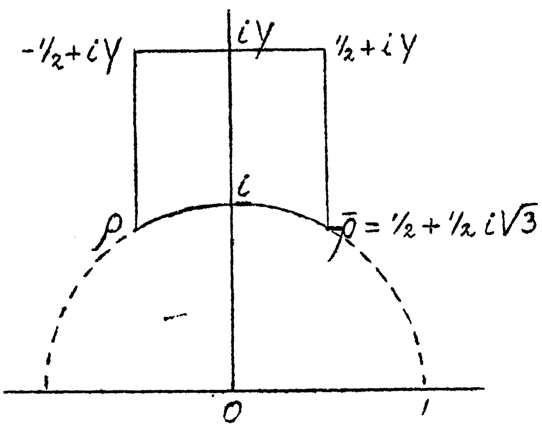
$$\vartheta_{00}(\tau) \sim 1, \quad \vartheta_{01}(\tau) \sim 1, \quad \vartheta_{10}(\tau) \sim 2q^{\frac{1}{4}} = 2e^{\frac{1}{4}\pi i\tau},$$

hence

$$24f_2(\tau) \sim 2, \quad 256h(\tau) \sim 2^8 r, \quad J(\tau) \sim \frac{1}{1728r}.$$

So $J(\tau)$ has a simple pole at $\tau = i\infty$, measured in the parameter r.

Let a be an arbitrary complex number $\neq J(i), J(\rho)$. In view of $\lim_{\tau \to i\infty} J(\tau) = \infty$ we can chose a real number $Y > 1$, such that we have $J(\tau) \neq a$ for $\mathrm{Im}\,\tau \geq Y$, i.e. $|r| \leq e^{-2\pi Y}$. Let in the τ-plane $R = R(Y)$ be a path, joining successively i, $-\bar{\rho} = \frac{1}{2}+\frac{1}{2}i\sqrt{3}$, $\frac{1}{2}+iY$, $-\frac{1}{2}+iY$, ρ, i



along the unit-circle and the straight lines $\mathrm{Re}\,\tau = \pm\frac{1}{2}$, $\mathrm{Im}\,\tau = Y$. If R contains some points τ with $J(\tau) = a$ - since R consists of pairwise equivalent segments and since $a \neq J(i), J(\rho)$, such points always occur in pairs of equivalent points - then in the neighbourhood of such "bad" points modify R by inserting pairs of equivalent small circles avoiding these points. Now we have $J(\tau)-a \neq 0$ on R, whereas the number N of zeros of $J(\tau)-a$, each zero being counted according to its multiplicity, in the interior of R is given by

$$N = \frac{1}{2\pi i} \int_R \frac{J'(\tau)}{J(\tau)-a}\, d\tau = \frac{1}{2\pi i} \int_R d \log\{J(\tau)-a\}.$$

Applying successively the substitutions $\tau' = \tau+1$, $\tau' = -\frac{1}{\tau}$, $r = e^{2\pi i\tau}$, we find for the different parts of this integral

$$\int_{\rho}^{-\frac{1}{2}+iY} d \log\{J(\tau)-a\} = \int_{-\bar{\rho}}^{+\frac{1}{2}+iY} d \log\{J(\tau)-a\},$$

$$\int_{\rho}^{i} d \log\{J(\tau)-a\} = \int_{-\bar{\rho}}^{i} d \log\{J(\tau)-a\},$$

$$\int_{\frac{1}{2}+iY}^{-\frac{1}{2}+iY} d\log\{J(\tau)-a\} = \int_C d\log\{g(r)-a\},$$

where $g(r)$ is defined by $g(e^{2\pi i\tau}) = J(\tau)$ and where the last integral is taken along the circle $C: |r| = e^{-2\pi Y}$ in the negative sense. In view of the simple pole of $g(r)$ at $r = 0$ we get $N = 1$. Now let D, E be small circles with radius ε, surrounding ρ, i respectively, D_1 the part of D contained in G, E_1 the part of E contained in G and D_2 the reflection of D_1 in the imaginary axis. Let a be equal to $J(\rho)$ or $J(i)$ and let $J(\tau)-a$ possess a p-tuple zero at $\tau = \rho$ (hence also at $\tau = -\bar{\rho}$) and a q-tuple zero at $\tau = i$. Noting the angles of G at the points ρ, i, $-\bar{\rho}$, we see (the circles being described in the negative sense)

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{D_1} \frac{J'(\tau)}{J(\tau)-a}\,d\tau = \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{D_2} \frac{J'(\tau)}{J(\tau)-a}\,d\tau = -\tfrac{1}{6}p,$$

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{E_1} \frac{J'(\tau)}{J(\tau)-a}\,d\tau = -\tfrac{1}{2}q.$$

By the above method we find

$$N = \frac{1}{2\pi i}\int_C d\log\{g(r)-a\} + \frac{1}{2\pi i}\int_{D_1+E_1+D_2} \frac{J'(\tau)}{J(\tau)-a}\,d\tau,$$

hence $\frac{1}{6}p+\frac{1}{2}q+\frac{1}{6}p = 1-N$. If $a = J(\rho)$, then we have $p \geq 3$ (see below), so $p=3$, $q = 0$, $N = 0$; if $a = J(i)$, we find $p = 0$, $q = 2$, $N = 0$. So we have found that for each complex number a there exists exactly one point $\tau \in G$ with $J(\tau) = a$. This proves the theorem.

If τ_0 is the zero of $J(\tau)$ in G, then on account of

$$J(\tau) = \{f_2(\tau)\}^3\{h(\tau)\}^{-1}, \quad h(\tau) \neq 0$$

we see that τ_0 is at least a triple zero of $J(\tau)$. Regarding the last part of the proof of theorem 2 and putting $J(i) = b$ we may conclude:

(3.8) $J(\rho) = 0$, $b = J(i) \neq 0$; ρ <u>is a triple zero of</u> $J(\tau)$;
i <u>is a double zero of</u> $J(\tau)-b$.

Since $J(\tau)-b$ has only double zeros and $h(\tau)$ has only a single zero at $\tau \to i\infty$, the function $\sqrt{\{J(\tau)-b\}h(\tau)}$ is regular for finite τ ($\mathrm{Im}\,\tau > 0$) and also for $\tau \to i\infty$. Moreover $\{J(\tau)-b\}h(\tau) \to \frac{1}{1728}$ if $\tau \to i\infty$. Hence the function $f_3(\tau)$, defined by [2])

(3.9) $\sqrt{27}\, f_3(\tau) = \sqrt{\{J(\tau)-b\}h(\tau)}$, $f_3(\tau)$ positive at $\tau = i\infty$,

is determined uniquely. The function $f_3^2(\tau)$ is an integral modular form of dimension -12. Describing the boundary of G from $\tau = i$ onwards in both directions, we see that $f_3(\tau)$ has the same value in equivalent points of the boundary of G. Thus we find

(3.10) $f_3(\tau)$ <u>is an integral modular form of dimension -6</u>.

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2) In § 4 we again shall find, in an independent way, an integral modular form of dimension -6, which referring to the proof of theorems 3 and 4 can serve the same purpose as $f_3(\tau)$.

Theorem 3. A modular function is a rational function of $J(\tau)$.

Proof. Let J_0 be an arbitrary complex number. The function $J(\tau)$ assumes the value J_0 in a set of equivalent points τ_0. In each of these points τ_0 the modular function $f(\tau)$ has the same value, $\bar{f}(J_0)$ say. Then $\bar{f}(J_0)$ is determined uniquely by J_0, whereas for each τ_0 of the above set we have $f(\tau_0) = \bar{f}(J(\tau_0))$. Thus $f(\tau)$ is a one-valued function $\bar{f}(J)$ of J.

Now assume $J_0 \neq 0$, b, ∞. Let τ_0 be a point with $J(\tau_0) = J_0$. In the neighbourhood of τ_0, in virtue of theorem 2, we can invert the function $J(\tau)$ into a regular function $\tau = \tau(J)$. Since we have $\bar{f}(J) = f(\tau(J))$ and $f(\tau)$ (or $\frac{1}{f(\tau)}$ if τ_0 is a pole of $f(\tau)$) is regular, we find that $\bar{f}(J)$, c.q. $\frac{1}{\bar{f}(J)}$, is regular at J_0. In the points 0, b, ∞ the (arbitrarily chosen) branches $\tau(J)$ and $f(\tau)$, hence also $\bar{f}(J)$ are continuous (or the reciprocals of these functions, as the case may be). We conclude, that at each point of the J-plane, finite or infinite, the function $\bar{f}(J)$ or its reciprocal is regular. Hence, by a well known result in the theory of complex functions, $\bar{f}(J)$ is a rational function. This proves the theorem.

Theorem 4. The integral modular forms of negative dimension -k constitute a ν-dimensional linear set of functions, where ν is given by

(3.11) $$\nu = \left[\frac{k}{12}\right] \text{ if } k \equiv 2 (\text{mod } 12); \quad \nu = \left[\frac{k}{12}\right] + 1 \text{ if } k \not\equiv 2 (\text{mod } 12).$$

Proof. We know already that k is an even integer. The six functions $f_2^{\mu}(\tau) f_3^{\lambda}(\tau)$ with $\mu = 0, 1$ or 2 and $\lambda = 0$ or 1, evidently are modular forms of dimension $4\mu + 6\lambda = 0, -4, -6, -8, -10, -14$. Multiplying a given modular form $f(\tau)$ of dimension -k with a suitable function $f_2^{\mu}(\tau) f_3^{\lambda}(\tau)$ we obtain a modular form of dimension $-k_1 = -(k + 4\mu + 6\lambda)$ with $k_1 \equiv 0 (\text{mod } 12)$. Then the function

$$f(\tau) f_2^{\mu}(\tau) f_3^{\lambda}(\tau) \left\{ h(\tau) \right\}^{-\frac{1}{12}k_1}$$

is a modular <u>function</u>, which is regular for finite τ and which has a $\frac{1}{12}k_1$-tuple pole at $\tau = i\infty$ (measured in the parameter r). Applying theorem 3 we find that this function is expressible as a polynomial in $J(\tau)$ of degree $\frac{1}{12}k_1$. Or, otherwise stated,

(3.12) $$f(\tau) f_2^{\mu}(\tau) f_3^{\lambda}(\tau) = \sum b_{lm} f_2^{l}(\tau) f_3^{m}(\tau),$$

where b_{lm} are constants and l, m are non-negative integers with $4l + 6m = k_1$. By (3.7), (3.8) and theorem 2 we know

$$f_2(\rho) = 0, \quad f_2(i) \neq 0;$$

by (3.9), (3.10) and theorem 2 we see

$$f_3(\rho) \neq 0, \quad f_3(i) = 0.$$

Hence, if $\mu > 0$, then the left hand member of (3.12) vanishes for $\tau = \rho$, hence also the right hand member. So the only possible term in the right hand member of (3.12) with $l = 0$ actually does not occur, i.e. this sum contains a factor $f_2(\tau)$; hence in (3.12) we can delete a factor $f_2(\tau)$. Repeating the argument if $\mu > 1$, or $\lambda > 0$ (in the last case with $f_3(\tau)$, i instead of $f_2(\tau)$, ρ), we find

(3.13) $f(\tau) = \sum c_{lm} \, f_2^l(\tau) f_3^m(\tau)$,

where c_{lm} are constants and l, m are non-negative integers with $4l+6m = k$.

This proves the theorem with a non-negative integer ν, equal to the number of solutions of the diophantine equation $4l+6m = k$; from the last fact (3.12) is an easy consequence. In particular there does not exist an integral modular form of dimension -2, whereas there exists essentially one integral modular form of dimension $-k$, if k has one of the values 4, 6, 8, 10, 14.

4. A generating function for modular forms. Eisenstein series.

In the foregoing section the theory of elementary modular forms was based upon the expressions (2.16), (2.17). However the connection between theta functions and modular forms is not exhausted by the considerations of § 3. We expand $\log \vartheta_{11}(z|\tau)$ as a power series in z [3]), starting from the infinite product for $\vartheta_{11}(z|\tau)$ (see (2.10)). We have

$$\log \vartheta_{11}(z|\tau) = \log 2 + \frac{\pi}{2}i + \frac{1}{8}\log r + \log \sin \pi z +$$
$$+ \sum_{n=1}^{\infty} \left\{ \log(1-r^n) + \log(1-r^n e^{2\pi i z}) + \log(1-r^n e^{-2\pi i z}) \right\}.$$

In view of

$$\log \sin \pi z = \log \pi z - \sum_{k=1}^{\infty} \frac{1}{k} s_{2k} \, z^{2k} \quad (0<|z|<1; \; s_{2k} = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}),$$

$$\log(1-r^n e^{2\pi i z}) + \log(1-r^n e^{-2\pi i z}) = -\sum_{m=1}^{\infty} \left\{ \frac{1}{m} r^{mn} e^{2m\pi i z} + \frac{1}{m} r^m e^{-2m\pi i z} \right\} =$$

$$= -2 \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi z)^{2k}}{(2k)!} \sum_{m=1}^{\infty} m^{2k-1} r^{mn}$$

we find

$$\log \vartheta_{11}(z|\tau) = -\sum_{k=0}^{\infty} \frac{1}{2k} (2\pi)^{2k} c_k(\tau) z^{2k} + \log \pi z \qquad (0<|z|<1), \tag{4.1}$$

where $c_0(\tau)$, $c_1(\tau)$,... are some functions of τ, which for $k \geq 2$ are given by [4])

$$(4.2) \quad \begin{cases} c_k(\tau) = \dfrac{2 s_{2k}}{(2\pi)^{2k}} + 4k \dfrac{(-1)^k}{(2k)!} \displaystyle\sum_{m,n=1}^{\infty} m^{2k-1} r^{mn} = \\ = \dfrac{2 s_{2k}}{(2\pi)^{2k}} + 4k \dfrac{(-1)^k}{(2k)!} \displaystyle\sum_{k=1}^{\infty} \sigma_{2k-1}(h) e^{2h\pi i \tau}, \quad \sigma_{2k-1}(h) = \displaystyle\sum_{m|h} m^{2k-1}. \end{cases}$$

Let us now apply the transformations $z, \tau \rightarrow z, \tau+1$ and $z, \tau \rightarrow \frac{z}{\tau}, -\frac{1}{\tau}$ to both members of (4.1). In view of (2.4) we find

$$-\sum_{k=0}^{\infty} \frac{1}{2k} (2\pi)^{2k} c_k(\tau+1) z^{2k} + \log \pi z = \log \vartheta_{11}(z|\tau+1) =$$

$$= \tfrac{1}{4}\pi i + \log \vartheta_{11}(z|\tau) = \tfrac{1}{4}\pi i - \sum_{k=0}^{\infty} \frac{1}{2k} (2\pi)^{2k} c_k(\tau) z^{2k} + \log \pi z,$$

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3) Cf. B. VAN DER POL, On a non-linear partial differential equation satisfied by the logarithm of the Jacobian theta-functions, with arithmetical applications, Proc.Kon.Ned.Akad.v.W. 54(1951), 261-284.

4) The explicit form of the functions $c_0(\tau)$, $c_1(\tau)$ is without interest for our purposes.

hence, equating coefficients of z^{2k},

(4.3) $\qquad c_k(\tau+1) = c_k(\tau)$ for $k = 1,2,\dots$.

On the other hand (2.5) yields

$$-\sum_{k=0}^{\infty} \frac{1}{2k}(2\pi)^{2k}\, c_k(-\tfrac{1}{\tau})(\tfrac{z}{\tau})^{2k} + \log \pi z - \log \tau =$$

$$= \log \vartheta_{11}(\tfrac{z}{\tau}|-\tfrac{1}{\tau}) = \log \sqrt{-i\tau} - \tfrac{1}{2}\pi i + \pi i \frac{z^2}{\tau} + \log \vartheta_{11}(z|\tau) =$$

$$= \log \sqrt{-i\tau} - \tfrac{1}{2}\pi i + \pi i \frac{z^2}{\tau} - \sum_{k=0}^{\infty} \frac{1}{2k}(2\pi)^{2k}\, c_k(\tau) z^{2k} + \log \pi z,$$

hence

(4.4) $\qquad c_k(-\frac{1}{\tau}) = \tau^{2k}\, c_k(\tau)$ for $k = 2,3,\dots$.

Resuming we have the following result.

For $k \geqslant 2$ the functions $c_k(\tau)$, given by (4.2), are integral modular forms of dimension $-2k$, whereas $\log \vartheta_{11}(z|\tau)$ is a generating function for these forms.

Theorem 5. The following relations exist between the functions $c_2(\tau)$, $c_3(\tau)$, $f_2(\tau)$, $f_3(\tau)$, $h(\tau)$

(4.6) $\qquad f_2(\tau) = 60c_2(\tau)$, $f_3(\tau) = 140c_3(\tau)$,

(4.7) $\qquad f_2^3(\tau) - 27f_3^2(\tau) = h(\tau)$, i.e. $b = J(i) = 1$.

Proof. By (2.18) and (4.5) both functions $f_2(\tau)$, $c_2(\tau)$ are integral modular forms of dimension -4. By theorem 4 there is essentially one integral modular form of dimension -4, so we have $f_2(\tau) = \kappa\, c_2(\tau)$ with a certain constant κ . Regarding the behaviour for $\tau \to i\infty$ we find $\frac{1}{12} = \kappa\, 2s_4(2\pi)^{-4}$, hence on account of

$$s_{2k} = (-1)^{k+1} \frac{1}{2(2k)!}(2\pi)^{2k}\, B_{2k} \qquad (k = 2,3,\dots),$$

$$B_4 = -\frac{1}{30},\quad B_6 = \frac{1}{42},\ \dots$$

we get $\kappa = \frac{1}{12}.\frac{1}{2}.90.2^4 = 60$. Arguing the same way we find $f_3(\tau) = \kappa' c_3(\tau)$ with a certain constant κ'. On account of (3.9) we have $f_3(\tau) \sim \frac{1}{\sqrt{27}}.\frac{1}{\sqrt{1728}} = 2^{-3}.3^{-3}$ for $\tau \to i\infty$, hence $2^{-3}.3^{-3} = \kappa'.2s_6(2\pi)^{-6} = 2\,\frac{1}{2.6!42}$, $\kappa' = 140$. So the relations (4.6) are proved.

By (3.7), (3.9) we have $bh(\tau) = f_2^3(\tau) - 27f_3^2(\tau)$. By (4.2) and (4.6) we have

(4.8) $\qquad f_2(\tau) = \frac{1}{12} + 20\sum_{1}^{\infty} \sigma_3(h)r^h$, $f_3(\tau) = \frac{1}{216} - \frac{7}{3}\sum_{1}^{\infty} \sigma_5(h)r^h$.

Hence

$$f_2^3(\tau) - 27f_3^2(\tau) = (\frac{1}{12^3} - 27.\frac{1}{216^2}) + (3.\frac{1}{12^2}.20 + 27.2.\frac{1}{216}.\frac{7}{3})r + \dots \sim r.$$

In the two-dimensional set of integral modular forms of dimension -12 there occurs exactly one function, which vanishes at $\tau = i\infty$ and moreover behaves as r (cf. theorem 4). Both members of the relation (4.7) have this property, so they are equal.

Formula (1.3) now easily can be proved. Consider the so called <u>Eisenstein series</u> G_k, defined by

(4.9) $$G_k(\omega_1,\omega_2) = \sum{}' (m\omega_1+n\omega_2)^{-2k} \quad (k = 2,3,\ldots),$$

where ω_1, ω_2 satisfy (1.1) and where the summation is extended over all integral pairs $m,n \neq 0,0$. The well known connection with the series $c_k(\tau)$, defined by (4.2), can be deduced as follows. Differentiating the first and the last member of the relation

$$\frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x+n} + \frac{1}{x-n}\right) = \pi\cot\pi x = -\pi i\left(1+2\sum_{n=1}^{\infty} e^{2n\pi i x}\right)$$

2k-1 times with respect to x, one finds

$$\sum_{n=-\infty}^{\infty}(x+n)^{-2k} = \frac{1}{(2k-1)!}(2\pi i)^{2k}\sum_{n=1}^{\infty} n^{2k-1} e^{2n\pi i x},$$

hence, putting $x = m\tau$ and summing over m,

$$\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(m\tau+n)^{-2k} = \frac{(-1)^k}{(2k-1)!}(2\pi)^{2k}\sum_{h=1}^{\infty}\sigma_{2k-1}(h)e^{2h\pi i\tau}.$$

Thus we obtain, in view of $\omega_1/\omega_2 = \tau$,

$$G_k(\omega_1,\omega_2) = \omega_2^{-2k}\left[2\sum_{n=1}^{\infty} n^{-2k} + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(m\tau+n)^{-2k}\right] =$$

$$= \left(\frac{2\pi}{\omega_2}\right)^{2k}\left[\frac{2s_{2k}}{(2\pi)^{2k}} + 4k\frac{(-1)^k}{(2k)!}\sum_{h=1}^{\infty}\sigma_{2k-1}(h)e^{2h\pi i\tau}\right],$$

hence

(4.10) $$G_k(\omega_1,\omega_2) = \left(\frac{2\pi}{\omega_2}\right)^{2k} c_k(\tau).$$

Putting $g_2 = 60G_2$, $g_3 = 140G_3$, $\Delta = g_2^3-27g_3^2$, by (4.10), (4.6), (2.16), (3.9), (4.7), (2.17), (2.20) we obtain

$$\begin{cases} g_2(\omega_1,\omega_2) = \left(\frac{2\pi}{\omega_2}\right)^4 f_2(\tau) = \frac{1}{24}\left(\frac{2\pi}{\omega_2}\right)^4\left\{\vartheta_{00}^8(\tau)+\vartheta_{01}^8(\tau)+\vartheta_{10}^8(\tau)\right\}, \\ g_3(\omega_1,\omega_2) = \left(\frac{2\pi}{\omega_2}\right)^6 f_3(\tau), \\ \Delta(\omega_1,\omega_2) = \left(\frac{2\pi}{\omega_2}\right)^{12} h(\tau) = \left(\frac{2\pi}{\omega_2}\right)^{12} r\prod_{n=1}^{\infty}(1-r^n)^{24}. \end{cases}$$

Theorem 4, applied in the proof of theorem 5, is a source of many other relations between modular forms. For instance each function $c_k(\tau)$ can be expressed as a polynomial in $c_2(\tau)$, $c_3(\tau)$, the coefficients can be determined in a finite number of steps by using the expansions (4.2) and equating a suitable number of coefficients. In view of the nature of the series (4.2) this gives rise to many relations between divisor functions. VAN DER POL in the paper cited above used a certain non-linear partial differential equation for $\log\vartheta_{11}(z|\tau)$ to establish other relations.

<u>Examples</u>.

I. $$-\vartheta_{00}^4(\tau)+\vartheta_{01}^4(\tau)+\vartheta_{10}^4(\tau) = 0.$$

If the left hand member were not identical zero, then on account of (2.16), (2.17) it should be an integral modular form of dimension -2, non identical zero. Since no such form exists, the above relation is established.

II. Putting $c_k^*(\tau) = \frac{(2\pi)^{2k}}{2s_{2k}} c_k(\tau) \quad (k = 2,3,\ldots),$

we have

$$c_2^*(\tau) = 1+240\sum_{h=1}^{\infty} \sigma_3(h)e^{2h\pi i\tau}, \quad c_3^*(\tau) = 1-504\sum_{h=1}^{\infty} \sigma_5(h)e^{2h\pi i\tau},$$

$$c_4^*(\tau) = 1+480\sum_{h=1}^{\infty} \sigma_7(h)e^{2h\pi i\tau}, \quad c_5^*(\tau) = 1-262\sum_{h=1}^{\infty} \sigma_9(h)e^{2h\pi i\tau},$$

on account of (4.6), (4.8), (4.2) and $(2\pi)^{-8}2s_8 = -\frac{1}{8!}B_8 = \frac{1}{8!30}$, $(2\pi)^{-10}2s_{10} = \frac{1}{10!}B_{10} = \frac{1}{10!}\cdot\frac{5}{66}$. Since there is essentially only one integral modular form of dimension -10, we may conclude [3])

$$c_5^*(\tau) = c_2^*(\tau).c_3^*(\tau).$$

III. In virtue of $h(\tau)\sim e^{2\pi i\tau}$ for $\tau\to i\infty$ and

$$c_2^*(\tau)c_4^*(\tau)-\left\{c_3^*(\tau)\right\}^2 \sim (720+1008)e^{2\pi i\tau} = 1728e^{2\pi i\tau},$$

we have [3])

$$1728h(\tau) = c_2^*(\tau)c_4^*(\tau)-\left\{c_3^*(\tau)\right\}^2.$$

IV. To give a somewhat different example I prove in a simple way the well known formula

$$(4.11)\qquad \vartheta_{00}^2(\tau) = 1+4\pi\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin\frac{n\pi}{2}.$$

Consider the two functions $\vartheta_{11}'(z|\tau)\vartheta_{01}(z|\tau)-\vartheta_{01}'(z|\tau)\vartheta_{11}(z|\tau)$ and $\vartheta_{00}(z|\tau)\vartheta_{10}(z|\tau)$, the accent denoting differentiation with respect to z. Reflecting the formulae

$$\vartheta_{gh}(z+1|\tau) = (-1)^g\vartheta_{gh}(z|\tau),$$

$$\vartheta_{gh}(z+\tfrac{1}{2}\tau|\tau) = q^{-\frac{1}{4}}e^{-\pi iz}\vartheta_{g+1,h}(z|\tau),$$

$$\vartheta_{11}'(z+\tfrac{1}{2}\tau|\tau) = q^{-\frac{1}{4}}e^{-\pi iz}\left[\pi i\vartheta_{01}(z|\tau)-\vartheta_{01}'(z|\tau)\right]$$

$$\vartheta_{01}'(z+\tfrac{1}{2}\tau|\tau) = q^{-\frac{1}{4}}e^{-\pi iz}\left[-\pi i\vartheta_{11}(z|\tau)+\vartheta_{11}'(z|\tau)\right],$$

we see that both functions are multiplied by -1 and $q^{-\frac{1}{2}}e^{-2\pi iz}$, if we apply the substitutions $z\to z+1$ and $z\to z+\frac{1}{2}\tau$ respectively. Inspection of the proof of (4.2) shows that we have the following Fourier expansion

$$(4.12)\quad\left\{\begin{aligned}\frac{\vartheta_{11}'(z|\tau)}{\vartheta_{11}(z|\tau)} &= \frac{d}{dz}\log\vartheta_{11}(z|\tau) = \\ &= \frac{d}{dz}\left[\log\sin\pi z - \sum_{n,m=1}^{\infty}\frac{1}{m}q^{2mn}(e^{2\pi izm}+e^{-2\pi izm})\right] = \\ &= \pi\cot\pi z-2\pi i\sum_{n,m=1}^{\infty}q^{2mn}(e^{2m\pi iz}-e^{-2m\pi iz}) = \\ &= \pi\cot\pi z+4\pi\sum_{m=1}^{\infty}\frac{q^{2m}}{1-q^{2m}}\sin 2m\pi z,\end{aligned}\right.$$

and in the same way

$$\frac{\vartheta_{00}'(z|\tau)}{\vartheta_{00}(z|\tau)} = \frac{d}{dz}\sum_{n=1}^{\infty}\left\{\log(1+q^{2n-1}e^{2\pi iz})+\log(1+q^{2n-1}e^{-2\pi iz})\right\} =$$

$$(4.13)\quad\left\{\begin{aligned} &= -\frac{d}{dz}\sum_{n,m=1}^{\infty}\frac{(-1)^m}{m}\,q^{m(2n-1)}(e^{2m\pi iz}+e^{-2m\pi iz}) = \\ &= 4\pi\sum_{n,m=1}^{\infty}(-1)^m q^{m(2n-1)}\sin 2m\pi z = \\ &= 4\pi\sum_{m=1}^{\infty}\frac{(-q)^m}{1-q^{2m}}\sin 2m\pi z. \end{aligned}\right.$$

Now the zeros of $\vartheta_{00}(z|\tau)\,\vartheta_{10}(z|\tau)$ are given by the points z, which are congruent with $\frac{1}{2}$ modulo 1, $\frac{1}{2}\tau$. Moreover, by (4.12) and (4.13) the point $z = \frac{1}{2}$ is a zero of $\vartheta'_{11}(z|\tau)\,\vartheta_{01}(z|\tau) - \vartheta'_{01}(z|\tau)\,\vartheta_{11}(z|\tau)$. We may conclude that the quotient of the two functions is a double periodic function without poles, hence a constant. By (2.9), etc. we find

$$\frac{\vartheta'_{11}(z|\tau)\,\vartheta_{01}(z|\tau) - \vartheta'_{01}(z|\tau)\,\vartheta_{11}(z|\tau)}{\vartheta_{00}(z|\tau)\,\vartheta_{10}(z|\tau)} =$$

$$= \left[\frac{\vartheta'_{11}(z|\tau)\,\vartheta_{01}(z|\tau)}{\vartheta_{00}(z|\tau)\,\vartheta_{10}(z|\tau)}\right]_{z=0} = \pi i\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2(1-q^{2n-1})^2}{(1+q^{2n-1})^2(1+q^{2n})^2} =$$

$$= \pi i\prod_{n=1}^{\infty}\frac{(1-q^n)^4}{(1-q^{2n})^2} = \pi i\,\vartheta_{01}^2(\tau).$$

Applying the substitution $\tau \to \tau+1$ and using (2.4) this becomes

$$\frac{\vartheta'_{11}(z|\tau)\,\vartheta_{00}(z|\tau) - \vartheta'_{00}(z|\tau)\,\vartheta_{11}(z|\tau)}{\vartheta_{01}(z|\tau)\,\vartheta_{11}(z|\tau)} = \pi i\,\vartheta_{00}^2(\tau).$$

In this formula take $z = \frac{1}{4}$. On account of $\vartheta_{00}(\frac{1}{4}|\tau) = \vartheta_{01}(\frac{1}{4}|\tau)$, $\vartheta_{11}(\frac{1}{4}|\tau) = i\,\vartheta_{10}(\frac{1}{4}|\tau)$ and the relations (4.12), (4.13) we finally obtai

$$\vartheta_{00}^2(\tau) = \frac{1}{\pi i}\,\frac{\vartheta_{00}(\frac{1}{4}|\tau)\,\vartheta_{11}(\frac{1}{4}|\tau)}{\vartheta_{01}(\frac{1}{4}|\tau)\,\vartheta_{10}(\frac{1}{4}|\tau)}\left\{\frac{\vartheta'_{11}(\frac{1}{4}|\tau)}{\vartheta_{11}(\frac{1}{4}|\tau)} - \frac{\vartheta'_{00}(\frac{1}{4}|\tau)}{\vartheta_{00}(\frac{1}{4}|\tau)}\right\} =$$

$$= \frac{1}{\pi}\left\{\pi\cot\frac{\pi}{4} + 4\pi\sum_{m=1}^{\infty}\frac{q^{2m}-(-q)^m}{1-q^{2m}}\sin\frac{m\pi}{2}\right\} =$$

$$= 1+4\sum_{m=1}^{\infty}\frac{q^{2m}+q^m}{1-q^{2m}}\sin\frac{m\pi}{2} = 1+\sum_{m=1}^{\infty}\frac{q^m}{1-q^m}\sin\frac{m\pi}{2}.$$

This proves the formula (4.11).

References:

R. Fricke, Die elliptischen Funktionen und ihre Anwendungen (1916).

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