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Geometric deduction of Markov's minimal forms

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1. In the following we shall consider indefinite binary quadratic forms. Such forms have the shape

$$q = q(x) = \alpha x_1^2 + \beta x_1 x_2 + \gamma x_2^2 \qquad (\alpha, \beta, \gamma \text{ real});$$

and have positive discriminant

$$d = d(q) = \beta^2 - 4\alpha\gamma .$$

In this report we shall be concerned with the lower bound of $|q(x)|$ for integral $x_1, x_2 \neq 0, 0$.

We shall denote points with integral coordinates by $u=(u_1,u_2)$, $v=(v_1,v_2)$, etc. In particular, we shall write $o=(0,0)$. For given q, we put

$$\mu(q) = \inf_{u \neq o} |q(u)| . \tag{1}$$

Further, we shall call two forms q, q' equivalent, and write $q \sim q'$, if there is an integral unimodular transformation

$$x \rightarrow Ux = (u_{11}x_1 + u_{12}x_2, u_{21}x_1 + u_{22}x_2)$$

such that $q(Ux) = q'(x)$. Next, we shall write $q \approx q'$ if q is equivalent with a multiple $\sigma q'$ of q'. The following relations are trivial:

$$\mu(q) = \mu(q') \text{ and } d(q) = d(q') \text{ if } q \sim q' \tag{2}$$

$$\mu(q)/\sqrt{d(q)} = \mu(q')/\sqrt{d(q')} \quad \text{if} \quad q \approx q'. \tag{3}$$

The theorem of Markov, which we shall state below and for which we intend to give a geometric proof, gives detailed information concerning the quantity $\mu(q)/\sqrt{d(q)}$.

The geometry can be brought in as follows. Let S be the two-dimensional domain

(4) $$S : |x_1 x_2| < 1,$$

bounded by the two orthogonal hyperbolas $x_1x_2=\pm 1$, and let Y denote the lattice of points u in the plane with integral coordinates. The domain S is left invariant under the hyperbolic rotations

(5) $$T : x_1' = \tau x_1 , \quad x_2' = \tau^{-1} x_2 \quad (\tau \neq 0 \text{ and real})$$

and under the reflections with respect to the coordinate axes and the lines $x_1=\pm x_2$.

Further, if we subject Y to a nonsingular linear transformation

$$x \rightarrow Ax = (a_{11}x_1+a_{12}x_2, a_{21}x_1+a_{22}x_2),$$

then we get a general plane lattice Λ consisting of the points Au ($u \in Y$) enz. It is generated by the two points Ae,Af, where e=(1,0) and f=(0,1), and has determinant

$$d(\Lambda) = |\det A| .$$

Now consider an arbitrary form q(x). It can be written as the product of two linear factors, say

(6) $$q(x) = (a_{11}x_1+a_{12}x_2)(a_{21}x_1+a_{22}x_2);$$

if A is the matrix (a_{ij}) and $\overline{q}$ denotes the special form $\overline{q}(x)=x_1x_2$, then (6) reads

(6') $$q(x) = \overline{q}(Ax).$$

Suppose that $\mu(q)$ has a positive value μ. Then, by (1) and (6), each point $x \neq o$ of the form Au satisfies $|x_1x_2| \geqslant \mu$, i.e. the lattice $\Lambda = AY$, where A satisfies (6'), has no point $\neq 0$ in $\sqrt{\mu}\, S$. With the usual terminology, we say that Λ is admissible for $\sqrt{\mu}\, S$. More precisely, we have

(7) $$\inf_{x \neq o, x \in \Lambda} |x_1x_2| = \mu ,$$

where $\mu = \mu(q)$, $\Lambda = AY$, $q(x)=\overline{q}(Ax)$, so that Λ is admissible for $\sqrt{\mu}\, S$, but no longer for $\sqrt{\mu}\,' S$ as $\mu' \mu$.

We may denote the square root of the left hand member of (7) by

(8) $$\mu(\Lambda) = \mu(S, \Lambda) = \inf_{x \neq o, x \in \Lambda} |x_1x_2|^{\frac{1}{2}} .$$

We further note that the form (6) has discriminant $(\det A)^2$. Then, for the lattice Λ considered above,

$$(9) \qquad \mu(\Lambda) = \sqrt{\mu(q)}\ , \quad d(\Lambda) = \sqrt{d(q)}.$$

There is a correspondence between forms q and lattices Λ. It can be expressed by

$$(10) \qquad q(x) = \overline{q}(Ax)\ , \quad \Lambda = A\Upsilon\ .$$

But this correspondence is not one-to-one. On the other hand, any two lattices $A\Upsilon$ and $AU\Upsilon$ are identical, since $\Upsilon = U\Upsilon$. Thus, equivalent forms correspond with the same lattice. On the other hand, the form $\overline{q}$ is left invariant under all hyperbolic rotations T (which means that q(x) and $q(A^{-1}TAx)$ are identical for all T). The corresponding lattices are $\Lambda = A\Upsilon$, $\Lambda' = TA\Upsilon$. Such lattices are obtained from each other by means of a hyperbolic rotation of the plane, and one has the formulas (similar to (2)):

$$(11) \qquad \mu(\Lambda) = \mu(\Lambda') \text{ and } d(\Lambda) = d(\Lambda') \text{ if } \Lambda' = T\Lambda\ .$$

The right form of the correspondence between forms and lattices is given in our case by

$$(12) \qquad \{q_U\}_U \leftrightarrow \{\Omega^{\varepsilon} T\Lambda\}_T \qquad (\varepsilon = 0,1),$$

where q and Λ are connected by (10) and q_U means the form q(Ux).

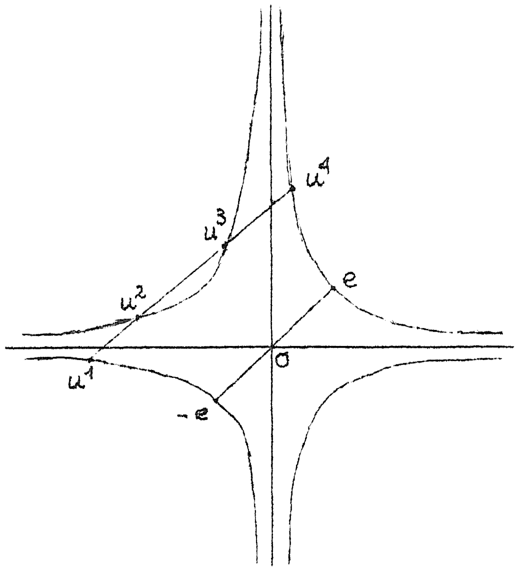
2. We proceed to find all lattices Λ with

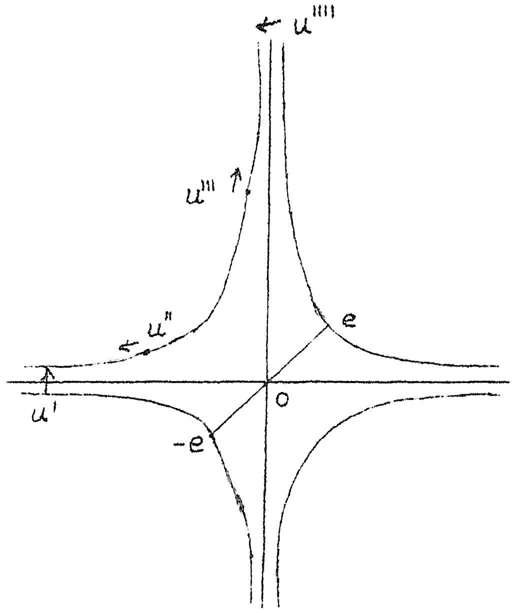
$$(13) \qquad \mu(S, \Lambda) = 1\ , \qquad d(\Lambda) < 3.$$

Let B denote the boundary of S, and let B_1, B_2, B_3 be the parts of B in the 1st,2nd,3rd quadrant respectively. Let x^0 be the point (1,1) on B_1. We shall first determine the lattices Λ satisfying (13) and <u>having a point on the boundary B</u>. It is no restriction to suppose that $x^0 \in \Lambda$; then Λ has a basis $\{e,f\}$, with

$$(14) \qquad e = x^0.$$

A generic point u_1e+u_2f of Λ may be denoted by $u=(u_1,u_2)$; accordingly, from now on coordinates will always be taken with respect to a (suitably chosen) basis $\{x^0,f\}$ of Λ. The lattice Λ is determined completely if we know three points e,u,v of Λ on the boundary B; likewise, the corresponding quadratic form q is determined uniquely by its values in e,u,v. We now construct a certain denumerable set of lattices and then prove that these lattices are all admissible and give all lattices satisfying our conditions. We repeat that we always take $e=x^0$.





The first lattice, say $\Lambda_1=A_1Y$, is such that $u^2=(-2,1)\in B_2$ and $u^3=(-1,1)\in B_2$. Put

(15) $$q_1(x) = \det(x,V_0x), \quad \text{where} \quad V_0 = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}.$$

Then $q_1(u) = 1,-1,-1$ for $u=e,u^2,u^3$ respectively, and so $\overline{q}(A_1x)=q_1(x)$. For arbitrary A, we have

(16) $$\det(Ax,Ay) = \det A \cdot \det(x,y).$$

Hence, $q_1(V_0x) = q_1(x)$, i.e. V_0 is an automorphism of $q_1(x)$. Hence, $u^1=-V_0^{-1}e=(-3,1)$ lies on B_3 and $u^4=V_0e=(0,1)$ lies on B_1.

The second lattice, say $\Lambda_2=A_2Y$, is obtained from Λ_1 by moving u^2 along B_2 until u^4 reaches B_2. Put

(17) $$q_2(x) = \tfrac{1}{2}\det(x,U_0x), \text{ where } U_0 = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Then $q_2(u) = 1,-1,-1$ for $u=e,u^2,u^4$, and so $\overline{q}(A_2x)=q_2(x)$. Further, U_0 is an automorphism of $q_2(x)$, so that $-U_0^{-1}e=(-5,2)$ lies on B_3 and $U_0e=(1,2)$ lies on B_1. As is easily verified, the matrices U_0, V_0 satisfy the following relation, which will be important in the sequel:

(18) $$U_0V_0 = V_0KU_0, \quad \text{with } K=\begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix}.$$

The third lattice, say $\Lambda_3 = A_3 Y$, has the points $u^4 = V_0 e = (0,1)$ and $-U_0^{-1}e = (-5,2)$ on B_2. We put

(19) $$q_5(x) = \frac{1}{5}\det(x, U_0 V_0 x).$$

We have $q_5(e)=1$. By repeated application of (16) and (18) and by using $Ke=-e$ we find that $q_5(V_0 e) = q_5(-U_0^{-1}e) = -q_5(e) = -1$. So $\overline{q}(A_3 x) = q_5(x)$. Further, $U_0 V_0$ is an automorphism of $q_5(x)$. So we find that Λ_3 has a.o. the following four points on B:

(20) $$u' = -(U_0 V_0)^{-1}e, \quad u'' = -U_0^{-1}e, \quad u''' = U_0 V_0 e.$$

These points are connected with each other by

(21) $$V_0 u' = u'', \quad U_0 V_0 u'' = u''', \quad U_0 u''' = u''''.$$

The above procedure can be continued indefinitely. It should be noted that Λ_1 and Λ_2 are symmetric with respect to the bisectrices of the axes, but not Λ_3. So from Λ_3 we can obtain two different lattices, for which respectively u', u''' and u'', u'''' lie on B_2. We now introduce the following notations.

- $\mathfrak{M}$: set of pairs of integral matrices (U,V) such that
  - a) $(U_0, V_0) \in \mathfrak{M}$
  - b) if $(U,V) \in \mathfrak{M}$, then $(UV, V) \in \mathfrak{M}$ and $(U, UV) \in \mathfrak{M}$
- $|\mathfrak{M}|$: set of matrices in $\mathfrak{M}$
- $\Lambda(U,V)$: lattice through x^0 with $-U^{-1}e \in B_2$ and $Ve \in B_2$
- $\mathcal{L}$: set of the lattices $\Lambda_1, \Lambda_2, \Lambda(U,V)$ $\quad ((U,V) \in \mathfrak{M})$
- $q(U,V;x)$: form q with $q(e)=1$, $q(-U^{-1}e) = q(Ve) = -1$
- $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad L = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ $\quad$ (so that $K = -L^2$).

Then, if (U,V) is a pair of $\mathfrak{M}$ and W is an arbitrary matrix of $|\mathfrak{M}|$, the following seven properties hold:

I $\quad UV = VKU$

II $\quad (WL)^2 = -I$

III $\quad q(U,V;x) = \frac{1}{m}\det(x, Wx)$, where $W = (w_{ij}) = UV$ and $m = w_{21}$

IV $\quad q(U,V;x)$ is invariant under the transformation UV and is transformed into $-q(U,V;x)$ by VL

V $\Lambda(U,V)$ has four points u', u'', u''', u'''' on B_3, B_2, B_2, B_1 respectively, such that

(22) $$UVu' = -e,\ Uu'' = -e,\ Ve = u''',\ UVe = u'''',$$

(23) $$Vu' = u'',\ UVu'' = u''',\ Uu''' = u''''$$

VI W has the form $\begin{pmatrix} k & 1 \\ m & 3m-k \end{pmatrix}$, so that the corresponding form $\frac{1}{m}\det(x,Wx) = \frac{1}{m}\{mx_1^2+(3m-2k)x_1x_2-1x_2^2\} = q_m(x)$, say, has discriminant $d_m = d(q_m) = 9-4m^{-2}$

VII the lattices in $\mathcal{L}$ are admissible for S.

<u>Proof</u>. Properties I and II are easily proved by induction.

Property III can be verified as follows:

$$\det(e,We) = m,$$
$$\det(U^{-1}e, UVU^{-1}e) = \det(U^{-1}e, VKe) = -\det(e,UVe) = -m,$$
$$\det(Ve, UVVe) = \det(e, KUVe) = -m.$$

The first clause in IV is a consequence of III and the second one is proved as follows:

$$\det(VLx, UVVLx) = \det(VLx, -VL^2UVLx) = \det(x, -LUVLx)$$
$$= \det(UVx, -(UVL)^2x) = \det(UVx, x) = -\det(x, UVx).$$

Property V is proved by induction as follows. We saw already that V holds for $\Lambda_3 = \Lambda(U,V)$. Suppose V holds for $\Lambda(U,V)$, and consider $\Lambda(UV,V)$. Let $v^{(i)}$ $(1 \leq i \leq 4)$ be the points with

$$UVVv' = -e,\ UVv'' = -e,\ Ve = v''',\ UVVe = v''''.$$

Then $v'' = u'$, $v''' = u'''$, and so

$$Vv' = -(UV)^{-1}e = v'',\ UVVv'' = UVVu' = UVu'' = u''' = v''',$$
$$UVv''' = UVVe = v''''.$$

Further, by IV and the proof of III, the $v^{(i)}$ lie on B. Hence V holds for $\Lambda(UV,V)$. Similarly for (U,UV).

As for the proof of VI, let $W=UV$ and let the $u^{(i)}$ be given by (22). By II, $(WL)^2e = -e$ or $LWLe = -W^{-1}e$, i.e. $Lu'''' = u'$. Now $u'''' = We$ has second coordinate $w_{21} = m$, and so u'''' has the form (k,m). Then $u' = Lu'''' = (k-3m, m)$. Then, since $u'''' = We$ and $u' = W^{-1}e$, W has the form stated. Since $\det W = 1$, we have

(24) $$k^2+1 = m(3k-1).$$

Finally, $d_m = m^{-2}\{(3m-2k)^2+41m\} = 9-4m^{-2}$. Property VI also holds for $W = U_0, V_0$.

Property VII = lemma 10 in [2], chapter II; the proof is based on IV and VI.

It is now easy to prove the following

Theorem 1. The lattices Λ satisfying (13) and passing through x^0 are just given by the lattices in $\mathcal{L}$.

Proof. Let Λ be such a lattice. Let G_1 be the set of points outside S lying in the 1st or 3rd quadrant, and let G_2 denote the set of points outside S in the 2nd quadrant. We may suppose that (-2,1) and (-1,1) belong to G_2 (see the figure). If (-3,1) and (0,1) belong to G_1, then necessarily $\Lambda = \Lambda_1$. If not, then for reasons of symmetry we may suppose that $(0,1) \in G_2$. Then necessarily $\Lambda = \Lambda_2$, if $(-5,2) \in G_1$ and $(1,2) \in G_1$ (see figure). If not, then we may suppose that $(-5,2) \in G_2$. Then $-U_0^{-1}e$ and V_0e lie in G_2.

We now use the following property, which is geometrically clear: if Λ satisfies our conditions, if $u^{(i)}$ are defined by (22) and if u'', u''' lie in G_2 and u', u'''' in G_1, then necessarily $\Lambda = \Lambda(U,V)$. Suppose that, for some pair $(U,V) \in \mathfrak{M}$,

$$-U^{-1}e \in G_2 \, , \quad Ve \in G_2. \tag{25}$$

Then there are four possibilities:

a) $-(UV)^{-1}e$, $UVe \in G_1$; then $\Lambda = \Lambda(U,V)$

b) $-(UV)^{-1}e, \in G_2$; then (25) holds for the pair (UV,V)

c) $UVe \in G_2$; then (25) holds for the pair (U,UV)

d) $-(UV)^{-1}$, $UVe \in G_2$; it is easy to prove that in this case $d(\Lambda) > 3$.

We now note that $d_m = 9-4m^{-2}$ and that m is a positive integer increasing for each step. It follows that after finitely many steps the possibility a) occurs, since $d(\Lambda) < 3$. This proves theorem 1.

In order to solve our problem completely we must consider lattices which do not have a point on B. But here we have the following

Lemma. Each lattice satisfying (13) has points on B.

The proof of this lemma may be sketched as follows. Let Λ be any lattice with $\mu(S, \Lambda) = 1$, which does not have any point on the boundary. Then there is a sequence of points $x^r (r=1,2,\ldots)$ of with $|\bar{q}(x^r)| \to 1$. Suppose $\bar{q}(x^r) \to 1$. By applying suitable hyperbolic rotations T_r we get a sequence of lattices $\Lambda_r = T_r\Lambda$ with $\mu(S, \Lambda_r) = 1$ $(r=1,2,\dots)$, $\Lambda_r \ni y^r$, $y^r \to x^0$. A suitable subsequence of the sequence $\{\Lambda_r\}$ converges to some lattice $\bar{\Lambda}$. This lattice $\bar{\Lambda}$ contains the point x^0 and has determinant $d(\bar{\Lambda}) = d(\Lambda)$. Further, each point of $\bar{\Lambda}$ is the limit of a point of Λ_r and so $\bar{\Lambda}$ is admissible for S. Hence, by theorem 1, if Λ satisfies (13), then $\bar{\Lambda}$ belongs to the set $\mathcal{L}$. Finally, using the corresponding automorphism W, one can find a neighbourhood N of $\bar{\Lambda}$ and a positive number δ with the following property:

(26) if $\Lambda' \in N$ and Λ' is not homothetic with $\bar{\Lambda}$, then

$$\mu(S, \Lambda') < 1 - \delta$$

(see theorem I in [2], chapter II). This contradicts the properties of the Λ_r, and so proves the lemma.

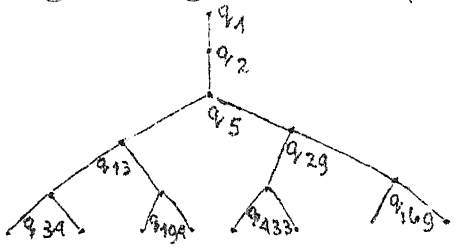
The lattices Λ in $\mathcal{L}$ have points on B_1 as well as B_2. Hence theorem 1 and the lemma together give the following

Theorem 2. Each lattice Λ' with $\mu(S, \Lambda') = 1$, $d(\Lambda') < 3$ is of the form $\Lambda' = T\Lambda$, where T is a hyperbolic rotation of S and $\Lambda \in \mathcal{L}$.

This theorem can immediately be put in the following arithmetic form:

Theorem 2'. Let $q = q(x)$ be an indefinite binary quadratic form, of discriminant d. Then $\mu(q) > \frac{1}{3}\sqrt{d}$, if and only if $q \approx q_m = \frac{1}{m}\det(x, Wx)$ for some $W \in |\mathfrak{M}|$ $(m = w_{21})$. Further, $\mu(q) = \sqrt{d/d_m}$, with $d_m = 9 - 4m^{-2}$, if $q \approx q_m$.

The pairs (U,V), hence also the forms q_m, can be represented by a genealogical tree (see figure). The numbers m associated with



three matrices U,V,W=UV, say m_1, m_2, m, satisfy the famous equation of Markov:

(27) $m_1^2 + m_2^2 + m^2 = 3m_1m_2m$.

This relation is easily proved by induction. Cohn deduces it from property I and a general relation for the traces of 2×2-matrices. As is well known, one can deduce from theorem 2' a corresponding theorem for the approximation of inationals by rationals.

Final remark. Probably the method of this report can be extended to the more general domain $-1 < x_1x_2 < k$ (k a positive integer).

However, even the second minimum of this region is not yet known (see [4]).

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