STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING ZUIVERE WISKUNDE

ZW 1968-008

On the continuity of fixed points of contractions

by

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May 1968

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0. Introduction. Throughout this report

(i) (X, ρ) is a metric space,

(ii) Λ is an index set supplied with a topology.

This report is motivated by the following three well known theorems.

1. [BANACH] (c.f. [1] p. 2 , [2] p. 190, [3] p. 54.).
Let (X, ρ) be complete and let $\phi : X \to X$ be a strong contraction [1)] on X. Then ϕ has precisely one fixed point $\hat{x}$ $(= \phi(\hat{x}))$.

2. If (X, ρ) is complete and if for each $\lambda \in \Lambda$ $\phi_\lambda : X \to X$ is a strong contraction on X, then the fixed point $\hat{x}_\lambda$ of ϕ_λ is a continuous function of λ provided that the following conditions are satisfied:

(i) There exists a constant k $(0 \leq k < 1)$
such that

$$\rho(\phi_\lambda(x_1), \phi_\lambda(x_2)) \leq k.\rho(x_1, x_2)$$

for each $\lambda \in \Lambda$ and all $x_1, x_2 \in X$,

(ii) for each triple ε, x_0, λ_0, where $\varepsilon > 0$, $x_0 \in X$ and $\lambda_0 \in \Lambda$, there exists a neighbourhood $T_{\lambda_0} = T_{\lambda_0}(x_0, \varepsilon)$ of λ_0 such that

$$\rho(\phi_{\lambda_0}(x_0), \phi_\lambda(x_0)) < \varepsilon \quad \text{for all } \lambda \in T_{\lambda_0}.$$

For a proof we refer to [1] p. 6.

Remark. The continuity condition (ii) may be briefly formulated as

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1) This means that there exists a (contraction) constant k $(0 \leq k < 1)$ such that

$$\rho(\phi(x_1), \phi(x_2)) \leq k.\rho(x_1, x_2)$$

for all $x_1, x_2 \in X$.

$$(\forall\varepsilon > 0)(\forall x_0 \in X)(\forall \lambda_0 \in \Lambda)(\exists T_{\lambda_0} = T_{\lambda_0}(x_0, \varepsilon))(\lambda \in T_{\lambda_0} \to$$

$$\rho(\phi_{\lambda_0}(x_0), \phi_\lambda(x_0)) < \varepsilon).$$

<u>3</u>. If $\phi : X \to X$ is a weak contraction [2] on X (X need not be complete) such that the total image $\phi(X)$ of X under ϕ is pre-compact [3] in X, then ϕ has precisely one fixed point $\hat{x}$.
A proof of this theorem can be found in [1] p. 15.

Suppose now, that for each $\lambda \in \Lambda$, $\phi_\lambda : X \to X$ is a weak contraction on X such that $\phi_\lambda(X)$ is pre-compact in X.
Since each ϕ_λ has a unique fixed point $\hat{x}_\lambda$, one may ask under what conditions $\hat{x}_\lambda$ will be a continuous function of λ.
In section 1 we will show that the following condition is sufficient:
For each $x_0 \in X$ and each $\lambda_0 \in \Lambda$, there exists a neighbourhood $U_{x_0} = U_{x_0}(\lambda_0)$ of x_0 such that for each $\varepsilon > 0$ there exists a neighbourhood $T_{\lambda_0} = T_{\lambda_0}(x_0, \varepsilon)$ of λ_0 with the property that $\rho(\phi_{\lambda_0}(x), \phi_\lambda(x)) < \varepsilon$ for all $x \in U_{x_0}$ and all $\lambda \in T_{\lambda_0}$. This condition may also be formulated as follows:

$$(\forall x_0 \in X)(\forall \lambda_0 \in \Lambda)(\exists U_{x_0} = U_{x_0}(\lambda_0))(\forall\varepsilon > 0)(\exists T_{\lambda_0} = T_{\lambda_0}(x_0, \varepsilon))$$

$$(x \in U_{x_0} \wedge \lambda \in T_{\lambda_0} \to \rho(\phi_{\lambda_0}(x), \phi_\lambda(x)) < \varepsilon).$$

Furthermore, it will be proved that if X is locally compact, the following weaker condition is sufficient:
$\phi_\lambda(x)$, as a function of the two variables λ and x, is continuous on $\Lambda \times X$. To show the difference between this condition and the previous one, we restate this continuity condition as follows:

$$(\forall\varepsilon > 0)(\forall x_0 \in X)(\forall \lambda_0 \in \Lambda)(\exists U_{x_0} = U_{x_0}(\lambda_0, \varepsilon))(\exists T_{\lambda_0} = T_{\lambda_0}(x_0, \varepsilon))$$

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[2] This means that $\rho(\phi(x_1), \phi(x_2)) < \rho(x_1, x_2)$ for all $x_1, x_2 \in X$ such that $x_1 \neq x_2$.

[3] This means that the closure $\overline{\phi(X)}$ of $\phi(X)$ is compact in X.

$$(x \in U_{x_0} \wedge \lambda \in T_{\lambda_0} \rightarrow \rho(\phi_{\lambda_0}(x_0), \phi_\lambda(x)) < \varepsilon).$$

In section 2 we will show by means of an example that in the last case $\hat{x}_\lambda$ need not be a continuous function of λ if we omit the condition that X is locally compact.

1. Throughout this section we will assume that for each $\lambda \in \Lambda$, $\phi_\lambda : X \to X$ is a weak contraction on X such that $\phi_\lambda(X)$ is pre-compact in X.

Theorem 1.1. If for each $x_0 \in X$ and each $\lambda_0 \in \Lambda$ there exists a neighbourhood $U_{x_0} = U_{x_0}(\lambda_0)$ of x_0 such that for each $\varepsilon > 0$ there exists a neighbourhood $T^0_{\lambda_0} = T^0_{\lambda_0}(x_0, \varepsilon)$ of λ_0 with the property that

$$\rho(\phi_{\lambda_0}(x), \phi_\lambda(x)) < \varepsilon \quad \text{for all } x \in U_{x_0} \text{ and all } \lambda \in T_{\lambda_0},$$

then $\hat{x}_\lambda$ is a continuous function of λ.

Proof: Let λ_0 be any point of Λ; for $\hat{x}_{\lambda_0}$ and λ_0 there exists a neighbourhood U_0 of $\hat{x}_{\lambda_0}$ such that for each $\alpha > 0$ there exists a neighbourhood $T_{\lambda_0}(\alpha)$ of λ_0 with the property that

$$\rho(\phi_{\lambda_0}(x), \phi_\lambda(x)) < \alpha \quad \text{for all } x \in U_0 \text{ and all } \lambda \in T_{\lambda_0}(\alpha).$$

Let B be a closed ball with center $\hat{x}_{\lambda_0}$ and radius r ($0 < r \leqq \varepsilon$) which is contained in U_0.
Since ϕ_{λ_0} is a contraction and B is a ball with center the fixed point $\hat{x}_{\lambda_0}$ of ϕ_{λ_0}, it is clear that $\phi_{\lambda_0}(B) \subset B$. Since ϕ_{λ_0} is a weak contraction, we can not say that $\phi_{\lambda_0}(B)$ is contained in a ball B_1 with center $\hat{x}_{\lambda_0}$ and radius $r_1 < r$. To overcome this difficulty we consider the mapping $\phi_\lambda^2 : X \to X$, where $\phi_\lambda^2(x) = \phi_\lambda(\phi_\lambda(x))$ for all $x \in X$. It is easily seen that ϕ_λ^2 is a weak contraction on X and that $\phi_\lambda^2(X)$ is pre-compact. Hence ϕ_λ^2 has a unique fixed point $\tilde{x}_\lambda$ which is easily seen to be equal to $\hat{x}_\lambda$.
We will show that $\phi_{\lambda_0}^2(B)$ is contained in a ball B_1 with center $\hat{x}_{\lambda_0}$ and radius $r_1 < r$.
Since $\phi_{\lambda_0}^2(B) \subset \phi_{\lambda_0}(\overline{\phi_{\lambda_0}(B)})$ it is sufficient to show that $\phi_{\lambda_0}(\overline{\phi_{\lambda_0}(B)})$ is contained in such a ball B_1. In order to do this we consider the continuous

function $\rho(\hat{x}_{\lambda_0}, \phi_{\lambda_0}(x))$ on the compact set $\overline{\phi_{\lambda_0}(B)}$. Let the maximum of this function be $r_1 (\leq r)$.

It is clear that $\phi_{\lambda_0}(B) \subset B$; it is also clear that $\phi_{\lambda_0}(B)$ does not contain any point of the boundary of B. From this it is easily seen that the compact set $\phi_{\lambda_0}\overline{(\phi_{\lambda_0}(B))}$ is contained in B and has no points in common with the boundary of B. Thus for the point y_0 in which the function $\rho(\hat{x}_{\lambda_0}, \phi_{\lambda_0}(x))$ has its maximum, we have $\rho(\hat{x}_{\lambda_0}, \phi_{\lambda_0}(y_0)) < r$ and hence $\phi_{\lambda_0}\overline{(\phi_{\lambda_0}(B))}$ is contained in a ball B_1 with center $\hat{x}_{\lambda_0}$ and radius $r_1 < r$.

We now consider the images of B under the mappings ϕ_λ^2. If $x \in B$, then we have (because of $\phi_{\lambda_0}(B) \subset B \subset U_0$) for each $\lambda \in T_{\lambda_0}(\frac{r-r_1}{2})$,

$$\rho(\phi_{\lambda_0}^2(x), \phi_\lambda^2(x)) \leq \rho(\phi_{\lambda_0}(\phi_{\lambda_0}(x)), \phi_\lambda(\phi_{\lambda_0}(x))) + \rho(\phi_\lambda(\phi_{\lambda_0}(x)), \phi_\lambda(\phi_\lambda(x))) <$$

$$< \frac{r - r_1}{2} + \frac{r - r_1}{2} = r - r_1,$$

so that $\phi_\lambda^2(B) \subset B$ for all $\lambda \in T_{\lambda_0}(\frac{r-r_1}{2})$.

Since $\phi_\lambda^2(B)$ is pre-compact in X, B is closed and $\phi_\lambda^2(B) \subset B$ for $\lambda \in T_{\lambda_0}(\frac{r-r_1}{2})$, it follows that $\phi_\lambda^2(B)$ is also pre-compact in the subspace B of X.

From this it is clear that the unique fixed point $\hat{x}_\lambda$ of ϕ_λ must be contained in B for all $\lambda \in T_{\lambda_0}(\frac{r-r_1}{2})$ so that $\hat{x}_\lambda$ is continuous at $\lambda = \lambda_0$.

<u>Theorem 1.2</u>. Let (X,ρ) be locally compact. If for each $\varepsilon > 0$, each $x_0 \in X$ and each $\lambda_0 \in \Lambda$ there exist neighbourhoods $U_{x_0} = U_{x_0}(\lambda_0, \varepsilon)$ and $T_{\lambda_0} = T_{\lambda_0}(x_0, \varepsilon)$ of x_0 and λ_0, respectively, such that

$$\rho(\phi_{\lambda_0}(x_0), \phi_\lambda(x)) < \varepsilon \text{ for all } x \in U_{x_0} \text{ and all } \lambda \in T_{\lambda_0},$$

then $\hat{x}_\lambda$ is a continuous function of λ.

Proof: it is sufficient to prove that the continuity condition of theorem 1.1 is satisfied.

Let x_0 be any point in X, λ_0 any point in Λ and C any compact neighbourhood of x_0. For each point $p \in C$ and each $\varepsilon > 0$ there exists an open neighbourhood $U_p = U_p(\lambda_0, \varepsilon)$ of p and an open neighbourhood $T_{\lambda_0}(p, \frac{\varepsilon}{2})$

of λ_0 such that

$$\rho(\phi_{\lambda_0}(p), \phi_\lambda(x)) < \frac{\varepsilon}{2} \quad \text{for all} \quad x \in U_p \text{ and all } T_{\lambda_0}(p, \frac{\varepsilon}{2}).$$

Since $p \in C$ we have $C \subset \bigcup_{p \in C} U_p$. The compactness of C implies that there is a finite number of points $p_i \in C$ ($i = 1, 2, 3, \ldots, n$) such that
$C \subset \bigcup_{i=1}^{n} U_{p_i}$.
We define $T_0(\varepsilon) = \bigcap_{i=1}^{n} T_{\lambda_0}(p_i, \frac{\varepsilon}{2})$.
Each $x \in C$ is contained in at least one U_{p_i} and hence

$$\rho(\phi_{\lambda_0}(x), \phi_\lambda(x)) \leqq \rho(\phi_{\lambda_0}(x), \phi_{\lambda_0}(p_i)) + \rho(\phi_{\lambda_0}(p_i), \phi_\lambda(x)) <$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for all } x \in C \text{ and all } \lambda \in T_0(\varepsilon).$$

It follows that theorem 1.2. is a particular case of theorem 1.1.

2. In this section we will show by means of an example that theorem 1.2. is not generally true if one omits the condition that X is locally compact.

Let X be the subset of the x-y plane which may be described as follows: Connect the origin O(0, 0) with the points A_i ($i = 1, 2, 3, \ldots$) on the circle $x^2 + y^2 = 1$, where the points A_i are chosen such that

(i) A_i lies in the first quadrant
(ii) $\tan A_iOP = \frac{1}{i}$, where P is the point (1, 0).

On X we define the following metric ρ: if $w_1 \in X$ and $w_2 \in X$ are on the same radius OA_i, then $\rho(w_1, w_2)$ is the usual Euclidian distance between w_1 and w_2; in case w_1 and w_2 are on two different radii, then

$$\rho(w_1, w_2) = \rho(w_1, O) + \rho(w_2, O).$$

It is well known that (X, ρ) is a complete metric space.
For Λ we take the set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\}$ supplied with the usual topology.
The contractions $\phi_\lambda : X \to X$ will be defined by

(i) if $\lambda = 0$, then $\phi_\lambda(X) = O$
(ii) if $\lambda = \frac{1}{i}$ ($i = 1, 2, 3, \ldots$), then

a. in case $\rho(w, A_\lambda) \geq \frac{i+1}{i}$ then $\phi_\lambda(w) = 0$

b. in case $\rho(w, A_\lambda) < \frac{i+1}{i}$ then $\phi_\lambda(w)$ is the point with distance $\frac{i}{i+1}\rho(w, A_\lambda)$ (< 1) from A_λ.

It is easily verified that $\phi_\lambda : X \to X$ is a strong contraction for each $\lambda \in \Lambda$, such that $\phi_\lambda(X)$ is pre-compact. Furthermore, the continuity condition of theorem 1.2. is satisfied.

However, X is not locally compact since the origin 0 has no compact neighbourhoods. The contraction ϕ_0 has the fixed point 0, where $\phi_{\frac{1}{i}}$ has the fixed point A_i.

Consequently, because $\rho(A_i, 0) = 1$ for all $\lambda \neq 0$, $\hat{x}_\lambda$ is discontinuous at $\lambda = 0$.

3. In this section we will consider two additional theorems concerning strong contractions on complete metric spaces.

Throughout this section (X, ρ) will be complete and for each $\lambda \in \Lambda$, $\phi_\lambda : X \to X$ will be a strong contraction on X.

We will not assume that the least upper bound of all contraction constants k_λ is smaller than 1.

Theorem 3.1. The fixed point $\hat{x}_\lambda$ of ϕ_λ is a continuous function of λ provided that the continuity condition of theorem 1.1. is satisfied.

Proof: Let λ_0 be any point in Λ. For $\hat{x}_{\lambda_0}$ and λ_0 there exists a neighbourhood U_0 of $\hat{x}_{\lambda_0}$ such that for each $\alpha > 0$ there exists a neighbourhood $T_{\lambda_0}(\alpha)$ of λ_0 with the property that

$$\rho(\phi_{\lambda_0}(x), \phi_\lambda(x)) < \alpha \quad \text{for all } x \in U_0 \text{ and all } \lambda \in T_{\lambda_0}(\alpha).$$

Let B be a closed ball with center $\hat{x}_{\lambda_0}$ and radius r $(0 < r \leq \varepsilon)$ which is contained in U_0.

Since $\hat{x}_{\lambda_0} \in B$ and ϕ_{λ_0} is a strong contraction, $\phi_{\lambda_0}(B)$ is contained in a ball B_1 with center $\hat{x}_{\lambda_0}$ and radius $r_1 = k_{\lambda_0} \cdot r < r$.

If we take $\alpha = r - r_1$ then we have

$$\rho(\phi_{\lambda_0}(x), \phi_\lambda(x)) < r - r_1 \text{ for all } x \in B \subset U_0 \text{ and all } \lambda \in T_{\lambda_0}(r-r_1).$$

From this it follows that $\phi_\lambda(B) \subset B$ for all $\lambda \in T_{\lambda_0}(r - r_1)$. Since B itself is a complete subspace of X and for each $\lambda \in T_{\lambda_0}(r - r_1)$ the strong contraction ϕ_λ maps B into itself, we have that $\hat{x}_\lambda \in B$, because of the uniqueness of the fixed point of ϕ_λ.
From this it is clear that $\hat{x}_\lambda$ is a continuous function of λ.

Theorem 3.2. If (X, ρ) is locally compact and the continuity condition of theorem 1.2. is satisfied, then $\hat{x}_\lambda$ is a continuous function of λ.
This theorem may be proved in the same way as theorem 1.2.

Literature

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