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On the existence of well distributed sequences in compact spaces

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1. Introduction.

The concept of a well distributed sequence was introduced by E. HLAWKA [6] and independently by G.M. PETERSEN [10]. Every well distributed sequence is uniformly distributed; but whereas there are many uniformly distributed sequences (almost every sequence in a compact Hausdorff space satisfying the second axiom of countability and furnished with a normed Borel measure μ is μ-uniformly distributed, E. HLAWKA [6,7]), the set of well distributed sequences is much smaller: it was shown by G. HELMBERG and A.B. PAALMAN - DE MIRANDA [5] that almost no sequence is well distributed. This leads to the question (posed in the colloquium on uniform distribution at the Mathematical Centre in Amsterdam, 1963/1964) whether well distributed sequences exist at all in every compact Hausdorff space satisfying the second countability axiom and for every normed Borel measure. (cf. also [1], where several results on almost well distributed sequences in such spaces are obtained under the explicit assumption that the space admits at least one well distributed sequence).

In the present paper we show that the answer to this question is affirmative: if X is an arbitrary non-void compact Hausdorff space satisfying the second axiom of countability and if μ is an arbitrary normed Borel measure on X, there exists a μ-well distributed sequence in X. In the proof we apply (in a modified form) a construction used by the second author in [3] in order to show the existence of uniformly distributed sequences.

It was pointed out to us by G. HELMBERG that our construction used for proving lemma 4 (the special case of a non-atomic measure) is closely related to the method used by P.R. HALMOS in exhibiting the isomorphism between an arbitrary separable, non-atomic, normalized measure algebra and the measure algebra of the unit segment I ([2], section 41). In fact this isomorphism theorem can easily be derived from our results; but we need more then a correspondence between measure algebra's: we need and construct a point-by-point map, in order to be able to lift a well-distributed sequence from the unit interval (where they are known to exist, cf. [6,10]) to the space under consideration.

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2. Notation.

By N we will denote the set of natural numbers. The unit segment [0,1], furnished with its usual topology, is designated by I, while λ is used for Lebesgue measure on I.

If X is a compact Hausdorff space, and if μ is a Borel measure on X, the restriction of μ to a Borel subset of X will also be denoted by μ. Moreover, if ϕ is an integrable function on X, we write $\mu(\phi)$ synonymously with $\int_X \phi(x)\, d\mu$.

The interior of a subset A of a topological space is denoted by A^o, its boundary by b(A). If X is a metric space and $A \subset X$, then d(A) designates the diameter of A measured in the given metric of X.

If X is a set, then $\mathcal{P}(X)$ will denote the power set (set of all subsets) of X.

Every topological space X occurring in the sequal is assumed to be non-void.

3. The special case of non-atomic Borel measures.

Definition 1. Let X be a compact metric space, μ a Borel measure on X, and ε a positive real number. An (X,μ,ε)-quasicover is a finite collection $\mathcal{C} = \{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{P}(X)$ with the following properties:

(Q1) C_i is compact, and $d(C_i) < \varepsilon$, for $1 \leq i \leq n$.

(Q2) $\mu C_i > 0$ and $\mu b(C_i) = 0$, for $1 \leq i \leq n$.

(Q3) $C_i^o \cap C_j^o = \emptyset$ if $i \neq j$, $1 \leq i,j \leq n$.

(Q4) $\mu(X \setminus \bigcup \mathcal{C}) = 0$.

Remark. It follows from (Q2) that $C_i^o \neq \emptyset$ for all i.

Lemma 1. Let X be a compact metric space, μ a Borel measure on X, and ε > 0. There exists an (X,μ,ε)-quasicover.

Proof.

Let ρ denote the metric of X. If $x \in X$ and δ > 0, we denote by $V_\delta(x)$ and $U_\delta(x)$ the following sets:

(1) $$V_\delta(x) = \{y \in X: \rho(x,y) < \delta\} ;$$

(2) $$U_\delta(x) = \{y \in X: \rho(x,y) = \delta\} .$$

Let $a_1, a_2, \ldots, a_m$ be a finite number of points in X such that $X \subset \bigcup_{i=1}^{m} V_{\varepsilon/2}(a_i)$. For each i, $1 \leq i \leq m$, let δ_i be a real number such that $\frac{\varepsilon}{2} < \delta_i < \varepsilon$ and $\mu U_{\delta_i}(a_i) = 0$. (Such a δ_i certainly exists, for if each of the uncountably many disjoint sets $U_\delta(a_i)$, $\frac{\varepsilon}{2} < \delta < \varepsilon$, would have a positive measure, the measure of the compact space X could not be finite).

Now consider all sets of the form $A_1 \cap A_2 \cap \ldots \cap A_m$, where each A_i is either $\overline{V_{\delta_i}(a_i)}$ or $X \setminus V_{\delta_i}(a_i)$. Let $C_1, C_2, \ldots, C_n$ be those among these sets which have a positive measure. Then $C = \{C_1, C_2, \ldots, C_n\}$ is an (X,μ,ε)-quasicover.

In the description of our constructions below it will be useful to work with certain partially ordered index sets. We will call them D-sets; they are obtained in the following way.

Let Σ be the set of all finite sequences $n_1 n_2 \ldots n_k$ of non-negative integers. The length of a sequence $\sigma \in \Sigma$ will be denoted by L(σ). If $\sigma_1 \in \Sigma$ and $\sigma_2 \in \Sigma$, say $\sigma_1 = n_1 n_2 \ldots n_k$ and $\sigma_2 = m_1 m_2 \ldots m_k$, we put

(3) $$\sigma_1 \leqq \sigma_2 \quad \text{if } k \leqq h \ \underline{\text{and}}\ n_i = m_i \ \text{ for } 1 \leqq i \leqq k.$$

The relation $\leqq$ defined in this way partially orders Σ. Each $\sigma \in \Sigma$ of length L(σ) > 1 has exactly one immediate predecessor, denoted by p(σ), and denumerably many immediate successors, constituting a set S(σ).

A <u>D-set</u> is a subset Δ of Σ enjoying the following properties:

(D1) $0 \in \Delta$, and 0 is the only element of Δ of length 1.

(D2) If $\sigma \in \Delta$ and $L(\sigma) > 1$, then $p(\sigma) \in \Delta$.

(D3) If $\sigma \in \Delta$, then $0 \neq \text{card}\,(S(\sigma) \cap \Delta) < \aleph_0$.

If Δ is a D-set and $k \in N$, we will denote by Δ_k the set

(4) $$\Delta_k = \{\sigma \in \Delta: L(\sigma) = k\}.$$

<u>Definition 2</u>. Let X be a compact metric space, μ a normed Borel measure on X, and Δ a D-set. A Δ-sieve on X is a map $\Phi: \Delta \to \mathcal{P}(X)$ with the following properties:

(S1) $\Phi 0 = X$.

(S2) If $\sigma \in \Delta$, and if $S(\sigma) \cap \Delta = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$, $L(\sigma) = n$, then $\{\Phi\sigma_1, \Phi\sigma_2, \ldots, \Phi\sigma_k\}$ is a $(\Phi\sigma, \mu, \frac{1}{2^n})$-quasicover.

<u>Lemma 2</u>. Let X be a compact metric space and μ a Borel measure on X. There exists a Δ-sieve Φ on X, for a suitable D-set Δ.

<u>Proof</u>.

We will construct successively Δ_1 and $\Phi | \Delta_1$, Δ_2 and $\Phi | \Delta_2$, etc. Let $\Delta_1 = \{0\}$ and $\Phi(0) = X$. Suppose now that Δ_k and $\Phi | \Delta_k$ are already defined. For each $\sigma \in \Delta_k$ there exists, by lemma 1, a $(\Phi\sigma, \mu, \frac{1}{2^k})$-quasicover $(C_{\sigma 0}, C_{\sigma 1}, \ldots, C_{\sigma n_\sigma})$. Let Δ_{k+1} consist of all sequences σi with $\sigma \in \Delta_k$ and $0 \leq i \leq n_\sigma$, and let $\Phi | \Delta_{k+1}$ be defined by putting $\Phi(\sigma i) = C_{\sigma i}$. Clearly $\Delta = \bigcup_{k=1}^{\infty} \Delta_k$ is a D-set, and the map Φ is a Δ-sieve on X.

<u>Lemma 3</u>. Let X be a compact metric space, μ a normed Borel measure on X, and Φ a Δ-sieve on X. There exists a map $\Psi: \Delta \to \mathcal{P}(I)$ enjoying the following properties:

(i) $\Psi(\sigma)$ is a segment, for each $\sigma \in \Delta$, and $\Psi(0) = I$.

(ii) $\bigcup\{\Psi(\tau): \tau \in \Delta \cap S(\sigma)\} = \Psi(\sigma)$, for each $\sigma \in \Delta$.

(iii) $\lambda\Psi(\sigma) = \mu\Phi(\sigma)$, for each $\sigma \in \Delta$.

Proof.

We define $\Psi \mid \Delta_1$ by putting $\Psi(0) = I$. Suppose $\Psi \mid \Delta_i$ is already defined for $1 \leq i \leq k$ in such a way that the requirements (i), (ii), (iii) are met, and let $\sigma \in \Delta_{k+1}$. Writing τ for $p(\sigma)$, we know then that $\Psi\tau$ is a subsegment $[a,b]$ of I and that $b-a = \mu(\Phi\tau)$. Suppose

$$S(\tau) = \{\tau n_1, \tau n_2, \ldots, \tau n_r\}, \tag{5}$$

with $n_1 < n_2 < \ldots < n_r$; say $\sigma = \tau n_i$. We put

$$\Psi_\sigma = \begin{cases} [a,\ a + \mu(\Phi\sigma)] & \text{in case } i=1\ ; \\ \left[a + \sum_{j=1}^{i-1} \mu\Phi(\tau n_j),\ a + \sum_{j=1}^{i} \mu\Phi(\tau n_j)\right] & \text{if } 2 \leq i \leq r. \end{cases} \tag{6}$$

The mapping Ψ defined in this way satisfies the conditions.

Lemma 4. Let X be a compact metric space, μ a normed Borel measure on X, and suppose in addition that μ is non-atomic (i.e. $\mu(\{x\}) = 0$ for every $x \in X$). Then there exists in X a μ-well distributed sequence.

Proof.

Let Φ be a Δ-sieve on X, and let $\Psi\colon \Delta \to \mathcal{P}(I)$ meet the requirements of lemma 3. We put

$$X_1 = \bigcap_{k=1}^{\infty} \left(\bigcup \{(\Phi\sigma)^o \colon \sigma \in \Delta_k\} \right) ; \tag{7}$$

clearly $\mu X_1 = \mu X = 1$.

If $x \in X_1$, then for every natural number k there is exactly one $\sigma = \sigma_k \in \Delta$ such that $L(\sigma_k) = k$ and $x \in \Phi\sigma_k$ (by condition (Q3) in definition 1); moreover, we know that $\Phi\sigma_{k+1} \subset \Phi\sigma_k$ for all k. As μ is non-atomic, $\mu\Phi\sigma_k \to 0$ for $k \to \infty$ (we use here the fact that every Borel measure on a separable locally compact space is a Baire measure and hence is regular; cf. [2] chapter X). It follows that $\Psi\sigma_{k+1} \subset \Psi\sigma_k$ for all k, and that $\lambda\Psi\sigma_k \to 0$ for $k \to \infty$ (as $\lambda\Psi\sigma_k = \mu\Phi\sigma_k$; lemma 3, (iii)).

Consequently there exists exactly one $y \in I$ such that $y \in \Psi\sigma_k$ for all k; this y we will denote by $f(x)$.

Let $I_1 = fX_1 = \{fx: x \in X_1\}$. We assert that I_1 is dense in I. Assume to the contrary that $I \setminus I_1$ contains an open set U. As every $y \in I$ is the common point of a descending chain of sets $\Psi\sigma$, $\sigma \in \Delta$, there exists a $\sigma \in \Delta$ such that $\Psi\sigma \subset U$. By (S2) and (Q2), $\mu\Phi\sigma > 0$; hence, as $\mu(X \setminus X_1) = 0$, there exists an $x \in X_1 \cap \Phi\sigma$. For this x we have $fx \in I_1$ and on the other hand $f(x) \in \Psi\sigma \subset U$, which is impossible.

Now it is well known that there exist λ-well distributed sequences in I (e.g. the sequences of the form $(n\theta - [n\theta])_{n \in N}$, θ irrational), and it follows easily that every dense subset of I contains a sequence which is λ-well distributed in I (cf. [6,9]; see also [4]). Let $(y_n)_{n \in N}$ be a λ-well distributed sequence in I such that $y_n \in I_1$ for all n; for each $n \in N$, let $x_n \in X_1$ be chosen in such a way that $fx_n = y_n$. We assert that $(x_n)_{n \in N}$ is μ-well distributed in X.

Let ϕ be an arbitrary real-valued continuous function on X, and let $\varepsilon > 0$. We have to show that there exists an $M_0 = M_0(\varepsilon, \phi)$, such that

$$\left| \frac{1}{M} \sum_{m=1}^{M} \phi(x_{m+k}) - \mu(\phi) \right| < \varepsilon \tag{8}$$

for all $M \geqq M_0(\varepsilon,\phi)$, uniformly in $k \in N$.

As ϕ is uniformly continuous on the compact space X, there exists a $\delta > 0$ such that $|\phi x - \phi y| < \frac{\varepsilon}{4}$ whenever $\rho(x,y) < \delta$ (ρ denoting the metric of X). We fix an $r \in N$ such that $\frac{1}{2^r} < \delta$, and for each $\sigma \in \Delta_r$ we choose an arbitrary but fixed $z_\sigma \in \Phi\sigma$.

Let $\eta = \frac{\varepsilon}{4} \left(1 + \sum_{\sigma \in \Delta_r} |\phi(z_\sigma)|\right)^{-1}$.

If $\sigma \in \Delta_r$, then

$$\left| \mu\Phi\sigma - \frac{1}{M} \sum_{m=1}^{M} \chi_{\Phi\sigma}(x_{m+k}) \right| = \left| \lambda\Psi\sigma - \frac{1}{M} \sum_{m=1}^{M} \chi_{\Psi\sigma}(y_{m+k}) \right| ; \tag{9}$$

hence, as the sequence $(y_n)_{n \in N}$ is λ-well distributed in I, there exists an $M_1 = M_1(\varepsilon, \sigma)$, independent of k, such that

$$(10) \qquad \left| \mu\Phi\sigma - \frac{1}{M} \sum_{m=1}^{M} \chi_{\Phi\sigma}(x_{m+k}) \right| < \eta$$

for all $M \geq M_1$. Let $M_o = \max_{\sigma \in \Delta_r} M_\sigma$; M_o depends on ε (and on ϕ), but not on k, and

$$(11) \qquad \left| \sum_{\sigma \in \Delta_r} \mu\Phi\sigma . \phi(z_\sigma) - \frac{1}{M} \sum_{m=1}^{M} \sum_{\sigma \in \Delta_r} \chi_{\Phi\sigma}(x_{m+k})\, \phi(z_\sigma) \right| <$$

$$< \eta . \sum_{\sigma \in \Delta_r} |\phi(z_\sigma)| < \frac{\varepsilon}{4}$$

whenever $M \geq M_o$.

Next, we consider the expression

$$(12) \qquad \left| \sum_{\sigma \in \Delta_r} \chi_{\Phi\sigma}(x_{m+k})\, \phi(z_\sigma) - \phi(x_{m+k}) \right| .$$

As each x_n $(n \in N)$ is contained in exactly one $\Phi\sigma$, $\sigma \in \Delta_r$, (12) may be reduced to the form

$$(13) \qquad \left| \phi(z_{\sigma_o}) - \phi(x_{m+k}) \right|$$

with $x_{m+k} \in \Phi\sigma_o$. Consequently, its value is at most $\frac{\varepsilon}{4}$ (by the choice of r), and we find that, for all M,

$$(14) \qquad \left| \frac{1}{M} \sum_{m=1}^{M} \sum_{\sigma \in \Delta_r} \chi_{\Phi\sigma}(x_{m+k})\, \phi(z_\sigma) - \frac{1}{M} \sum_{m=1}^{M} \phi(x_{m+k}) \right| < \frac{\varepsilon}{4} .$$

Finally we remark that

$$(15) \qquad \left|\mu\phi - \sum_{\sigma\in\Delta_r} \mu\Phi\sigma\cdot\phi(z_\sigma)\right| =$$

$$= \left|\sum_{\sigma\in\Delta_r}\left\{\int_{\Phi\sigma}\phi(x)\,d\mu - \phi(z_\sigma)\,\mu\Phi\sigma\right\}\right| < 2\frac{\varepsilon}{4}.$$

Combining (11), (14) and (15) we arrive at (8).

4. Proof of the main result.

Theorem. Let X be a compact space satisfying the second axiom of countability, and let μ be a normed Borel measure on X. There exists in X a μ-well distributed sequence.

Proof.

It follows from Urysohn's metrization theorem (cf. e.g. [8] chapter 4) that the space X is metrizable. Therefore the assertion follows from lemma 4 in case the set

$$(16) \qquad X_o = \{x \in X: \mu(\{x\}) > 0\}$$

is empty. Let us suppose now that $X_o \neq \emptyset$. We remark that X_o, being countable, is a Borel set.

We first assume $\mu X_o \neq 1$. In that case we define a new normed Borel measure ν on X in the following way: if B is an arbitrary Borel set in X, then

$$(17) \qquad \nu B = \frac{\mu(B\setminus X_o)}{\mu(X\setminus X_o)}.$$

According to lemma 4, there exists in X a ν-well distributed sequence $(x_n)_{n\in N}$.

Let $\{z_1, z_2, z_3, \ldots\}$ be an enumeration of X_o, and let

$I_o = [0,1-\mu X_o|$, $I_1 = [1-\mu X_o, 1-\mu X_o+\mu(\{z_1\})]$,
$I_2 = [1-\mu X_o+\mu(\{z_1\}), 1-\mu X_o+\mu(\{z_1\})+\mu(\{z_2\})]$, etc.
Let $(y_n)_{n\in N}$ be a λ-well distributed sequence in I such that no y_n is an endpoint of one of the intervals $I_o, I_1, I_2, \ldots$, and let $(y_{n_i})_{i\in N}$ be its subsequence consisting of all $y_n \in I_o$.

We now define a new sequence $(u_n)_{n\in N}$ in X as follows. If $n = n_i$, for some i, we put $u_n = x_i$; if $n \neq n_i$ but, say, $y_n \in I_k$, we put $u_n = z_k$. We will show that this sequence $(u_n)_{n\in N}$ is μ-well distributed in X.

For arbitrary $n,k,M \in N$ we define

$$j_n(k,M) = \sum_{m=1}^{M} \chi_{I_n}(y_{m+k}) \quad . \tag{18}$$

Let $\varepsilon > 0$ be arbitrary, and let ϕ be a continuous real-valued function on X. As $\sum_n \mu(\{z_n\}) < 1 < \infty$, there exists an $n_o \geqq 1$, depending only on ε and ϕ, such that

$$(1 + \max_{x\in X} |\phi(x)|) \sum_{n>n_o} \mu(\{z_n\}) < \frac{\varepsilon}{8} \quad . \tag{19}$$

As $(y_n)_{n\in N}$ is λ-well distributed, there exists an M_1, depending only on ε and ϕ, such that, for all $n \leqq n_o$ and all $M \geqq M_1$,

$$\left| \frac{1}{M} j_n(k,M) - \lambda I_n \right| < \varepsilon\cdot(1 + 4n_o\cdot\max_{x\in X} |\phi(x)|)^{-1} \quad , \tag{20}$$

uniformly in k. In particular it follows that $j_n(k,M) \to \infty$ if $M \to \infty$, for all $n \leqq n_o$ and uniformly in k, and also that there exists a $K > 0$ such that

$$\left| \frac{j_n(k,M)}{M} \right| < K \tag{21}$$

for all $n \leqq n_o$, all k and all M.

Let $J = \bigcup_{n > n_o} I_n$; by (19), $\lambda J < \frac{\varepsilon}{8}(1 + \max_{x \in X} |\phi(x)|)^{-1}$.
Applying the good distribution of $(y_n)_{n \in N}$ to χ_J we find that there exists an $M_2(\varepsilon)$ such that

$$(22) \qquad \left| \frac{1}{M} \sum_{n > n_o} j_n(k,M) - \lambda J \right| < \frac{\varepsilon}{8}(1 + \max_{x \in X} |\phi(x)|)^{-1},$$

whenever $M \geqq M_2$ (again, uniformly in k). It follows that

$$(23) \qquad \left| \sum_{n > n_o} \frac{1}{M} j_n(k,M) \right| < \frac{\varepsilon}{4}(1 + \max_{x \in X} |\phi(x)|)^{-1}$$

for all $M \geqq M_2$ and all k.

As $(x_n)_{n \in N}$ is ν-well distributed in X, there exists an $M_3(\varepsilon)$ such that

$$(24) \qquad \left| \frac{1}{M} \sum_{m=1}^{M} \phi(x_{m+k}) - \nu(\phi) \right| < \frac{\varepsilon}{8} K^{-1},$$

uniformly in k, if $M \geqq M_3(\varepsilon)$. Let M_o be such that $M_o \geqq \max(M_1, M_2, M_3)$ while moreover $j_o(k,M) > M_3(\varepsilon)$ for all $M \geqq M_o$, uniformly in k.

Such an M_o exists, and it depends only on ε and ϕ but not on k. Then - if we write $\sum_s^i$ for sums over s consecutive values of the parameter i - we have:

$$\left| \frac{1}{M} \sum_{m=1}^{M} \phi(u_{k+m}) - \mu\phi \right| = \left| \left(\frac{j_o(k,M)}{M} \frac{1}{j_o(k,M)} \sum_{j_o(k,M)}^{i} \phi(x_i) - \int_{X \setminus X_o} \phi(x) d\mu \right) + \right.$$

$$+ \sum_{n=1}^{n_o} \left(\frac{j_n(k,M)}{M} \phi(z_n) - \int_{\{z_n\}} \phi(x)\, d\mu \right) + \sum_{n > n_o} \frac{j_n(k,M)}{M} \phi(z_n) +$$

$$\left. - \sum_{n > n_o} \int_{\{z_n\}} \phi(x)\, d\mu \right| \leqq$$

$$\leq \left|\frac{j_o(k,M)}{M}\right| \left|\frac{1}{j_o(k,M)} \sum_{j_o(k,M)}^{i} \phi(x_i) - \nu(\phi)\right| +$$

$$+ \left|\frac{j_o(k,M)}{M} - \lambda I_o\right| \left|\nu(\phi)\right| + \sum_{n=1}^{n_o} \left|\frac{j_n(k,M)}{M} - \lambda I_n\right| \left|\phi(z_n)\right| +$$

$$+ \left|\sum_{n > n_o} \frac{j_n(k,M)}{M}\right| \max_{x \in X} |\phi(x)| + \sum_{n > n_o} \mu(\{z_n\}) \max_{x \in X} |\phi(x)| \leq$$

$$\leq \frac{\varepsilon}{8} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} = \varepsilon ,$$

whenever $M \geq M_o(\varepsilon,\phi)$, uniformly in k.

There remains the case $\mu X_o = 1$. In this case we can use the same proof as outlined above; all terms containing $j_o(k,M)$ must, however, be omitted.

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