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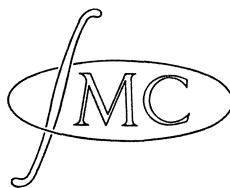
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Mappings commuting with the hat

by

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1. Introduction

In [1] the continuous map f_2 of the unit interval $[0,1]$ into itself, such that

$$f_2(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

called the hat, was studied. It was shown that f_2 commutes with all multihats f_n , where $f_n(x) = nx$ for $0 \leq x \leq \frac{1}{n}$, $f_n(x) = 2-nx$ if $\frac{1}{n} \leq x \leq \frac{2}{n}$, while f_n is periodic with period $\frac{2}{n}$ (in particular, $f_0(x) = 0$ for all x). Furthermore it was proved that the semigroup of all f_n , $n = 0,1,2,\dots$, is a maximal commutative semigroup: every continuous map of $[0,1]$ into itself commuting with all f_n is one of the f_n .

Experiments lead us to the conviction that every continuous map commuting with f_2 is either a multihat f_n or the map h such that $h(x)$ equals the fixed point $\frac{2}{3}$ of f_2 (the function f_0 maps every x on the other fixed point of f_2 , zero). We are as yet unable to stave this conviction with proof; but in this report we will show that in any case every open continuous map $g : [0,1] \rightarrow [0,1]$ commuting with f_2 is of necessity one of the multihats. In fact, an assumption much weaker than openness suffices: a continuous map g commuting with f_2 is a multihat as soon as $g(0) = 0$ while between two consecutive points where g equals zero there is located one and exactly one point where g takes the value 1.

As was pointed out to us by Z. Hedrlín, the results of [1] and of the present report give information about the Chebichev polynomials T_n , considered as transformations of the interval $[-1,1]$ into itself:

$$T_n(x) = \cos(n \cdot \arccos x).$$

For the map $\varphi(x) = \cos \pi x$ is a homeomorphism of $[0,1]$ onto $[-1,1]$, and $T_n = \varphi \circ f_n \circ \varphi^{-1}$, for $n = 0,1,2,\dots$. Hence it

follows from the results of [1] that these polynomials constitute a maximal commutative semigroup of commutative mappings of $[-1,1]$ into itself, while the main result of the present report is equivalent to the following theorem.

Theorem. Every open continuous map $g : [-1,1] \rightarrow [-1,1]$ that commutes with the Chebychev polynomial T_2 is one of the Chebychev polynomials T_n .

2. Functions commuting with f_2

A map $g : [0,1] \rightarrow [0,1]$ is open if it maps open subsets of the topological space $[0,1]$ onto open subsets; i.e. if $g((a,b))$ is open for $0 \leq a < b \leq 1$, $g([0,a))$ is open for $0 < a \leq 1$, and $g((b,1])$ is open for $0 \leq b < 1$. From this readily follows:

Lemma 1. A continuous map $g : [0,1] \rightarrow [0,1]$ is open if and only if it is of the following type: there is a finite set of points $\{a_0, a_1, \dots, a_n\}$, $0 = a_0 < a_1 < \dots < a_n = 1$ such that for each ν the value $g(a_\nu)$ equals either 0 or 1, while $g|_{[a_\nu, a_{\nu+1}]}$ is monotone, i.e. is a homeomorphism $[a_\nu, a_{\nu+1}] \rightarrow [0,1]$.

Remark. It follows that open maps coincide with full maps as defined by H. Cohen [2].

If we call a point x with $g(x) = a$ an a -point of g , then it follows that 0-points and 1-points of an open continuous map g alternate.

Any map g commuting with f_2 maps the fixed point 0 of f_2 into another fixed point: $g(0) = 0$ or $g(\frac{2}{3})$. If moreover $g(0) = 0$, the only possibility is that $g(0) = 0$. Hence every open continuous map g commuting with f_2 is of type I, where

Definition. A map $g : [0,1] \rightarrow [0,1]$ is of type I if there exists a finite set $\{a_1, a_2, a_3, \dots, a_n, b_0, b_1, b_2, \dots, b_m\}$ such that

- (i) $0 = b_0 < a_1 < b_1 < a_2 < b_2 < a_3 < \dots$; either $a_n = 1$ or $b_m = 1$;
- (ii) $g^{-1}(0) = \{b_0, b_1, \dots, b_m\}$, $g^{-1}(1) = \{a_1, a_2, \dots, a_n\}$.

In the remainder of this section g is assumed to be a continuous map of type I commuting with f_2 . Simplifying the notation, we write f instead of f_2 .

Lemma 2. $b_{2k} = 2b_k$, for $0 \leq 2k \leq m$;
 $b_{2k-1} = 2a_k$, for $1 \leq 2k-1 \leq m$.

Proof.

$0 = g(b_j) = gf(\frac{1}{2}b_j) = fg(\frac{1}{2}b_j) \Rightarrow g(\frac{1}{2}b_j) = 0$ or $1 \Rightarrow \frac{1}{2}b_j$ is either an $a_k \leq \frac{1}{2}$ or a $b_k \leq \frac{1}{2}$. Conversely, if x is either an $a_k \leq \frac{1}{2}$ or a $b_k \leq \frac{1}{2}$, then $g(2x) = gf(x) = fg(x) = 0$.

Lemma 3. If $g(1) = 0$, then $m = n$ and

$$a_{n-k} = 1 - a_{k+1} \quad \text{for } 0 \leq k \leq n-1;$$

$$b_{n-k} = 1 - b_k \quad \text{for } 0 \leq k \leq n.$$

Proof.

Clearly $m = n$. Let $x \leq \frac{1}{2}$. Then $g(x) \in \{0,1\} \Leftrightarrow g(2x) = fg(x) = 0 \Leftrightarrow g(1-x) \in \{0,1\}$.

Similarly:

Lemma 4. If $g(1) = 1$, then $m = n-1$ and

$$b_{n-k} = 1 - a_k, \quad \text{for } 1 \leq k \leq n.$$

Proposition 1. Let g be a continuous map of type I, commuting with f . Then

$$a_k = \frac{2k-1}{2n}, \quad b_k = \frac{2k}{2n} \quad (1 \leq k \leq n)$$

if $g(1) = 0$, and

$$a_k = \frac{2k-1}{2n-1}, \quad b_k = \frac{2k}{2n-1} \quad (1 \leq k \leq n)$$

if $g(1) = 1$.

Proof.

If $n = 1$, the assertions are trivial. Hence assume $n \geq 2$. We distinguish four cases, according to the parity of n and of $g(1)$.

Case 1: $g(1) = 0$, n even.

The inhomogeneous linear system of equations

$$(1) \quad \begin{aligned} x_k + x_{n-k} &= 1 & (1 \leq k \leq \frac{n}{2}), \\ 2x_k - x_{2k} &= 0 & (1 \leq k \leq \frac{n}{2}), \end{aligned}$$

admits the solution $x_k = b_k$ ($1 \leq k \leq n$), by lemma's 2,3. But obviously the following is also a solution: $x_k = \frac{k}{n}$ ($1 \leq k \leq \frac{n}{2}$). Hence it suffices to show that the system (1)

has a non-zero determinant, for then we must have:

$$b_k = \frac{k}{n}, \text{ while } a_k = \frac{1}{2}b_{2k-1} = \frac{2k-1}{2n}, \text{ by lemma 3.}$$

The form of the matrix M of (1) is indicated on page 5.

If one subtracts the $(n-k)$ -th column from the k -th one, for $1 \leq k \leq \frac{n}{2} - 1$, the first $n/2$ rows are all left with exactly one non-zero entry. Moreover, the last column has only one non-zero entry to begin with. Hence if we next strike out the first $\frac{n}{2}$ rows, the last row, and the last $\frac{n}{2} + 1$ columns, we are left with a matrix M_1 the determinant of which differs from $\det(M)$ only by a factor ± 1 .

$$M = \begin{pmatrix}
 \begin{array}{cccc|cccc}
 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0
 \end{array} &
 \begin{array}{cccc|cccc}
 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0
 \end{array} \\
 \hline
 \begin{array}{cccc|cccc}
 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 0 & 2 & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 0 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -1
 \end{array} &
 \begin{array}{cccc|cccc}
 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
 \end{array}
 \end{pmatrix}$$

$\leftarrow \text{---} n/2 \text{---} \rightarrow$ $\leftarrow \text{---} n/2 \text{---} \rightarrow$

$$M_1 = \begin{pmatrix}
 \begin{array}{cccc|cccc}
 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
 0 & 2 & 0 & -1 & \cdots & 0 & 0 & 0 \\
 0 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 2
 \end{array}
 \end{pmatrix}$$

$n/2 - 1$

The determinant of M_1 is easily seen to be equal to $2^{\frac{n}{2}-1}$ (reduce M_1 to triangular form); hence $\det(M) \neq 0$.

Case 2: $g(1) = 0$, n odd.

The $(n-1) \times (n-1)$ inhomogeneous linear system

$$(2) \quad \begin{aligned} x_k + x_{n-k} &= 1 & (1 \leq k \leq \frac{n-1}{2}) \\ 2x_k - x_{2k} &= 0 & (1 \leq k \leq \frac{n-1}{2}) \end{aligned}$$

admits the solutions $x_k = \frac{k}{n}$ ($1 \leq k \leq \frac{n-1}{2}$) and $x_k = \frac{k}{n}$ ($1 \leq k \leq \frac{n-1}{2}$). Using a reduction as in case 1 we easily find that the determinant of (2) differs only by a factor ± 1 from the determinant of a matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & 1 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 2 \end{pmatrix}.$$

Evaluating this matrix modulo 2, we see that its determinant is an odd integer, hence differs from 0.

Case 3: $g(1) = 1$, n even.

Use the square system

$$(3) \quad \begin{aligned} x_k + x_{2n-k-1} &= 1 & (1 \leq k \leq n-1); \\ 2x_k - x_{n+2k-2} &= 0 & (1 \leq k \leq \frac{n}{2}); \\ 2x_{n+k-1} - x_{n+2k-1} &= 0 & (1 \leq k \leq \frac{n}{2}-1). \end{aligned}$$

One solution is given by $x_k = a_k, x_{n+k-1} = b_k$ ($1 \leq k \leq n-1$), another by $x_k = \frac{2k-1}{2n-1}, x_{n+k-1} = \frac{2k}{2n-1}$ ($1 \leq k \leq n-1$). The determinant of (3) is easily seen to differ only by a factor ± 1 from the determinant of a matrix of the form

$$\begin{pmatrix} 2 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 2 & \cdots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 2 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}$$

If one considers the coefficients of this matrix modulo 2 one arrives at a matrix each row and column of which has only one non-zero entry, this entry being equal to ± 1 . The determinant of such a matrix equals ± 1 , hence the determinant of (3) is odd, thus certainly non-zero, proving that $a_k = \frac{2k-1}{2n-1}, b_k = \frac{2k}{2n-1}$, for $1 \leq k \leq n-1$.

Case 4: $g(1) = 1, n$ odd.

The proof is similar to the one in the cases 2 and 3, using the system

$$\begin{aligned} x_k + x_{2n-k-1} &= 1 & (1 \leq k \leq n-1); \\ 2x_k - x_{n+2k-2} &= 0 & (1 \leq k \leq \frac{n-1}{2}); \\ 2x_{n+k-1} - x_{n+2k-1} &= 0 & (1 \leq k \leq \frac{n-1}{2}), \end{aligned}$$

which is satisfied by $x_k = a_k, x_{n+k-1} = b_k$ ($1 \leq k \leq n-1$).

Proposition 2. Let n be a natural number, and let g be a continuous map $[0,1] \rightarrow [0,1]$ commuting with f_2 such that

$$g^{-1}(0) = \left\{ \frac{2k}{n} : k \text{ integer and } 0 \leq k \leq \frac{n}{2} \right\};$$

$$g^{-1}(1) = \left\{ \frac{2k-1}{n} : k \text{ integer and } 0 \leq k \leq \frac{n+1}{2} \right\}.$$

Then $g = f_n$.

Proof.

Suppose $g(x) = f_n(x)$ for all $x \in [0,1]$ such that $2^m f_n(x)$ is an integer (by assumption this is true if $m = 0$). We will show that also $g(x) = f_n(x)$ if $2^{m+1} f_n(x)$ is integer.

Indeed, let $f_n(x) = \frac{k}{2^{m+1}}$, k integer. If $x \leq \frac{1}{2}$, then $f_n(2x) = f_n f_2(x) = f_2 f_n(x) = \frac{k}{2^m}$, k' integer; hence $f_n(2x) = g(2x)$, which implies

$$(5) \quad f_2 f_n(x) = f_2 g(x).$$

Similarly we find that (5) holds for $x \geq \frac{1}{2}$. It then follows that either $f_n(x) = g(x)$ - in which case we are ready - or $f_n(x) = 1-g(x)$. But this is impossible except if $f_n(x) = g(x) = 1$, as one verifies by considering the largest $x_1 < x$ and the least $x_2 > x$ such that $2^m f_n(x_i)$ is integer (taking into account the continuity of g).

Hence g and f_n coincide on a dense subset of $[0,1]$. As both maps are continuous, it follows that $g = f_n$.

3. The main results

From propositions 1 and 2 we conclude at once:

Theorem 1. Every continuous map $g : [0,1] \rightarrow [0,1]$ of type I that commutes with f_2 is a multihat f_n .

As every open continuous map that commutes with f_2 is of type I, we have in particular:

Theorem 2. Every open continuous map $g : [0,1] \rightarrow [0,1]$ that commutes with f_2 is a multihat f_n .

By the observations made in the introduction, this theorem 2 is equivalent to

Theorem 3. Every open continuous map $g : [-1,1] \rightarrow [-1,1]$ that commutes with the Chebychev polynomial T_2 is itself a Chebychev polynomial T_n .

Remark. It seems that the methods of this report can be generalized to functions of type I that commute with f_3 . However, we do not see at the moment how to arrive at a general description that fits all $f_n, n \geq 2$.

In any case it seems probable to us that the results of this note hold true more generally, i.e. that every continuous function of type I, in particular every open continuous map, that commutes with a multihat $f_n, n \geq 2$, is itself a multihat. This would mean that every open continuous map commuting with a Chebychev polynomial $T_n, n \geq 2$, is itself a Chebychev polynomial T_n .

References

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