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A property of positive definite matrices

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In this paper the following theorem is proved

<u>Theorem</u>. Let $Z_1, \dots, Z_k$ denote arbitrary n x m matrices $(m \leq n)$ and let A denote a positive definite n x n matrix. Then for the k x k matrices $Z_r' A\, Z_s \quad (r,s=1,\dots,k)$

we prove

$$\det_{r,s} (\det (Z_r' A\, Z_s)) \geq 0.$$

For the proof of the theorem a process is introduced which will be called m-compoundification.

If B is an arbitrary p x q matrix and m a positive integer $\leq p$ and $\leq q$, then the m-compound matrix $B^{(m)}$ of B is a $\binom{p}{m} x \binom{q}{m}$ matrix, the elements of which are all possible minors of B of order m; thus any element $b_{ij}^{(m)}$ of $B^{(m)}$ is an m x m minor of B in which elements of the $i_1^{th}, \dots, i_m^{th}$ row (with $i_1 < \dots < i_m$) and $j_1^{th}, \dots, j_m^{th}$ column (with $j_1 < \dots < j_m$) of B occur. The elements $b_{ij}^{(m)}$ are ordered in such a way that $i < k$ if the set $i_1, i_2, \dots, i_m$ preceeds $k_1, k_2, \dots, k_m$ lexicographically and $j < l$ if $j_1, \dots, j_m$ preceeds $l_1, \dots, l_m$ lexicographically [1].

Consequences of this procedure are

1^o: In the elements $b_{ii}^{(m)}$ only elements of the $i_1^{th}, \dots, i_m^{th}$ row and of the $i_1^{th}, \dots, i_m^{th}$ column of B occur, hence $b_{ii}^{(m)}$ is a principal minor of B and may be denoted by $(i_1, \dots, i_m)$.

2^o: If one of the integers p and q, say p, is equal to m, then $B^{(m)}$ is a $1 \times \binom{q}{m}$ matrix hence a vector. In this case the elements of $B^{(m)}$ can be denoted by $b_i^{(m)}$ $(i=1,\dots,\binom{q}{m})$.

Not every vector with $\binom{q}{m}$ components can be considered as an m^{th} compound of an m x q matrix. This is the case when and only when its components satisfy the wellknown p-relations.[2]

We now prove

<u>Lemma</u> 1. If A is a positive definite p x p matrix, then its m-compound

[1] Confer A.C.Aitken, Determinants and matrices, Chapter V, p.90.
[2] Confer R.Weitzenbock, Invariantentheorie, p.116,117,85.

$A^{(m)}$ (where $m \leq p$) is also a positive definite matrix.

Proof: We prove the lemma by induction on p.
For p=1 the lemma is obvious, since then m=1 and $A^{(m)}=A$.
Now suppose the lemma holds for p-1 and arbitrary $m \leq p-1$.
We then prove it for p, i.e. we show that the m-compound $A^{(m)}$ of a p x p matrix $A^{(m)}$ is positive definite. It is sufficient to show that the principal minors $A_h^{(m)}$ of $A^{(m)}$ $(h=1,\ldots,\binom{p}{m})$ are > 0; here $A_h^{(m)}$ is the principal minor containing elements of the

$\binom{p}{m}-h+1^{th}, \binom{p}{m}-h+2^{th}, \ldots, \binom{p}{m}^{th}$ row(and column) of $A^{(m)}$.

Now two cases are considered

1^o: $h \leq \binom{p-1}{m}$. Then, due to the lexicographical order of the elements of $A^{(m)}$ in the principal minor $A_h^{(m)}$ of $A^{(m)}$, in all elements only elements of the $2^{nd},\ldots,p^{th}$ row (and column) of A occur, hence $A_h^{(m)}$ is a principal minor of $A_{11}^{(m)}$; here A_{11} denotes the minor of a_{11} in A. Since A is positive definite also the principal minor A_{11} of A is positive definite. This minor A_{11} being of order p-1 by the induction hypothesis we infer that $A_{11}^{(m)} > 0$.

2^o: $h > \binom{p-1}{m}$. Then those m x m minors of A which occur in the complementary minor of $A_h^{(m)}$ contain elements of the first row (and column) of A. Now by Franke's theorem [3]) the identity

$$A_h^{(m)} = |a_{rs}|^c M \quad \text{holds, where } c = h-\binom{p-1}{m}$$

and where M is the complementary minor in the adgugate of $A^{(m)}$.

By the above remark the elements of M are minors of A in which neither elements of the first row nor elements of the first column of A occur. Hence M is a principal minor of $A_{11}^{(p-m)}$ and as before by induction hypothesis we have $M > 0$. Since A is positive definite we get

$|a_{rs}| > 0$, hence $A_h^{(m)} > 0$.

Lemma 2. If V is an n x m matrix $(m \leq n)$ and if A is an n x n matrix, then for the m-compounds $V^{(m)}$ and $A^{(m)}$ one has

$$\det (V'A\,V) = V'^{(m)}A^{(m)}V^{(m)}.$$

Proof. By a theorem of Binet-Cauchy [4]) on compound matrices from C = AB it follows that $C^{(m)} = A^{(m)}B^{(m)}$ Hence

$$V'^{(m)}A^{(m)}V^{(m)} = (V'AV)^{(m)} = \det (V'AV),$$ because V'AV is an m x m matrix.

[3]) Confer A.C.Aitken, loc.cit.,p.100
[4]) Confer A.C.Aitken, loc.cit.,p.93

We now proceed to prove the above theorem on matrices $A, Z_1, \ldots, Z_k$. Since A is positive definite, by lemma 1 also $A^{(m)}$ is positive definite hence for any $\binom{n}{m} \times 1$ matrix V one has

$$V' A^{(m)} V \geqq 0.$$

Taking $\quad V = \sum_{r=1}^{k} \lambda_r Z_r^{(m)}$

one gets for all real $\lambda_1, \ldots, \lambda_k$

$$\sum_{r,s=1}^{k} \lambda_r \lambda_s \, Z_r'^{(m)} \, A^{(m)} \, Z_s^{(m)} \geqq 0,$$

hence $\quad \det(Z_r'^{(m)} \, A^{(m)} \, Z_s^{(m)}) \geqq 0.$

Then by lemma 2 one obtains

$$\det_{r,s}(\det(Z_r' \, A \, Z_s)) \geqq 0.$$

It is not without interest to investigate some cases in which the last relation is an equality. Now since in lemma 1 the matrix $A^{(m)}$ was proved to be positive definite, this can only occur if V=0, i.e. if

(1) $$\sum_{r=1}^{k} \lambda_r \, Z_r^{(m)} = 0 \qquad (\lambda_1, \ldots, \lambda_k \text{ not all } =0)$$

Now we denote the m columns (m x 1 submatrices) of Z_r by $z_{r1}, \ldots, z_{rm}$ (r=1,...,k), which we interprete as points in a projective space G_m of m-1 dimensions. The linear space generated by $z_{r1}, \ldots, z_{rm}$ will be denoted by X_r (r=1,...,m).

Now obviously the relation (1) is equivalent to the relation

(2) $$\sum_{r=1}^{k} \lambda_r \, (z_{r1} \cdot \cdot \cdot z_{rm} \, u_{m+1} \cdot \cdot \cdot u_n) = 0 \, ;$$

here $u_{m+1}, \ldots, u_n$ denote arbitrary points of G_m, further $(z_{r1} \cdot \cdot \cdot z_{rm} \, u_{m+1} \cdot \cdot \cdot u_n)$ denotes the determinant the columns of which are $z_{r1} \cdot \cdot \cdot z_{rm} \, u_{m+1} \cdot \cdot \cdot u_n$.

We further remark that if the relation $(w_1 \cdot \cdot \cdot w_h \, u_{h+1} \cdot \cdot \cdot u_n) = 0$ holds for arbitrary $u_{h+1} \cdot \cdot \cdot u_n$, the points $w_1, \ldots, w_h$ belong to a G_{h-1}, and conversely.

We discuss the relation (2) for some values of k.

I. k = 1. Then $Z_1^{(m)} = 0$, hence all m x m minors of Z_1 are = 0, hence the points $z_{11}, \ldots, z_{1m}$ are linear dependent and so belong to a G_{m-1}. Conversely if these m points belong to a G_{m-1}, then $Z_1^{(m)} = 0$.

II. k = 2. We may suppose that both X_1 and X_2 are G_m's, for otherwise we are in case I. Hence $\lambda_1 \lambda_2 \neq 0$.

In the case $n > m$ in (2) with $k=2$ we put $u_{m+1} = z_{1\mu}$ ($\mu = 1,\dots,m$). Then it follows that $z_{1\mu} \in X_2$, hence
$X_1 \subset X_2$, consequently $X_1 = X_2$. In the case $n = m$ one has $X_1 = X_2$, since both are equal to the whole space G_n.
Conversely if the G_m's X_1 and X_2 are equal, then there exists an $m \times m$ matrix T such that

$$Z_2 = Z_1 T,$$

hence by lemma 2

$$Z_2^{(m)} = Z_1^{(m)} T^{(m)},$$

where $T^{(m)}$ is a scalar this proves (1).
III. $k = 3$. We may suppose that X_1, X_2 and X_3 are G_m's and moreover that $X_1 \neq X_2$, $X_2 \neq X_3$, $X_3 \neq X_1$, for otherwise we are either in case I or in case II.
Hence $\lambda_1\lambda_2\lambda_3 \neq 0$ and $n > m$. Since $X_3 \neq X_1$ there exists a point x in X_3 which does not belong to X_1.
Substituting $u_{m+1} = x$ in (2) we infer

$$(3) \qquad \lambda_1(z_{11}\cdot\,\cdot\,\cdot z_{1m}\; x\; u_{m+2}\cdot\,\cdot\,\cdot u_n) + \lambda_2(z_{21}\cdot\dots\cdot z_{2m}\; x\; u_{m+2}\cdot\,\cdot\,\cdot u_n) = 0$$

for arbitrary $u_{m+2}\cdot\,\dots u_n$. If x would belong to X_2 then we would get

$$\lambda_1(z_{11}\cdot\,\cdot\,\cdot z_{1m}\; x\; u_{m+2}\dots u_n) = 0,$$

hence $x \in X_1$ on account of $\lambda_1 \neq 0$. Thus $x \notin X_2$.
Then applying case II on the relation (3) used with $m+1$ in stead of m we find that the two G_{m+1}'s generated by x and X_1 and by x and X_2 are equal. Consequently by a well known theorem the intersection S of X_1 and X_2 is a G_{m-1}. Substituting $u_{m+1} = s$ in (2), where s is an arbitrary point of S we infer

$$\lambda_3(z_{31}\cdot\,\cdot\,\cdot z_{3m}\; s\; u_{m+2}\cdot\dots, u_n) = 0$$

for arbitrary $u_{m+2}\cdot\,\cdot\,\cdot u_n$, hence $s \in X_3$ on account of $\lambda_3 \neq 0$.
Consequently $S \subset X_3$.
The above G_{m+1} contains S and moreover x. Obviously $x \notin S$. Consequently the intersection of this G_{m+1} and the G_m generated by x and S is a G_m and this $G_m = X_3$ since $x \in X_3$, $S \subset X_3$.
So in this case III we find that the intersection of X_1, X_2 and X_3 is a G_{m-1} and their union is a G_{m+1}.
Conversely if X_1, X_2 and X_3 are G_m's and are mutually different and if their intersection S is a G_{m-1} and their union R is a G_{m+1}, then (1) holds with $k = 3$.
To prove this result we choose $m-1$ linear independent points $s_1,\dots,s_{m-1}$ in S. Let a_1 belong to X_1 but not to S, let a_2 belong to X_2, but not to S. Since the G_2 generated by a_1 and a_2 and X_3 (which is a G_m) belong

to R(which is a G_{m+1}) the intersection of G_2 and X_3 is a G_1. Hence there exists a point a_3 in X_3 such that

$$\mu_1 a_1 + \mu_2 a_2 + \mu_3 a_3 = 0 \quad , \quad \mu_1 \mu_2 \mu_3 \neq 0.$$

Let A_r denote the matrix with columns $s_1 \ldots s_{m-1}\, a_r$ (r=1,2,3). Then we have

$$\mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3 = 0.$$

Further there exist nonsingular matrices T_r such that

$$A_r = Z_r T_r \ (r=1,2,3).$$

Hence

$$\mu_1 Z_1 T_1 + \mu_2 Z_2 T_2 + \mu_3 Z_3 T_3 = 0.$$

Again using lemma 2 and the fact that $T_r^{(m)}$ is a scalar $\neq 0$ (r=1,2,3) we get

$$\lambda_1 Z_1^{(m)} + \lambda_2 Z_2^{(m)} + \lambda_3 Z_3^{(m)} = 0,$$

where

$$\lambda_r = \mu_r T_r^{(m)} \neq 0 \quad (r=1,2,3).$$