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SCHUBERT METHODS WITH AN APPLICATION TO ALGEBRAIC CURVES

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Introduction

These notes have grown out of a seminar about algebraic curves at the Mathematical Centrum in the spring of 1971. I have tried to follow the spirit of the spirit of the original program of Jacobi, Abel and Riemann. This program was to generalize the striking facts discovered about ellyptic integrals to the case of integrals of other algebraic expressions. The fundamental idea is "inverting the integrals". Thus, an ellyptic curve is viewed as an one-dimensional abelian variety. More generally, a Riemann surface is studied in terms of its Jacobian and objects on the Jacobian associated to the surface.

No integrals appear in my treatment. This is due to the fact, commonly referred to as Abel's theorem, concerning the relationship between integrals and meromorphic functions on the curve. In effect, the Jacobian can be replaced by a Picard variety of divisor classes. A general account of the theory of Picard schemes may be found in Grothendieck's Bourbaki talks. The basic facts about the Picard varieties of good curves can be found in Serre's book [14].

The main result on algebraic curves in these notes goes back in some form to Riemann. It says that a linear system of dimension k of degree d must exist on any Riemann surface if d and k are positive and $0 \leq (k+1)(k-d+g) \leq$ the genus of the surface. Essentially, the same proof, given here, was simultaneously discovered by Laksov and Kleiman [5]. In this proof, the existence question is reduced to a simple computation by Schubert methods. One advantage of this proof is that it does not depend on the theory of moduli.

The reader interested only in algebraic curves should begin reading in section 3. Schwartzenberger has suggested a different application of Porteus' formula to the theory of curves [12]. Section 4 deals with curves. The techniques appear in Mattuck's and Schwartzenberger's work.

The rest of these notes deals with Schubert conditions. A convenient reference is Hodge and Pedoe's book. Section 1 is an attempt to begin a study of the cohomology of invertible sheaves on homogeneous spaces in arbitrary characteristic. A Kodaira-type vanishing theorem is proved by induction. Section 2 gives an application of section 1 to demonstrate some recent results of Eagon and Hochster. They have generalized a famous result of Macaulay to an arbitrary Schubert condition. Also, Laksov has some further results in this direction.

In the following, I work over a fixed algebraically closed ground field. Everything is assumed to be algebraic; e.g. all sheaves are algebraic coherent sheaves. A variety is a reduced and irreducible shceme of finite type.

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Prerequisities

In general, I have assumed that the reader knows some of the basic facts about the cohomology of coherent sheaves on algebraic varieties. The basic references are FAC [13] and EGA, Chapter III, [2].

Section 1 assumes familiarity with the Leray spectral sequence and the fiber criterian for vanishing of direct images, EGA, III-7.9.8.

Section 2 assumes familiarity with Chapter III of FAC. Results from Chapter III, §1-3, are used without specific reference. Theorem 2 on page 269 of FAC is the essential point in the proof of the Cohen-Macaulayness of Schubert varieties. The reader should also consult Serre's notes on local algebr and Rees [10]. In the proof of lemma 1, I have used the finiteness theorem, EGA, III-3.2.1. In the projective case, EGA, III-2.2.1 would be enough.

In section 4, I have tried to be more explicit when applying the cohomological machinery. Mumford [9] has a discussion of the variation of cohomology groups in a flat and proper family. The reader may find this more palatable than the discussion in EGA.

Section 1. An introduction to the cohomology of Schubert manifolds.

For a given positive integer n, let G denote the group of invertible n×n matrices with values in some algebraically closed field. Furthermore, let B and T be the subgroups of G consisting of upper triangular matrices and of diagonal matrices. Then G is a reductive group with a maximal torus T and B is a Borel subgroup of G containing T. Let W be the Weyl group of T in G.

G has a Bruhat decomposition into double B-cosets. $G = \coprod_{w \in W} BwB$. Consider the closure of the cells BwB as w runs through W. All of these subvarieties are varieties of matrices satisfying rank conditions on certain submatrices. The most interesting of these subvarieties are smooth.

Let $(a) = (a_1,\dots,a_n)$ be an integral n-vector satisfying

$$a_1 \geq a_2 \geq \dots \geq a_n \quad \text{and} \quad n \geq a_i+i \quad \text{for all } i \in [1,\dots,n].$$

Denote the set of such (a)'s by A. If $(a) \in A$, let M(a) be the subvariety of G consisting of matrices (M^i_j) satisfying the Σa_i equations, $M^i_j = 0$ if $j > n-a_i$. One may check that the M(a)'s are among the smooth $\overline{BwB}$'s. The codimension of M(a) in G is Σa_i. G is $M(0,\dots,0)$ and B is $M(n-1,\dots,1,0)$.

For any $(a) \in A$, let S(a) be the Schubert manifold M(a)/B. These are smooth subvarieties of the flag manifold $F = S(0,\dots,0) = G/B$ and have codimension Σa_i in F. The purpose of this section is to prove the vanishing of cohomology groups of some invertible sheaves on the Schubert manifolds.

An easy way to describe the invertible sheaves on F is this. Let $X: B \to G_m$ be a character of B; i.e. an homomorphism of B into the multiplicative group. Denote the coset morphism $G \to G/B = F$ by π. Define L_X to be the invertible O_F-module such that a section of L_X over some open U in F is a function f on $\pi^{-1}(U)$ satisfying $f(g.b) = f(g) \cdot X(b)$ for all $b \in B$ and $g \in \pi^{-1}(U)$.

Any character X of B can be written in the form $\prod_{i=0}^{n} (M^i_i)^{p_i}$ for some integral n-vector $(p) = (p_1,\dots,p_n)$ where M^i_i is the i-th diagonal entree of a matrix. Abbreviate L_X by L(p). Note that if S(b) is a divisor in

$S(a)$, then

$$O_{S(a)}(S_{(b)}) = L((b)-(a))|_{S(a)}.$$

This follows from the functional equation for the M^i_j that defines $S(b)$ in $S(a)$.

Lemma 1. Let $(a) \in A$ and $(p) = (p_1,\ldots,p_n)$ be an integral n-vector. Assume, for some $i \in [1,\ldots,n-1]$, that $1 + p_i = p_{i+1}$ and $a_i = a_{i+1}$. Then

$$H^r(S(a),L(p)|_{S(a)}) = 0 \qquad \text{for all } r.$$

Proof. Let $(a') = (n-1,\ldots,1,0) - (\delta^i)$ where (δ^i) is the n-vector with zero entrees except for an one in the i-th place. Note that $M(a')$ is a group and that $S(a') = M(a')/B$ is the projective line $\mathbb{P}^1$. $L(p)|S(a')$ is of degree minus one because $1 + p_i = p_{i+1}$. Because $a_i = a_{i+1}$, $M(a)$ is stable under multiplication on the right by the group $M(a')$. Therefore the morphism $\alpha = M(a)/B \to M(a)/M(a')$ has fibers isomorphic to $\mathbb{P}^1 \approx M(a')/B$. The restriction of $L(p)$ to one (and hence all) fiber of α has degree minus one. Therefore $R^r\alpha_*(L(p)|S(a)) \approx 0$ for all r and the lemma follows by the Leray spectral sequence

$$H^i(M(a)/M(a'),R^j\alpha_*(-)) \Longrightarrow H^{i+j}(M(a)/B,-).$$

Lemma 2. Let $(p) = (p_1,\ldots,p_m)$ be an integral n-vector such that $1 + p_i \geq p_{i+1}$ for all $i \in [1,\ldots,n-1]$. Then

$$H^r(S(a),L(p)|_{S(a)}) = 0 \qquad \text{for } r \geq 1 \text{ and all } (a) \in A.$$

Proof. As a point has no higher cohomology, we may inductively assume that the dimension of $S(a)$ is greater than zero and that the lemma is true for the Schubert manifolds of smaller dimension. Let j be the largest number less than n such that $a_j < n-j$ and let k be the smallest number such that $a_k = a_j$. Then $(b) = (a) + (\delta^k) \in A$ and $a_k = a_{k+1}$.

Lemma 1 implies lemma 2 if $1 + p_k = p_{k+1}$. Hence we may inductively assume that $p_k \geq p_{k+1}$ and that the lemma is true for $(p') = (p) - (\delta^k)$.

Recalling that $L(\delta^k)|_{S(a)} \approx O_{S(a)}(S(b))$, we find an exact sequence of sheaves

$$K^*: 0 \to L(p')|_{S(a)} \to L(p)|_{S(a)} \to L(p)|_{S(b)} \to 0.$$

By the first induction $L(p)|_{S(b)}$ has no higher cohomology groups and by the second induction the same is true of $L(p')|_{S(a)}$. Therefore, the lemma follows from the long exact sequence of cohomology for K^*.

A partial summary of this result is the following.

<u>Theorem</u>. Let $(p) = (p_1,\ldots,p_n)$ be an integral n-vector such that $p_i \geq p_{i+1}$ for all $i \in [1,\ldots,n-1]$. Then for all $(a) \in A$,

1) $H^r(S(a),L(p)|_{S(a)}) = 0$ for $r > 0$ and

2) the natural homomorphism

$$H^0(F,L(p)) \to H^0(S(a),L(p)|_{S(a)})$$

is surjective.

Proof. 1) is obviously contained in lemma 2.
To see 2) we need only check that, if S(b) is a divisor in S(c), then

$$H^0(S(c),L(p)|_{S(c)}) \to H^0(S(b),L(p)|_{S(b)})$$

is surjective. Using the analogous exact sequence as in the proof of lemma 2, we find this result because $H^r(S(c),L((p)+(c)-(b))) = 0$ for $r \geq 1$ by lemma 2.

Section 2. The singularities of Schubert varieties.

Let p be a positive number less than n. Let $Y_{n,p}$ be the Grassmann manifold of p-dimensional affine subspaces of the affine space, $(x_1,\ldots,x_n)$. Let k be an integral-valued function of $[1,\ldots,p]$ such that $1 \leq k(1) < k(2) < \ldots < k(p) \leq n$. The points of $Y_{n,p}$ which correspond to subspaces y such that for all $r \in [1,\ldots,p]$

$$\dim(y \cap (x_1,\ldots,x_{k(r)},0,\ldots,0)) \geq r$$

form a Schubert variety. Denote this Schubert variety by Z(k).

A Schubert variety has a nice resolution of its singularities. Let $P_{n,p}$ be the partial flag manifold of nested, 1 through p-dimensional affine subspaces of the affine space, $(x_1,\ldots,x_n)$. A point of $P_{n,p}$ is a sequence $y_1 \subset y_2 \subset \ldots \subset y_p$ of subspaces such that the dimension of y_i is i for all $i \in [1,\ldots,p]$. There is a morphism of $P_{n,p}$ to $Y_{n,p}$ sending $(y_1,\ldots,y_p)$ to y_p. Now consider the subvariety X(k) of $P_{n,p}$ made up of the points $(y_1,\ldots,y_p)$ such that

$$y_r \subset (x_1,\ldots,x_{k(r)},0,\ldots,0) \quad \text{for all } r \in [1,\ldots,p].$$

The composite morphism $X(k) \hookrightarrow P_{n,p} \longrightarrow Y_{n,p}$ has image Z(k) and the morphism $X(k) \to Z(k)$ is easily seen to be bi-rational.

X(k) can be described in the group theoretic language of section 1. Let $(a) = (a_1,\ldots,a_n)$ be the integral n-vector satisfying $a_i = 0$ if $i \in [p+1,\ldots,n]$ and $a_i = n - k(i)$ if $i \in [1,\ldots,p]$. Then $(a) \in A$. Let P_1 and P_2 be the subgroups of G described as $M(n-1,\ldots,n-p,0,\ldots,0)$ and $M(\overbrace{n-p,\ldots,n-p}^{p \text{ times}},0,\ldots,0)$. M(a) is invariant on the right by P_1 and there are compatible isomorphisms

$$\begin{array}{ccccc} X(k) & \subset & P_{n,d} & \to & Y_{n,d} \\ \Vert & & \Vert & & \Vert \\ M(a)/P_1 & \subset & G/P_1 & \to & G/P_2 \end{array} .$$

In particular, this shows that X(k) is smooth.

A nice property of the resolution $\tau: X(k) \to Z(k)$ is that
1) the homomorphism $O_{Z(k)} \to \tau_* O_{X(k)}$ is an isomorphism and
2) $R^i\tau_* O_{X(k)} \approx 0$ for $i \geq 1$.
First we need some lemmas.

<u>Lemma 1</u>. Let $f: S \to T$ be a morphism of complete varieties. Assume that L is an ample invertible sheaf on T and $H^i(S, f^*L^{\otimes n}) = 0$ for all positive i and n. Then $R^i f_* O_s = 0$ for $i \geq 1$.

Proof. Because L is ample and $R^i f_* O_S$ is coherent, $H^j(T, R^i f_* O_S \otimes L^{\otimes n}) \approx 0$ for $j \geq 1$ and $n >> 0$. Because L is locally free,

$$R^i f_*(f^* L^{\otimes n}) \approx R^i f_* O_S \otimes L^{\otimes n}.$$

Hence $H^j(T, R^i f_*(f^* L^{\otimes n})) \approx 0$ for $j \geq 1$ and $n >> 0$. So the Leray spectral sequence

$$H^j(T, R^i f_*(f^* L^{\otimes n})) \Longrightarrow H^{i+j}(S, f^* L^{\otimes n})$$

degenerates for $n >> 0$. This gives isomorphisms

$$H^o(T, R^i f_*(f^* L^{\otimes n})) \approx H^i(S, f^* L^{\otimes n}) \quad \text{for } n >> 0,$$

but $H^i(S, f^* L^{\otimes n}) \approx 0$ for $i \geq 1$ and $n >> 0$. In other words, we have

$$H^o(T, R^i f^* O_S \otimes L^{\otimes n}) \approx 0 \quad \text{for } i \geq 1 \text{ and } n >> 0.$$

Therefore $R^i f^* O_S \approx 0$ for $i \geq 1$.

<u>Lemma 2</u>. With the assumptions of lemma 1, assume that f factors

$$S \longrightarrow R \xrightarrow{\;g\;} T, \qquad S \xrightarrow{\;f\;} T$$

where

$$R^i g_* O_R \approx \begin{cases} O_T & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

If the homomorphisms

$$H^0(R,g^*L^{\otimes n}) \to H^0(S,f^*L^{\otimes n})$$

are surjective for positive n, then the homomorphism $O_T \to f_*O_S$ is surjective and, hence, f_*O_S is the structure sheaf of the image variety of T under f.

Proof. By the degenerateness of the Leray spectral sequences, the horizontal arrows in the commutative square

$$\begin{array}{ccc} H^0(T,L^{\otimes n}) & \xrightarrow{\approx} & H^0(R,g^*L^{\otimes n}) \\ \downarrow & & \downarrow \\ H^0(T,f_*O_S\otimes L^{\otimes n}) & \xrightarrow{\approx} & H^0(S,f^*L^{\otimes n}) \end{array}$$

are isomorphisms. Hence, for positive n,

$$H^0(T,L^{\otimes n}) \to H^0(T,f_*O_S\otimes L^{\otimes n})$$

is surjective and, therefore, $O_T \to f_*O_S$ must be surjective.

Now, by combining the results of section 1 and these lemmas, one can easily see that the above niceness properties are verified. Let

$$\begin{array}{ccc} S & \longrightarrow & R \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\ & & T \end{array} \qquad \text{be} \qquad \begin{array}{ccc} X(k) & \longrightarrow & P_{n,d} \\ & \searrow & \downarrow \\ & & Y_{n,d} \end{array}$$

To prove that

$$R^i g_* O_{P_{n,d}} = \begin{cases} O_{Y_{n,d}} & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}$$

notice that g is flat. Hence we need only prove that

$$H^i(g^{-1}(y),O_{g^{-1}(y)}) = \begin{cases} \text{ground field} & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in Y_{n,d}$. These fibers are flag manifolds and, by Theorem 1, these conditions are verified. Also to check the other assumptions of the lemma, it is enough to replace $X(k) \subset P_{n,d}$ by $S(a) \subset G/B$ by the same flat fibering argument and the Leray spectral sequence. As soon as we find an appropriate L, these assumptions will be verified by Theorem 1.

Recall the classical projective embedding of $Y_{n,d}$ is given by sending $y \in Y_{n,d}$ into the line

$$\text{"} \overset{p}{\wedge} y \text{"} \subset A^{n!/n-p!p!}.$$

In the group theoretic language the corresponding ample invertible sheaf an $Y_{n,d} = G/P_2$ is defined by the character of P_2 which sends $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ into $\det(A)$ (here A is a p×p matrix). When we pull this invertible sheaf to G/B, it is equal $L(\underbrace{1,\ldots,1}_{p \text{ times}},0,\ldots,0)$. Therefore, Theorem 1 applies to this sheaf.

<u>Theorem 2</u>. The Schubert varieties $Z(k)$ are Cohen-Macaulay varieties.

Proof. In [FAC] Serre has given a criterion for $Z(k) \subset \mathbb{P}^{n!/n-p!-1}$ to be Cohen-Macaulay. We need only check that, if L is the restriction of $O_{\mathbb{P}}(-1)$ to $Z(k)$, then

$$H^i(Z(k), L^{\otimes -n}) \approx 0$$

for all $i \subset$ dimension of $Z(d)$ and $n >> 0$. By using the nice resolution $\tau: X(k) \to Z(k)$, the above is equivalent to $H^i(X(k), \tau^* L^{\otimes -n}) \approx 0$ for all $i <$ dimension of $X(k)$ and $n >> 0$. By duality, this is the same condition as

$$H^i(X(k), \Omega_X \otimes \tau^* L^{\otimes n}) \approx 0 \text{ for } i \geq 1 \text{ and } n >> 0$$

where Ω_X is the sheaf of dualizing differentials on $X(k)$. This will follow from lemma 2, section 1, applied to $\Omega_X \otimes \tau^* L^{\otimes n}$ pull back to $S(a)$ if we check the following fact.

A character giving Ω_X pulled back to $S(a)$ is determined by an integral n-vector $(p_1,\ldots,p_n)$ where $1 + p_i \geq p_{i+1}$ for $i \in [1,\ldots,p-1]$ or for $i \in [p+1,\ldots,n-1]$.

A character giving $\Omega_{S(0,\ldots,0)/P_1}$ pull back to $S(0,\ldots,0)$ is the sum $(-n+1,-n+2,\ldots,-n+p,0,\ldots,0) + (0,1,\ldots,p-1,p,\ldots,p)$. If V is a smooth variety with smooth divisor D, it is well-known that $\Omega_D \approx \Omega_V(D)|_D$. Therefore, the desired fact is true because, for the (a)'s we are considering, we have $a_0 > a_1 > \ldots > a_p$.

Section 3. Determinantial subschemes and the formula of Porteus.

Let S be a smooth quasi-projective variety. Assume that there are two locally free sheaves on S, F and G. Let the rank of F be f and the rank of G be g. Given any O_S-homomorphism n: F → G and any integer l with $0 \leq l \leq f$, define $Z_l(n)$ to be the largest subscheme of S where $\Lambda^{f-l+1}n$ is zero. We will refer to such subschemes as determinantial subschemes. More explicitly, assume that F and G are free with respect to some choice of basis (locally this is the case). Then n determines a g × f matrix of regular functions, (N^i_j). $Z_l(n)$ is the closed subscheme with equations, the determinants of all the f-l+1 minors of (N^i_j). A point s of S is contained in $Z_l(n)$ if and only if the vector space $\Gamma(S,\mathrm{Ker}(F\otimes k(s) \to G\otimes k(s))$ has dimension greater than or equal to l.

The study of determinantial subschemes is quite old. Some of the Schubert varieties in a Grassmannian are determinantial subschemes in the above sense. Macaulay established some nice properties of some of these subschemes. Recently, more general results have been obtained by Eagon and Hochster [1]. They have proved.

Theorem. Assume $g-f+l \geq 0$. If the codimension of $Z_l(n)$ in S is at least $l(g-f+l)$, then $Z_l(n)$ is a Cohen-Macaulay scheme of pure codimension $l(g-f+l)$ if it is non-empty. This theorem can be deduced from the theorem in section 2.

Early geometers determined the degrees of some determinantial subschemes and the Schubert calculus was developed. Today, these developments can be viewed in light of Chern's characteristic classes. The following theorem has grown out of Thom's study of singularities of mappings [15 and 16]. Recall that the cycle associatied with a subscheme is the formal sum of the component of the subscheme with the appropriate multiplicities.

Theorem. Assume $g-f+l \geq 0$. If the codimension of $Z_l(n)$ in S is at least $l(g-f+l)$, then the cycle associated to $Z_l(n)$ is rationally equivalent to

$$\Delta^l_{g-f+l}\ (c_tG/c_tF),$$

where c_t denotes the Chern polynomial of a locally free sheaf and Δ^1_{g-f+1} is a polynomial function of the coefficients of the above power series in t.

We need to know precisely what the polynomials Δ are. These polynomials were determined by Porteus. Let $p = \sum p_i t^i$ be a formal power series in t. For two non-negative integers, a and b, define $\Delta^a_b(p)$ to be the determinant of the a×a matrix with (r,s)-entree p_{b+r-s}. There, if b = 0, this determinant is one.

Remark. The above matrix represents the linear transformation sending a polynomial f of degree less than a into the terms of degree b,...,b+a-1 in the power series f.p.

An essential point of the demonstration of the theorem is the

Identity. $\Delta^a_b\left(\prod_{i=1}^{b} 1+X_i t \,/\, \prod_{j=1}^{a} 1+Y_j t\right) = \prod_{i=1}^{b} \prod_{j=1}^{a} (X_i-Y_j).$

Section 4. The existence of some invertible sheaves on a curve.

Let G be a complete curve of genus g. For any invertible sheaf L on G, we have theweak Riemann-Roch formula, $h^0(G,L) - h^1(G,L) = \deg L - g + 1$. There are well-known vanishing theorems;

$$h^0(G,L) = 0 \quad \text{if } \deg L < 0 \quad \text{and}$$

$$h^1(G,L) = 0 \quad \text{if } \deg L > 2g-2.$$

The obvious question is "For non-negative d, when does there exist an L such that deg L = d and $h^0(G,L) \geq l$, provided that $l \geq 0$ and $l-d+g-1 \geq 0$?" Riemann discovered the following answer when G is smooth. If $l(l-d+g-1) \leq g$, then such an L exists! To prove this we will show that the set of all such L is non-empty.

Let P_d be a Picard variety parameterizing the isomorphism classes of invertible sheaves on G of degree d. P_d can be equipped with an invertible sheaf M on $G \times P_d$. M is uniquely determined up to tensor multiple by a sheaf of the form p_2^*N where N is an invertible sheaf on P_d. If p is a point of P_d, let M_p denote the sheaf on G, $M|_{G\times\{p\}}$. As p varies M_p runs through the family of all isomorphism classes of degree d. P_d is a smooth quasi-projective variety of dimension g and is isomorphic to the generalized Jacobian of G.

For non-negative integers l and d, let E_d^l be the subset of P_d consisting of the points p where $h^0(G,M_p) \geq l$. By the upper semi-continuity of the function $p \to h^0(G,M_p)$, E_d^l is a closed subset of P_d. Also, we have the well-known

Lemma. Any general point p of a component of E_d^l satisfies the condition that $h^0(G,M_p) = l$, if $l-d+g-1 \geq 0$.

Proof. Let g be a point of E_d^l such that $h^0(G,M_q) > l$. Hence, $h^1(G,M_q) = h^0(G,M_q) - d+g-1 > l-d+g-1 \geq 0$. That is to say that M_p is special. For general point c_1 and c_2 on G, $h^0(G,M_q(c_1-c_2)) = h^0(G,M_q) - 1$.

For better results, we need to study more closely how the cohomology groups $H^0(G,M_p)$ vary with p. The main result about the variation is

Theorem 1. There exists an homomorphism $n\colon \underline{F} \to \underline{G}$ between two locally free sheaves, $\underline{F}$ and $\underline{G}$, on P_d satisfying the following condition. For all morphisms $t\colon S \to P_d$ from a scheme S, then there are isomorphisms (compatible with changing S)

$$\mathrm{Xer}(n_t) \approx R^0\, p_{2*}(G\times t)^*\, M \quad \text{and}$$

$$\mathrm{Cok}(n_t) \approx R^1\, p_{2*}(G\times t)^*\, M \quad \text{where}$$

n_t is the homorphism $\underline{F}\otimes O_S \to \underline{G}\otimes O_S$ induced by n and t.

Proof. The fact that such $n\colon \underline{F} \to \underline{G}$ exist locally on P_d follows because the sheaf M on $G \times P_d$ is flat with respect to p_2 and G is complete of dimension one [see 9]. We give a construction of $n\colon \underline{F} \to \underline{G}$ global below (see *).

Corollary. With the above notations,

$$\mathrm{Ker}(n) \approx R^0\, p_{2*}\, M \quad \text{and} \quad \mathrm{Cok}(n) \approx R^1\, p_{2*}\, M.$$

For any point p of P_d,

$$\Gamma(P_d,\mathrm{Ker}(n\otimes k(p))) \approx H^0(G,M_p) \quad \text{and}$$

$$\Gamma(P_d,\mathrm{Cok}(n\otimes k(p))) \approx H^1(G,M_p).$$

Furthermore, $-\operatorname{rank} \underline{G} + \operatorname{rank} \underline{F} = d - g + 1$.

By this theorem, we may regard E_d^l as the support of a determinantial subscheme, denoted with the same letter. One may even check that this subscheme structure does not depend on the choice of n.

Proposition. If E_d^l has codimension at least $l(l-d+g-1)$, then the cycle associated to this subscheme will be rationally equivalent to

$$\Delta^l_{l-d+g-1}(c_t\underline{G}/c_t\underline{F}) = \Delta^l_{l-d+g-1}(c_tR^1p_{2*}M/c_tR^0p_{2*}M).$$

Proof. This clearly follows from section 3 and the above theorem. Note also that we do not need to use the Cohen-Macanlayness theorem because the general point of any component of E_d^l is so nice.

The proposition always applies when $l = 1$ and $0 \leq d \leq g$. Denote E_d^1 by W^d.

<u>Theorem 2</u>. (Mattuck [6], Schartzenberger [11]). If $0 \leq d \leq g$, W^d is a variety of dimension d and its cycle is rationally equivalent to the g-d chern class of $R^1\, p_{2*}\, M_d$, where M_d is an M, as before, on $G \times P_d$.

Sketch of Proof. a) Let C^* be the smooth part of G. There is a morphism $C^* \times \ldots \times C^* \to P_d$ sending $(c_1,\ldots,c_d)$ into the invertible sheaf $O_G(c_1+\ldots+c_d)$. W^d is the closure of the image of this morphism. Therefore, W^d has dimension d.

b) The scheme W^d is Cohen-Macanlay by section 2. In particular, it has no embedded components and an easy infinitesimal calculation at a general point of W^d shows that W^d is reduced and, therefore, a variety.

c) By the above proposition, the cycle of W^d is rationally equivalent to $\Delta^1_{g-d}(c_t R^1 p_{2*} M_d / c_t R^0 p_{2*} M_d)$. By definition of Δ, we need only check that $R^0\, p_{2*}\, M_d \approx 0$. By the above corollary, $R^0\, p_{2*}\, M_d$ is contained in a free module, $\underline{F}$. Hence, we need only note that $R^0\, p_{2*}\, M_d$ is torsion. Let p be a general point of P_d such that $H^0(G, M_{d,p}) \approx 0$. Then, by the corollary, $n(p)\colon \underline{F}\otimes k(p) \to \underline{G}\otimes k(p)$ is injective. Therefore, $n\colon \underline{F} \to \underline{G}$ is injective in a neighborhood of p, but $R^0\, p_{2*}\, M_d \approx \mathrm{Ker}(n)$.

Pick a fixed point e on the smooth part of G. Let i and j be integers such that $i \geq j$. The exact sequence of sheaves on G,

$$0 \to O_G((j-i)e) \to O_G \to O_{(i-j)e} \to 0,$$

induces an exact sequence on $G \times P_i$,

$$0 \to M \otimes p_1^*\, O_G((j-i)e) \to M \to M \otimes p_1^*\, O_{(i-j)e} \to 0.$$

By projecting this sequence by p_2, we have an exact sequence on P_i,

$$(+)\quad \begin{array}{l} 0 \to R^0\ p_{2*}(M\otimes p_1^* O_G((j-i)e)) \to R^0\ p_{2*}\ M \to R^0\ p_{2*}(M\otimes p_1^* O_{(i-j)e}) \\ \qquad\qquad\qquad\qquad \big\downarrow \delta \\ 0 \leftarrow R^1\ p_{2*}\ M \leftarrow R^1\ p_{2*}(M\otimes p_1^* O_c(j-i)e)) \end{array}$$

This last 0 is because $R^1\ p_{2*}\ (M\otimes p_1^* O_{(i-j)e}) \approx 0$ as the support of $M \otimes p_1^*\ O_{(i-j)e}$ is $\{e\} \times P_i$.

(*) End of the proof of Theorem 1.

For fixed i and sufficiently negative j, $R^0\ p_{2*}(M\otimes p_1^* O_G((j-i)e)) \approx 0$ and $R^1\ p_{2*}$ (the same) is locally free of finite rank. From the theory of base extension, it follows that

$$\delta\colon R^0\ p_{2*}(M\otimes p_1^* O_{(i-j)e}) \to R^1\ p_{2*}(M\otimes p_1^* O_c((j-i)e))$$

is a good choice for $n\colon \underline{F} \to \underline{G}$.

As usual, the fixed point e can be used to construct isomorphisms, $\tau_j^i\colon P_i \to P_j$, by sending an invertible sheaf L of degree i into L((j-i)e). Let M^i be the sheaf, as before, on $G \times P_i$ and assume that M^i is normalized by the condition that

$$M^i|_{\{e\}\times P_i} \approx O_{\{e\}\times P_i}.$$

Do this for all i. Therefore,

$$M^i \approx (\tau_j^i)^*\ M^j \otimes p_1^*\ O_G((i-j)e).$$

<u>Lemma</u>. $c_t\ R^1\ p_{2*}\ M^i\ /\ c_t\ R^0\ p_{2*}\ M^i = (\tau_j^i)^*[c_t R^1 p_{2*} M^j / c_t R^0 p_{2*} M^j]$.

Proof. This follows from the exact sequence (+) when we see that $R^0 p_{2*}(M^i \otimes p_1^* O_{(i-j)e})$ has trivial chern classes. Note that $O_{(i-j)e}$ has composition factors O_e. So, $R^0\ p_{2*}(M^i\otimes p_1^* O_{(i-j)e})$ has composition factors

$$R^0\ p_{2*}(M^i|_{\{e\}\times P_i}) \approx R^0\ p_{2*}\ O_{\{e\}\times P_i} \approx O_{P_i}.$$

The main result of this section is this.

Theorem 3. If E^l_d has codimension at least $l(l-d+g-1)$, the cycle associated to the scheme E^l_d is rationally equivalent to

$$\Delta^l_{l-d+g-1}\left(\sum_{i=0}^{g} (\tau^d_i)^* W^i t^{g-i}\right).$$

Proof. This follows directly from the last proposition, theorem 2 and the lemma.

Corollary. If G is smooth and E^l_d has codimension at least $l(l-d+g-1)$, the cycle associated to the scheme E^l_d is numerically equivalent to

$$\prod_{0\le i\le l-1} \frac{i!}{(l-d+g-1+i)!} (\tau^d_{g-1})^* \underbrace{(W^{g-1},\ldots,W^{g-1})}_{\text{intersection } l(l-d+g-1) \text{ times}}$$

In particular, E^l_d is non-empty if $l(l-d+g-1) \le g$.

Proof. The last remark follows because this cycle is non-zero by the formula and the fact that W^{g-1} is an ample divisor on the complete variety P_{g-1}.

In the special case when $l = 1$. This is the known relation,

$$W^{g-i} \equiv \frac{1}{i!} (\tau^i_{g-1})^* \underbrace{(W^{g-1},\ldots,W^{g-1})}_{i \text{ times}}.$$

[See, for instance 8 or 4].

Using the special case and the theorem, the corollary follows once we establish the

Lemma. Let a and b be non-negative integers. Let exp t denote $\sum_{0\le i} \frac{1}{i!} t^i$. Then

$$\Delta^b_a(\exp t) = \prod_{0\le i\le b-1} \frac{i!}{(a+i)!}$$

Proof. $\Delta^b_0(\exp t) = 1$ by definition.

We may inductively assume that $a \ge 1$ and that the lemma is true for smaller a. Let f be a polynomial of degree less than b. The relation

$$\frac{d}{dt}(f \times \exp t) = (f + \frac{d}{dt} f) \exp t,$$

implies $$\Delta^b_{a-1}(\exp t) = \prod_{0 \leq j \leq b-1} (a+j)\, \Delta^b_a(\exp t)$$

(See the remark following the definition of Δ.)

<u>Appendix</u>. With the above notation, consider $R^1 p_{2*} M^{-1}$. This is a locally free sheaf of rank g. It has very simple chern classes. Is it possible to write it more explicitly by analytic methods in the Riemann surface case?

This locally free sheaf has other interesting properties and determines the curve as follows.

<u>Proposition</u>. Let i be a negative integer and G smooth. Let L be an invertible sheaf on P_i, which is algebraically equivalent to zero. Then

$$H^0(P_i, R^1 p_{2*}(M^i \otimes p_2^* L) \approx \text{ground field}$$

if $L^{-1} \approx M^i|_{\{f\}\times P_i}$ for some point f on G. Otherwise, it is zero.
Proof. $R^0 p_{2*}(M^i \otimes p_2^* L) \approx 0$ because $i < 0$. A generate Laray spectral sequence gives isomorphism

$$H^j(P_i, R^1 p_{2*}(M^i \otimes p_2^* L)) \approx H^{j+1}(G \times P_i, M^i \otimes p_2^* L).$$

Because G is smooth, P_i is isomorphic to an abelian variety. Let Q be a Picard variety of P_i and N a Poincaré sheaf on $Q \times P_i$ [9]. Now, regard $M^i \otimes p_2^* L$ as a family of invertible sheaves on P_i, algebraically equivalent to zero. This family induces an unique morphism $\psi_L : G \to Q$

such that $(\psi_L \times P_i)^* N \approx M^i \otimes p_2^* L \otimes p_1^* H$ for some invertible

sheaf H on G. By the autoduality of the Jacobian, ψ_L is essentially the canonical immersion of G into its Jacobian.

The proof of the proposition will use the following theorem of Mumford [9], that says that $R^k p_{1*}(N) \approx 0$ unless $k = g$ and $R^g p_{1*}(N) \approx O_{\{0\}}$ where $O_{\{0\}}$ is the structure sheaf of the variety with support at the identity of Q. Let O_G crudely denote the structure sheaf of the image of ϕ_L. From Mumford's result we can conclude that

$$R^{g-i}\ p_{1*}(N\otimes p_1^*O_G) \approx \mathrm{Tor}_i^{O_Q}(O_G, O_{\{0\}})$$

[see (9) again]. $R^{g-i}\ p_{1*}(N\otimes p_1^*O_G)$ is isomorphic to

$$R^{g-i}\ p_{1*}((\psi_L\times P_i)^*N) \approx R^{g-i}\ p_{1*}(M^i\otimes p_2^*L\otimes p_1^*H).$$

As the last sheaf has support on at most one point,

$$R^{g-i}\ p_{1*}(M^i\otimes p_2^*L\otimes p_1^*H) \approx R^{g-i}\ p_{1*}(M^i\otimes p_2^*L)$$

and, also, another Leray spectral sequence generates giving isomorphisms

$$H^{g-i}(G\times P_i, M^i\otimes p_2^*L) \approx \Gamma(G, R^{g-i}p_{1*}(M^i\otimes p_2^*L)).$$

If $\phi_L(G)\cap\{0\}$ is empty, $R^{g-i}\ p_{1*}(M^i\otimes p_2^*L) \approx 0$ for all i and, hence, we are done. If $\phi_L(G)\cap\{0\}$ is not empty, let f be the unique point of G such that $\phi_L(f) = 0$. By the definition of ϕ_L, $\phi_L(f) = 0$ if and only if $L^{-1} \approx M^i|_{\{f\}\times P_i}$. In conclusion, $\mathrm{Tor}_{g-1}^{O_Q}(O_G, O_{\{0\}}) \approx O_{\{0\}}$ and thus

$$H^0(P_i, R^1p_{2*}(M^i\otimes p_2^*L)) \approx \Gamma(G, O_{\{f\}}).$$

Hence, we are done in this case.

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