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On pseudo-convergent sequences

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On pseudo-convergent sequences

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A sequence $\{a_i\}$ of elements of a non-Archimedean valuated field K is called pseudo-convergent if either $|a_{i+1} - a_i| < |a_i - a_{i-1}|$ for all $i \geq i_0$ or $a_i = a_{i+1}$ for all $i \geq i_1$. It follows that for such a sequence either i: $|a_i| = |a_{i+1}|$ for all $i \geq i_2$ or ii: $|a_i| > |a_{i+1}|$ for all $i \geq i_3$. The sequence is called (pseudo-convergent) of the first kind or second kind according as i or ii holds. The property ii is sufficient for the pseudo-convergency of a sequence, property i however is not. This concept is due to A.Ostrowski (Untersuchungen zur arithmetischen Theorie der Körper; Math. Zeitschr. 39, p. 269-404 (1934)), who proved the following theorem: If $\{a_i\}$ is a pseudo-convergent sequence ($a_i \in K$) and f(x) a polynomial with coefficients in K then the sequence $\{f(a_i)\}$ is also pseudo-convergent.

F. Loonstra (Pseudokonvergente Folgen in nichtarchimedisch bewerteten Körpern; Proc. Ned. Akad. v. Wet. XLV, p.913-917 (1942)) treated this problem without using algebraic extension of K, contrary to Ostrowski.

The lemma stated as "Satz IV"·) however is false, as is shown by the following counter example. Take for K the field of the rational numbers with the 2-adic valuation and let $\overline{K}$ be its completion i.e. the field of 2-adic numbers.

The polynomial $x^2 + 7$ has a zero in $\overline{K}$ and not in K(for the underlying theory see B.L. van der Waerden, Moderne Algebra I, 3e aufl. (1950) § 79).

Let $a = \sum_{j=0}^{\infty} 2^{\nu_j}$ (with $\nu_{j+1} > \nu_j$) be the 2-adic expansion of such a zero. Now put $a_i = \sum_{j=0}^{i} 2^{\nu_j}$, then $\{a_i\}$ is pseudo-convergent since $|a_{i+1} - a_i| = 2^{-\nu_{i+1}} < 2^{-\nu_i} = |a_i - a_{i-1}|$.

·) Satz IV. Sei $\{a_i\}$ eine pseudokonvergente Folge. Es gebe weiter in K kein Element α derart, dasz die pseudokonvergente Folge $\{a_i - \alpha\}$ von der 2. Art ist. Sei f(x) ein Polynom mit Koeffizienten aus K. Dann ist $|f(a_i)|$ konstant von einem gewissen i_0 an.

$\{a_i - \alpha\}$ is pseudo-convergent of the first kind for all $\alpha \in K$. For, if $\alpha = \sum_{j=0}^{\infty} 2^{\mu_j}$ (with $\mu_{j+1} > \mu_j$) and if j_0 is the smallest index such that $\mu_{j_0} \neq \nu_{j_0}$ (it always exists because $\alpha \notin K$), then for $i > j_0$ we have $|a_i - \alpha| = 2^{-\min(\mu_{j_0}, \nu_{j_0})}$. Take for $f(x)$ the polynomial $x^2 + 7$. The sequence $\{a_i^2 + 7\}$ is convergent with the limit 0, hence $|a_i^2 + 7| \rightarrow 0$ and since $|a_i^2 + 7| \neq 0$ we have a contradiction with "Satz IV".