

STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

ZW 1954 - 007

A minimum problem on matrices

G.W. Veltkamp, H.J.A. Duparc and W. Peremans



1954

ZW

A minimum problem on matrices.

by

G.W.Veltkamp, H.J.A.Duparc and W.Peremans.

In a previous report (Rapport ZW 1954-003) the question was put to investigate for which L, independent of E, of the expression

$$(1) \quad \tau^2 = E'(LP)^{-1} LCL'(P'L')^{-1} E$$

attains its minimal value,

here

C is a constant positive definite $m \times m$ matrix;

L is a variable $k \times m$ matrix;

P is a constant $m \times k$ matrix of rank k ;

E is a constant $k \times 1$ matrix;

$k \leq m$.

We prove that the minimum is equal to $\tau_0^2 = E'(P'C^{-1}P)^{-1} E$

and that the minimum is attained if and only if $L = VP'C^{-1}$, where V is an arbitrary non singular $k \times k$ matrix.

One has

$$(2) \quad \tau^2 - \tau_0^2 = E'(LP)^{-1} LCL'(P'L')^{-1} E - E'(P'C^{-1}P)^{-1} E = \\ = E'(LP)^{-1} L \{ C - P(P'C^{-1}P)^{-1} P' \} L'(P'L')^{-1} E.$$

We first prove the following

Lemma: If C is a positive definite $m \times m$ matrix and if A is an $m \times m$ matrix subject to the condition

$$(3) \quad A'C^{-1} A = A,$$

then $C-A$ is a positive semidefinite matrix.

Furthermore an $m \times 1$ matrix U satisfies

$$U'(C-A)U = 0$$

if and only if

$$(4) \quad (C-A)U = 0.$$

Proof. In virtue of (3) one has $A' = A'C^{-1} A = A$ and further

$$U'(C-A)C^{-1}(C-A)U = U'(C-A-A+AC^{-1}A)U = U'(C-A)U.$$

Consequently since C^{-1} is definite, $C-A$ is semidefinite and moreover

$$U'(C-A)U = 0 \quad \text{if and only if (4) holds.}$$

By means of this lemma we can discuss (2) by taking

$$A = P(P'C^{-1}P)^{-1} P'.$$

This $m \times m$ matrix satisfies (3) and so

$$(5) \quad U'(C - P(P'C^{-1}P)^{-1} P')U \cong 0;$$

in the last relation the equality sign holds if and only if U satisfies

$$(6) \quad (C - P(P'C^{-1}P)^{-1} P')U = 0.$$

Comparing (2) and (5) we infer $\tau^2 \cong \tau_0^2$ and $\tau^2 = \tau_0^2$ if (6) holds with

$$U = L'(P'L')^{-1} E.$$

Hence the necessary and sufficient condition for L to minimalise (1) is

$$CL'(P'L')^{-1} E - P(P'C^{-1}P)^{-1} E = 0.$$

This last relation holds identically in E if and only if

$$CL'(P'L')^{-1} = P(P'C^{-1}P)^{-1},$$

which -since C and $P'L'$ are non singular- is equivalent to

$$(7) \quad L' = C^{-1}P(P'C^{-1}P)^{-1} P'L'.$$

From (7) we infer that L' has the form

$$(8) \quad L' = C^{-1} P V',$$

where V is a non singular $k \times k$ matrix.

Conversely if L' is given by (8) where V is an arbitrary non singular $k \times k$ matrix, then obviously L' satisfies (7).

REMARK: In the above argument it appeared that the matrix

$$(9) \quad A = P(P'C^{-1}P)^{-1} P'$$

(where C is a positive definite $m \times m$ matrix and P is an $m \times k$ matrix of rank k) satisfies

$$(10) \quad A'C^{-1} A = A.$$

We now prove that conversely every $m \times m$ matrix A of rank $k \leq m$ which satisfies (10) has the form (9), where P is an $m \times k$ matrix the k columns of which are independent linear composita of the columns of A .

In fact let P be an $m \times k$ matrix which is subjected to the condition that its k columns are independent linear composita of the columns of A , but further arbitrarily chosen. Then obviously P has the rank k .

Now conversely every column of A is a linear combination of the k columns of P , so we have $A = PQ'$, where Q is a suitably chosen $m \times k$ matrix, which has the rank k since A is of the rank k . Substitution in (10) gives

$$QP'C^{-1} PQ' = PQ'.$$

Since Q is of rank k this relation is equivalent to

$$QP'C^{-1} P = P.$$

Further since also P is of rank k the $k \times k$ matrix $P'C^{-1} P$ has the rank k and therefore is non singular. Consequently

$$Q = P(P'C^{-1} P)^{-1}, \text{ thus } A = P(P'C^{-1} P)^{-1} P'.$$