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A combinatorial problem on finite Abelian Groups

by

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§0. Introduction

This report deals with the following problem:

Let G be a finite Abelian group and $a_1, a_2, \dots, a_k$ a finite sequence of elements taken from G. We ask for a minimal value for k such that each sequence contains a non empty subsequence with sum zero.
A conjecture of prof. P.C. Baayen states that for a group of the form $C_{d_1} \oplus C_{d_2} \oplus C_{d_3} \oplus \dots \oplus C_{d_k}$ with $d_k | d_{k-1}, \dots, d_2 | d_1$ this minimal value for k is equal to $d_1 + d_2 + \dots + d_k - k + 1$. It was brought to the attention of the authors that Erdös stated the same conjecture for the special case $C_p \oplus C_p$ where p is an arbitrary prime number.
We prove the conjecture for the following cases:

1) $G = C_{d_1}$ for any d_1
2) $G = C_{d_1} \oplus C_{d_2}$ where $d_2 = 2^k \cdot 3^l \cdot 5^m$ and $d_2 | d_1$
3) $G = C_3 \oplus C_3 \oplus C_3$
4) $G = (C_2)^N$ for any N.

In the proof of 2) we employ a induction method which gives us the following implication:
"If the conjecture is true for each group $C_p \oplus C_p \oplus C_p$ it is true for each group $C_{d_1} \oplus C_{d_2}$."
Recently during the preparation of this report it came to our knowledge that H.B. Mann and J.E. Olson are working on the same problem. Their results (see [1]) for the case $G = C_p \oplus C_p$ imply that for the special case of a sequence consisting of <u>different</u> non zero elements of G the minimal value for k is equal to 2p - 2. For a probabilistic method applied to a related problem by P. Erdös and A. Rényi see [2].

§1. Conventions and notations

A <u>structure</u> S is a pair $\{V, \mu\}$ where V is a non empty finite set and μ is a multiplicity-function which assigns to any element a of V a natural number $\mu(a)$ which is called the <u>multiplicity of a</u>.
A structure is represented by the usual notation; if for example $S = \{V, \mu\}$ where $V = \{a,b,c,d\}$ and $\mu(a) = 2$, $\mu(b) = 3$, $\mu(c) = 1$, $\mu(d) = 4$ we denote $S = \{a\ b\ a\ c\ d\ b\ d\ b\ d\ d\}$. In the latter symbol the letters

in between the accolades can be interchanged.

The length of a structure $S = \{V,\mu\}$ is the sum of the multiplicities taken over V. It is exactly the number of letters in between the accolades in the notation defined above.

A structure $S' = \{V',\mu'\}$ is called a substructure of $S = \{V,\mu\}$ if $V' \subset V$ and $\mu'(a) \leq \mu(a)$ for any $a \in V'$. Notation: $S' \subset S$.

We use the usual language:
The elements of V are called "the elements of the structure". The sentence "x appears 3 times in S" means $\mu(x) = 3$.

In the sequel $\{G,+,0\}$ denotes a finite Abelian group with elements represented by $\{a,b,c,\ldots,x,y\}$. A structure over G is a structure $S = \{V,\mu\}$ where V is a subset of G. The value of the structure is the sum of its elements taken in account of their multiplicities.
Notation: $|S| = \sum_{a \in V} \mu(a) \cdot a$.

An element $x \in G$ is called generated by the structure S if there exists a substructure $S' \subset S$ with $|S'| = x$. The structure S is called a zero-structure if $|S| = 0$. If 0 is generated by S we say "S contains a zero-structure".

§2. Finite Abelian groups

The main theorem on finite Abelian groups states that any such group consisting of at least two elements can uniquely be represented as:

$$G = C_{d_1} \oplus C_{d_2} \oplus \ldots \oplus C_{d_k}$$

where $2 \leq d_k$, $d_k | d_{k-1}$, $\ldots$, $d_2 | d_1$ and C_d is the cyclic group of d elements. The number k is called the dimension of the group G.

If needed we represent the elements of this group by columns of k numbers:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_k \end{pmatrix} \qquad \text{where } 0 \leq x_1 < d,\ 0 \leq x_2 < d,\ \ldots,\ 0 \leq x_k < d.$$

In this form we can perform addition coordinatewise:

$$\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_k \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_k \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_k \end{pmatrix} \quad \text{where} \quad \begin{matrix} x_1 + y_1 \equiv z_1 \bmod d_1 \\ x_2 + y_2 \equiv z_2 \bmod d_2 \\ \cdot \\ \cdot \\ \cdot \\ x_k + y_k \equiv z_k \bmod d_k \end{matrix} .$$

In the special case $G = \underbrace{C_p \oplus C_p \oplus C_p \oplus \dots \oplus C_p}_{k\text{-times}}$, p prime it is known that G is a vectorspace over the finite field C_p, and the notation defined above coincides with the usual one. For this case we can use the theory on vectorspaces.

§3. Primitive structures and primitive zero structures

Let G be a finite Abelian group and let S be a structure over G. S is called a primitive structure if 0 is not generated by S. If S is a zerostructure and none of its proper substructures generates 0, S is called a primitive zero-structure.
It is easy to see that a non primitive zero-structure contains at least two disjoint zero-structures.

The length of a primitive structure can not be chosen arbitrarily. Actually it can never exceed the order of the group. We have the following general theorem.

Theorem 1. Any structure of length n over a finite group of order n contains a zero-structure.

Proof: Let S be the structure $\{x_1, x_2, \dots, x_n\}$. Let $a_k = x_1 + x_2 + \dots + x_k$. If $a_i = a_j$ for some $i < j$ then $0 = a_j - a_i = x_{i+1} + x_{i+2} + \dots + x_j$. If $a_i \neq a_j$ for each pair i,j, $i \neq j$, then the n elements of the group $\{a_1, \dots, a_n\}$ are different, which implies that one of them is the zero element.

It is easy to see that this theorem is also valid for non Abelian groups.

Let k be the maximal length of a primitive structure over some Abelian group G and l be the maximal length of a primitive zerostructure over the same group G. Then we have the relation $l = k + 1$. For, let S be a primitive structure of length k, then we construct a primitive zerostructure by adjoining the element $-|S|$. Conversely, let S' be a primitive zerostructure of length l then each substructure of S' of length $l - 1$ is a primitive structure. It follows that $k + 1 \leq l$ and $l - 1 \leq k$ which fact proves the relation.

Solutions for the general problem to find a maximal length of a structure over a semigroup containing no structure consisting of <u>successive</u> elements with idempotent value, are given in [3], [4], [5].

§4. Cyclic groups

The cyclic group of order n is an example of a group for which the maximal length l of a primitive zerostructure is equal to the order of the group. For if a is a generator, then the structure $S = \{V,\mu\}$ with $V = \{a\}$ and $\mu(a) = n$ is a primitive zerostructure, hence $l \geq n$. From theorem 1 it is immediate that $l \leq n$. Combining these two we have $l = n$.

A converse of this statement is also valid.

<u>Theorem 2'</u>. If the maximal length l of a primitive zerostructure over an Abelian group G is equal to the order n of the group, then the group is cyclic.

<u>Proof</u>: Let S be a primitive zerostructure on G with length n. We shall first show that all the elements of S are equal.
Assume the contrary, and let $S = \{x_1, x_2, \ldots, x_n\}$ with $x_1 \neq x_2$. As S is a primitive zerostructure the n-1 elements a_k for $k = 1, 3, 4, \ldots, n$ defined by $a_k = x_1 + x_3 + \ldots + x_k$ are not equal to zero and $a_i \neq a_j$ for $i \neq j$.
The same holds for the elements $b_k = x_2 + x_3 + \ldots + x_k$, $k = 2, 3, \ldots, n$. The elements of both $\{a_k\}$ and $\{b_k\}$ represent the set of non zero elements of G.
As $x_2 \neq 0$ and $x_1 \neq x_2$ we have $x_1 \neq x_1 - x_2$ and $x_1 - x_2 \neq 0$. Therefore there is a element a_k with $k \neq 1$ and $a_k = x_1 - x_2$. But now $b_k = a_k - x_1 + x_2 =$ $= 0$ which gives the desired contradiction.

Now, if all the elements of a primitive zero-structure of length n are equal to some element x then this x is an element of order n. Consequently, our group is cyclic.

From theorem 2' it follows:

Theorem 2. The maximal length of a primitive zero-structure over a finite Abelian group equals the order of the group if and only if the group is cyclic.

§5. A conjecture on non cyclic groups

From theorem 2, it follows that for non cyclic groups the maximal length l of a primitive zero-structure is less than the order of the group. The problem is to compute this length in case the structure of the group is given (if for example the numbers $d_1, \ldots, d_k$ in the representation from §2 are known).

Let G be the group $G = C_{d_1} \oplus C_{d_2} \oplus \ldots \oplus C_{d_k}$, where $2 \leq d_k$, $d_k | d_{k-1}, \ldots$ $\ldots, d_2 | d_1$.

It is easy to construct a primitive zero-structure of length $d_1 + d_2 + \ldots$ $\ldots + d_k - k + 1$. For example consider the structure consisting of the elements:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} d_1-1 \text{ times}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} d_2-1 \text{ times}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} d_k-1 \text{ times, and } \begin{pmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}.$$

Prof. P.C. Baayen has raised the conjecture that this number $d_1 + d_2 + \ldots$ $\ldots + d_k - k + 1$ is also an upper limit for the length of a primitive zero-structure on G.

Let us denote the statement:

"The maximal length of a primitive zero-structure over the group $C_{d_1} \oplus C_{d_2} \oplus \ldots \oplus C_{d_k}$ equals $d_1 + d_2 + \ldots + d_k - k + 1$"

by the symbol: $(C_{d_1} \oplus C_{d_2} \oplus \ldots \oplus C_{d_k})!$

Theorem 2 can now be restated as the validity of $(C_n)!$ for any $n \in \mathbb{N}$.

Other cases for which the conjecture has been proved are:

Theorem 3. $(C_{d_1} \oplus C_{d_2})!$ is true for $d_2 = 2^\alpha 3^\beta 5^\gamma$ and $d_2 | d_1$.

Theorem 4. $(C_3 \oplus C_3 \oplus C_3)!$ is true.

Theorem 5. $((C_2)^k)!$ is true for any $k \in \mathbb{N}$.

No counter example of the conjecture is known to the authors.

§6. Other conjectures

The proof of theorem 3 is carried out by using some induction principle. However, the conjecture stated above itself is not strong enough to use a same procedure. Stronger conjectures that work can be defined in two different ways. We shall restrict ourselves to groups of dimension ≤ 2 (generalisations of these conjectures in case of dimension 3 are not generally true).

The strong-conjecture $(C_{d_1} \oplus C_{d_2})!!$ says:

"Any structure of length $kd_1 + d_2 - 1$ contains k disjoint zero-structures for any $k \in \mathbb{N}$".

The strongest conjecture $(C_{d_1} \oplus C_{d_2})!!!$ says:

" $(C_{d_1} \oplus C_{d_2})!$ is true.
Any structure of length $2d_1 + d_2 - 1$ contains a zero-substructure of length $\leq d_1$".

It is easy to prove the implications:

$$(C_{d_1} \oplus C_{d_2})!!! \implies (C_{d_1} \oplus C_{d_2})!! \implies (C_{d_1} \oplus C_{d_2})!$$

In the above conjectures the case $d_2 = 1$ is not excluded, and is easily seen to be valid.

We have also the following implication:

Theorem 6. $(C_n \oplus C_n \oplus C_n)! \implies (C_n \oplus C_n)!!!$

Proof: First, let there be given a structure of length 2n-1 over

$$C_n \oplus C_n: \left\{ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_{2n-1} \\ y_{2n-1} \end{pmatrix} \right\}.$$

We "border" this structure as follows:

$$\left\{ \begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x_{2n-1} \\ y_{2n-1} \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{n-1 \text{ times}} \right\}$$

$((C_n)^3)!$ implies the existence of a zero-substructure which does not contain one of the last n-1 elements (else the lowest coordinate of the value of this structure is unequal to zero). Hence this zero-substructure is contained in the original structure over $(C_n)^2$.
This proves $(C_n \oplus C_n)!$

Secondly let there be given a structure of length 3n-2 over

$$C_n \oplus C_n: \left\{ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \cdots \begin{pmatrix} x_{3n-2} \\ y_{3n-2} \end{pmatrix} \right\}$$

Now we "border" this structure as follows:

$$\left\{ \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} \cdots \begin{pmatrix} x_{3n-2} \\ y_{3n-2} \\ 1 \end{pmatrix} \right\}$$

From $((C_n)^3)!$ it follows that there exists a zero-substructure. This substructure must have a length that is divisible by n. If the length is n we have found a zero-substructure of length n over $C_n \oplus C_n$ and we are ready. But if the length is 2n we have found a zerostructure of length 2n over $C_n \oplus C_n$ which can not be primitive; hence it contains two disjoint zero substructures one of which has a length $\leq$ n. From this the statement $(C_n \oplus C_n)!!!$ follows.

§7. An induction procedure for groups of dimension 2

Theorem 7. If $(C_k \oplus C_k)!!$ and $(C_{d_1} \oplus C_{d_2})!!$ are true then $(C_{kd_1} \oplus C_{kd_2})!!$ is also true.

Proof: Let a structure of length $td_1k + d_2k - 1$ be given over $C_{kd_1} \oplus C_{kd_2} = G$. There exists a subgroup A' which is isomorfic to $C_{d_1} \oplus C_{d_2}$ such that $G/A' \cong C_k \oplus C_k$. We denote the natural mapping from G onto G/A' by ϕ.

The mapping ϕ transforms the given structure into a structure over $C_k \oplus C_k$ with length $(td_1 + d_2)k - 1$. $(C_k \oplus C_k)!!$ implies that it contains $td_1 + d_2 - 1$ disjoint zero-structures.
It follows that the same substructures of the original structure have their values in the subgroup A'. Now we interprete these $td_1 + d_2 - 1$ values as the elements of a new structure over $A' \cong C_{d_1} \oplus C_{d_2}$.
$(C_{d_1} \oplus C_{d_2})!!$ implies the existence of t disjoint zero-substructures.
This means that from the $td_1 + d_2 - 1$ disjoint substructures with values in A' we can construct t disjoint combinations such that the combinated substructures have value zero in $C_{d_1k} \oplus C_{d_2k}$. So we find that the original structure contains t disjoint zero structures which completes the proof.

Theorem 7'. $(C_k \oplus C_k)!!$ and $(C_{d_1} \oplus C_{d_2})! \implies (C_{kd_1} \oplus C_{kd_2})!$

Theorem 7' is easily proved by the methods employed in the proof of theorem 7.

As before the case $d_2 = 1$ is not excluded; we have the implication:
$(C_k \oplus C_k)!! \implies (C_{nk} \oplus C_k)!!$ for any $n \in \mathbb{N}$,
because of the fact that $(C_n \oplus C_1)!!$ is valid for any $n \in \mathbb{N}$.

Let us define in general:

$(C_{d_1} \oplus C_{d_2} \oplus \dots \oplus C_{d_n})!!$ means: "A structure of length $td_1 + d_2 + \dots$ $\dots + d_n - n + 1$ contains t disjoint zero-substructures".

Then the statement:
"$((C_k)^n)$!! and $(C_{d_1} \oplus C_{d_2} \oplus \dots \oplus C_{d_n})$!! ==> $(C_{kd_1} \oplus C_{kd_2} \oplus \dots$
$\dots \oplus C_{kd_n})$!! can be proved in the same way as theorem 7. Again the case that some of the d_j are equal to 1 is not excluded. However this generalisation of the strong conjecture does not hold: the statement $((C_2)^3)$!! is false.

The structure $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$

does not contain two disjoint zero-substructures.

§8. A proof of theorem 5

Let $S = \{x_1, x_2, \dots, x_{k+1}\}$ be a structure of k+1 elements over $(C_2)^k$. This group is the additive group of a vector space of dimension k over the field C_2 which contains the scalars 0 and 1.
From the theory of vector spaces it follows that the k+1 elements of the structure are linearly dependent. Consequently there exists a relation $\lambda_1 x_1 + \dots + \lambda_{k+1} x_{k+1} = 0$ in which not every λ_i is zero.
By erasing the terms with λ_i is zero we find a relation:
$x_{j_1} + x_{j_2} + \dots + x_{j_n} = 0$ $(1 \leq j_1 < j_2 < \dots < j_n \leq k+1)$. So, there is a zero-substructure.

§9. An equivalent formulation of $(C_3)^k$!. Proof of $(C_3)^3$!

The group $(C_p)^k$ can be considered as the additive group of a vector space over C_p. With respect to a given base $e_1, \dots, e_k$ we define the unit-cell U as the set of vectors $\lambda_1 e_1 + \dots + \lambda_k e_k$ where $\lambda_i = 0, 1$.

With respect to the canonical base $\begin{pmatrix}1\\0\\0\\ \cdot\\ \cdot\\ \cdot\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\ \cdot\\ \cdot\\ \cdot\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\0\\ \cdot\\ \cdot\\ \cdot\\1\end{pmatrix}$

the unit-cell contains exactly the vectors with 0 or 1 coordinates.
A cell-structure is a structure the value of which is a vector from the unit-cell.

With these notions it is possible to reformulate the conjecture $(C_3)^k$!

Theorem 8. The following are equivalent:

a) $(C_3)^k$! is true.

b) A structure of length k+1 over $(C_3)^k$ contains a cell-structure.

c) A structure of length k+2 over $(C_3)^k$ with value $\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$ is the union of two disjoint cell-structures.

Proof: a) → b). Suppose $(C_3)^k$! is true and let S be a structure of length k+2 over $(C_3)^k$ with value

$|S| = \begin{pmatrix}1\\1\\1\\\vdots\\1\end{pmatrix}$. Then we adjoin the set of k vectors

$$T = \left\{ \begin{pmatrix}2\\0\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\\\vdots\\0\end{pmatrix}, \ldots, \begin{pmatrix}0\\0\\0\\\vdots\\2\end{pmatrix} \right\}.$$

The resulting structure is a zero-structure of length 2k+2, and therefore it is not a primitive zero-structure. Hence it contains a proper zero-structure U.

Now $U = (U \cap S) \cup (U \cap T)$. It is clear that $(U \cap S)$ is a cell-structure. It is also true that $(U \cap S)$ is a proper substructure of S, for suppose on the contrary that $U \cap S = S$, then we have $|U \cap S| = \begin{pmatrix}1\\1\\1\\1\\\vdots\\1\end{pmatrix}$ which

implies $|U \cap T| = \begin{pmatrix}2\\2\\\vdots\\2\end{pmatrix}$ and therefore $U \cap T = T$, which contradicts the

fact that U is a proper substructure of $S \cup T$. Thus S is the union of the two disjoint cell-structures $S \cap U$ and $S \setminus U$.

c) → b). Let S be a structure of length k+1 over $(C_3)^k$. Now we adjoin the element $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ - $|S|$. The resulting structure of length k+2 with value

$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ is the disjoint union of two disjoint cell-structures one of which

is contained in S.

b) → a). Let S be a structure of length 2k+1 over $(C_3)^k$. We choose from S a maximal set consisting of linear independent vectors $\{e_1, e_2, \ldots, e_n\}$. It is possible to choose a base for $(C_3)^k$ such that:

$$e_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \ldots e_n = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 2 \\ \cdot \\ 0 \end{pmatrix} \; n^{th} \text{ place.}$$

As the other k + 1 + (k - n) vectors are linear combinations of $e_1, \ldots, e_n$ their n+1, ..., k^{th} coordinates are zero. The structure consisting of these vectors contains a cell-structure S' whose value has also zero as its n+1, ..., k^{th} coordinate. Now it is possible to complete S' with some of the e_j $1 \le j \le n$ to form a zero-structure.

Now we shall prove $(C_3)^3!$ by showing that any structure S over $(C_3)^3$ with length 5 and value $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ contains a proper cell-substructure.

We consider a 3 × 5 matrix with elements taken from C_3 such that the sum of the elements of each row equals 1.

I. If the third row contains three or more zeros we can extract three elements forming a structure over $C_3 \oplus C_3 \oplus C_3$ of the form

$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ 0 \end{pmatrix} \right\}$. It is easy to show that such a structure always contains a cell-substructure. The same holds if one of the other rows contains three or more zeros.

II. The combination of two elements in the same row with sum 2 is called a <u>conjunction</u>. If the number of conjunctions in the matrix is smaller than 10, then it is possible to find a pair of columns without a conjunction. This pair clearly forms a cell-structure.

III. The possible structures of length 5 over C_3 with value 1 and with at most 2 zeros are tabulated below, followed by the number of conjunctions that they contain.

	Structure	Number of conjunctions
P	1, 1, 1, 1, 0	6
Q	2, 1, 1, 0, 0	3
R	2, 2, 2, 1, 0	3
S	2, 2, 1, 1, 1	3
T	2, 2, 2, 2, 2	0

IV. From I it follows that we can restrict ourselves to 3×5 matrices in which each row has the form P, Q, R, S, or T. However, if the form P is not present we find at most 9 conjunctions, and the existence of a cell-substructure follows. Therefore we may suppose that the first row has the form P.

V. Suppose that there are two rows present of the form P. If the last row is not of the form T there exists an element in the structure that is contained in the unit cell. But if the last row has the form T, then the vector $\begin{pmatrix}1\\1\\2\end{pmatrix}$ or $\begin{pmatrix}1\\2\\1\end{pmatrix}$ appears at least 3 times. Hence the structure contains a zero-structure. Therefore we may suppose that only the first row has the form P.

VI. In the combinations $\begin{bmatrix}P\\T\\Q\end{bmatrix}$, $\begin{bmatrix}P\\T\\R\end{bmatrix}$ and $\begin{bmatrix}P\\T\\S\end{bmatrix}$ the number of conjunctions is equal to 9, and by II there exists a cell-substructure. In the combinations $\begin{bmatrix}P\\S\\S\end{bmatrix}$, $\begin{bmatrix}P\\Q\\R\end{bmatrix}$, $\begin{bmatrix}P\\Q\\S\end{bmatrix}$ and $\begin{bmatrix}P\\Q\\Q\end{bmatrix}$ the number of elements equal 2 in the matrix is at most equal 4. Hence the structure contains an element contained in the unit-cell.

Therefore we may restrict ourselves to the two remaining combinations:

$$\begin{bmatrix} P \\ R \\ R \end{bmatrix} \text{ and } \begin{bmatrix} P \\ R \\ S \end{bmatrix}.$$

VII. We may restrict ourselves to those matrices in which no unit-cell column is present. For both combinations $\begin{bmatrix} R \\ R \end{bmatrix}$ and $\begin{bmatrix} R \\ S \end{bmatrix}$ there exists a unique form for which each column contains an element equal 2, that is given below.

$$\begin{bmatrix} R \\ R \end{bmatrix} : \begin{pmatrix} 2 & 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 \end{pmatrix} \qquad \begin{bmatrix} R \\ S \end{bmatrix} : \begin{pmatrix} 2 & 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

For each of those 2×5 matrices it is possible to combine an arbitrary column with some other column such that their sum is contained in the unit-cell. From this it follows that the remaining combinations $\begin{bmatrix} P \\ R \\ R \end{bmatrix}$ and $\begin{bmatrix} P \\ R \\ S \end{bmatrix}$ always contain a cell-structure of length ≤ 2. This completes the proof.

§10. <u>The group $C_5 \oplus C_5$</u>

The group $C_5 \oplus C_5$ is the additive group of the two dimensional vectorspace over the field C_5. A <u>line</u> in the group $C_5 \oplus C_5$ will be a one-dimensional subspace of this vectorspace. There are six lines in $C_5 \oplus C_5$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad : b_1$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad : b_2$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad : l_1$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad : l_2$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad : l_3$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad : l_4$$

Each of these lines is a group isomorfic to C_5.

It is clear that any structure over $(C_5 \oplus C_5)$ containing 5 or more elements taken from a single line contains a zero-substructure of length ≤ 5. Therefore we exclude in the sequel the possibility that the structure we are considering contains a zero-structure formed by elements taken from a single line. This imposes a strong restriction on the possible structures.

We enumerate the possible combinations of taking 4, 3, or 2 elements from a line. Combinations that can be transformed into each other by a linear transformation will be treated as being identical.

4 elements:	set of generated elements
a) $\{1, 1, 1, 1\}$	$\{1, 2, 3, 4\}$
3 elements:	
b_1) $\{1, 1, 1\}$	$\{1, 2, 3\}$
b_2) $\{1, 1, 2\}$	$\{1, 2, 3, 4\}$
2 elements:	
c_1) $\{1, 1\}$	$\{1, 2\}$
c_2) $\{1, 2\}$	$\{1, 2, 3\}$
c_3) $\{1, 3\}$: can be transformed into c_2)	

In the sequel we shall try to construct a structure of length 9 over $C_5 \oplus C_5$ which contains no zero-substructure, resp., a structure of length 14 over $C_5 \oplus C_5$ that contains no zero-substructure of length ≤ 5 in order to prove that such a construction is not possible. From the enumeration given above it follows that if S is an arbitrary (resp. primitive) structure containing 2 or 3 elements within a single line, such that the combination b_2 or c_2 is realized, then there exists a structure S' containing 3 or 4 elements in the same line, such that the combination a or b_1 is realized, and such that S' generates the <u>same</u> elements (resp. S' is primitive too), but has a <u>greater</u> length. Moreover, if S' contains a zero-substructure with length ≤ 5, then this is also true for S. Therefore, we can exclude the combinations b_2 and c_2.

For any given structure S there is a suitable canonical base for $C_5 \oplus C_5$ such that the lines b_1 and b_2 are exactly the lines in which the structure has "many" elements. Hence, we may assume without loss of generality that the elements $\binom{0}{1}$ and $\binom{1}{0}$ appear in the structure with multiplicities greater than 1, and that the other elements of the structure are not contained in the lines b_1 and b_2.

The argument given above makes it possible to transform any structure (that contains no zero-substructure consisting of elements taken from a single line) into an "equivalent" structure of which a number of elements are "known". These elements create a "forbidden region"; this is the collection of elements x of $C_5 \oplus C_5$ such that -x is generated by the "known" elements, we call it "forbidden" because of the fact that if some element of it is generated by the "unknown" elements, then there exists a zero-structure. In the proof of $(C_5 \oplus C_5)!!!$ each element of the "forbidden region" possesses a "critical number"; this is the maximal number k with the property that if the element is generated by k or less "unknown" elements, then there exists a zero-substructure with length ≤ 5.

For the sake of clearness we shall draw for each treated case, a diagram in which this forbidden region is marked. The meaning of the 25 cells that appear in the diagrams is given in diagram 1.

$\binom{0}{4}$	$\binom{1}{4}$	$\binom{2}{4}$	$\binom{3}{4}$	$\binom{4}{4}$
$\binom{0}{3}$	$\binom{1}{3}$	$\binom{2}{3}$	$\binom{3}{3}$	$\binom{4}{3}$
$\binom{0}{2}$	$\binom{1}{2}$	$\binom{2}{2}$	$\binom{3}{2}$	$\binom{4}{2}$
$\binom{0}{1}$	$\binom{1}{1}$	$\binom{2}{1}$	$\binom{3}{1}$	$\binom{4}{1}$
$\binom{0}{0}$	$\binom{1}{0}$	$\binom{2}{0}$	$\binom{3}{0}$	$\binom{4}{0}$

Diagram 1.

For the proof of $(C_5 \oplus C_5)!$ we use the following property Q of the group C_5, the easy proof of which is omitted:

Q: Let V be the set $\{1,2\} \subset C_5$. Then we have the following implications:

I) If $a \in V$, b and c are not equal to zero, and a + b and a + c are elements of V, then b = c.

II) If $a \neq 0$, b and c are elements of V, and a + b and a + c are elements of V, then b = c.

§11. Proof of $(C_5 \oplus C_5)!$

Let S be a structure of length 9 over $C_5 \oplus C_5$. Excluding the possibility that there exists a zero-structure formed by elements taken from a single line, there remain four cases after having choosen a suitable canonical base:

A) The structure contains 4 times the element $\binom{1}{0}$.

B) The structure contains 3 elements within b_1 and 3 elements within b_2.

C) The structure contains 3 elements within b_1 and 2 elements within b_2.

D) The structure contains 2 elements within b_1 and 2 elements within b_2.

In each of these cases we may assume that the elements taken from the lines b_1 and b_2 always are equal $\binom{1}{0}$ and $\binom{0}{1}$, as has been shown in §10.

Case A. There are at least 5 elements of the form $\binom{x}{y}$ with $y \neq 0$. By looking at the second coordinate only, we conclude that the structure of these five elements generate an element of the form $\binom{z}{0}$. Now either z = 0 or the element $\binom{-z}{0}$ is generated by the "known" elements $\{ \binom{1}{0} , \binom{1}{0} , \binom{1}{0} , \binom{1}{0} \}$. This means that the complete structure does generate $\binom{0}{0}$.

Case B. The structure contains $\binom{1}{0}$ three times, $\binom{0}{1}$ three times and three unknown elements. The "forbidden region" is shown in diagram 2.

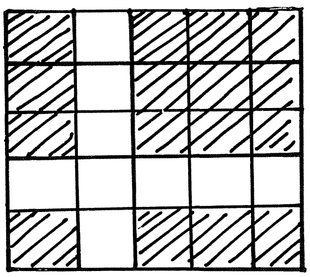

Diagram 2.

The three unknown elements have the form $\begin{pmatrix}1\\ \alpha\end{pmatrix}$ or $\begin{pmatrix}\beta\\ 1\end{pmatrix}$ with $\alpha,\beta \neq 0$. There are essentially two possibilities (with respect to the symmetrical properties of case B).

<u>Case B_1</u>. The three unknown elements are $\begin{pmatrix}1\\ \alpha\end{pmatrix}$ $\begin{pmatrix}1\\ \beta\end{pmatrix}$ and $\begin{pmatrix}1\\ \gamma\end{pmatrix}$.
Now either $\begin{pmatrix}2\\ \alpha+\beta\end{pmatrix} = \begin{pmatrix}2\\ \beta+\gamma\end{pmatrix} = \begin{pmatrix}2\\ \alpha+\gamma\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix}$ and hence $\alpha = \beta = \gamma = 3$ or an element inside the forbidden region is generated.
But if $\alpha = \beta = \gamma = 3$ then $\begin{pmatrix}3\\ 4\end{pmatrix}$ is generated which is an element of the forbidden region.

<u>Case B_2</u>. The three unknown elements are $\begin{pmatrix}1\\ \alpha\end{pmatrix}$ $\begin{pmatrix}1\\ \beta\end{pmatrix}$ and $\begin{pmatrix}\gamma\\ 1\end{pmatrix}$.
Now $\begin{pmatrix}1+\gamma\\ 1+\beta\end{pmatrix}$ is generated which is an element of the forbidden region.

<u>Case C</u>. The structure consists of $\begin{pmatrix}1\\ 0\end{pmatrix}$ three times, $\begin{pmatrix}0\\ 1\end{pmatrix}$ two times and four "unknown" elements.
The forbidden region is shown in Diagram 3.

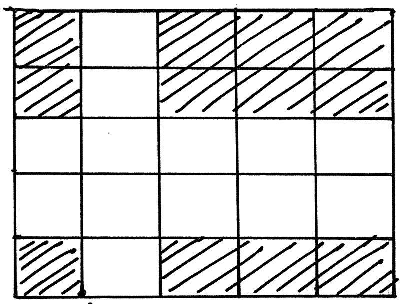

Diagram 3.

The four unknown elements must have the form $\begin{pmatrix}1\\ \alpha\end{pmatrix}$, $\alpha \neq 0$, or $\begin{pmatrix}\beta\\ \lambda\end{pmatrix}$ where $\beta \neq 0$ $\lambda \in \{1,2\} = V$.

There are 5 subcases:

<u>Case C_1</u>. The four unknown elements are: $\begin{pmatrix}1\\ a\end{pmatrix}$ $\begin{pmatrix}1\\ b\end{pmatrix}$ $\begin{pmatrix}1\\ c\end{pmatrix}$ and $\begin{pmatrix}1\\ d\end{pmatrix}$ where $a, b, c, d \neq 0$. Suppose no forbidden element is generated. Now we have $a + b \in V$, $a + b + c \in V$, $a + b + d \in V$, $b, c \neq 0$; hence QI (see §10) implies: $c = d$. In the same way we prove: $a = b = c = d$. But then the structure contains four identical elements; this situation was treated already in <u>case A</u>.

<u>Case C_2</u>. The four unknown elements are $\begin{pmatrix}1\\a\end{pmatrix}$ $\begin{pmatrix}1\\b\end{pmatrix}$ $\begin{pmatrix}1\\c\end{pmatrix}$ $\begin{pmatrix}d\\\lambda\end{pmatrix}$, where $a, b, c, d \neq 0$, $\lambda \in V$.

In order to avoid the generation of some forbidden element the conditions: $\lambda + a \in V$, $\lambda + b \in V$ and $\lambda + c \in V$ must be satisfied. As $a, b, c \neq 0$ and $\lambda \in V$, property QII implies $a = b = c$; hence there are three identical elements. See <u>case B</u>.

<u>Case C_3</u>. The four unknown elements are: $\begin{pmatrix}1\\a\end{pmatrix}$ $\begin{pmatrix}1\\b\end{pmatrix}$ $\begin{pmatrix}c\\\lambda\end{pmatrix}$ $\begin{pmatrix}d\\\mu\end{pmatrix}$, where $a, b, c, d \neq 0$ and $\lambda, \mu \in V$.

Now we find the following conditions:

$a + \lambda$, $b + \lambda \in V$: this implies $a = b$ by QI

$a + \lambda$, $a + \mu \in V$: this implies $\lambda = \mu$ by QII

$a + b \in V$. As $a = b$ it follows that $a = 1$ or $a = 3$. However $a = 3$ is impossible by $a + \lambda \in V$ and $\lambda \in V$; hence $a = b = 1$.

From λ, $a + \lambda \in V$ it follows that $\lambda = \mu = 1$.

Then we have $\begin{pmatrix}1 + 1 + c\\a + b + \lambda\end{pmatrix} = \begin{pmatrix}2 + c\\3\end{pmatrix}$; hence $c = 4$.

It follows similarly: $c = d = 4$.

Now the four unknown elements together generate the element $\begin{pmatrix}0\\4\end{pmatrix}$ which is contained within the forbidden region.

<u>Case C_4</u>. The four unknown elements are: $\begin{pmatrix}1\\a\end{pmatrix}$ $\begin{pmatrix}b\\\lambda\end{pmatrix}$ $\begin{pmatrix}c\\\mu\end{pmatrix}$ $\begin{pmatrix}d\\\nu\end{pmatrix}$, where $a, b, c, d \neq 0$, $\lambda, \mu, \nu \in V$.

Now we have: $a + \lambda$, $a + \mu$, $a + \nu$, $\in V$: this implies $\lambda = \mu = \nu$ by QII. As it is impossible that λ, $a + \lambda$, $a + 2\lambda \in V$ it follows: $1 + b + c = 1 + b + d = 1 + c + d = 1$. Therefore $b = c = d = 0$. <u>Contradiction</u>.

<u>Case C_5</u>. The four unknown elements are: $\begin{pmatrix}a\\\lambda\end{pmatrix}$ $\begin{pmatrix}b\\\mu\end{pmatrix}$ $\begin{pmatrix}c\\\nu\end{pmatrix}$ $\begin{pmatrix}d\\\xi\end{pmatrix}$ where $a, b, c, d \neq 0$ and $\lambda, \mu, \nu, \xi \in V$.

Suppose $\lambda = 2$ and $\mu = 1$. Then $a + b = 1$ and $a + b + c = 1$ follows; hence $c = 0$ and a contradiction arises. Therefore we may assume $\lambda = \mu = \nu = \xi = 1$ or $\lambda = \mu = \nu = \xi = 2$.

If $\lambda = 1$ then $a + b + c = a + b + c + d = 1$; hence $d = 0$ and a contradiction arises.

If $\lambda = 2$ then $a + b = a + c = a + d = b + c = b + d = c + d = 1$; hence $a = b = c = d = 3$. The structure contains four identical elements. See <u>case A</u>.

Case D. The structure consists of the elements $\binom{0}{1}$ $\binom{0}{1}$ $\binom{1}{0}$ $\binom{1}{0}$ and five other unknown elements. These other elements are all taken from l_1, l_2, l_3 or l_4. We may assume that at most two of them are taken from the same line. This implies that at most one of the four lines l_1, l_2, l_3, l_4 does not contain any element of the structure.

We prove that if l_2 contains an element of the structure, then some forbidden element is generated by the unknown elements. For reasons of symmetry it follows that the structure also does not contain an element taken from l_3, and a contradiction arises. The diagrams 4, 5, 6, 7 show the forbidden region for a structure consisting of $\binom{1}{0}$ $\binom{1}{0}$ $\binom{0}{1}$ $\binom{0}{1}$ and 5 unknown elements, resp. a structure consisting of $\binom{1}{0}$ $\binom{1}{0}$ $\binom{0}{1}$ $\binom{0}{1}$, one of the elements of l_2 and four unknown elements.
The fourth element of l_2 $\binom{4}{3}$ is already "forbidden" by $\binom{1}{0}$ $\binom{1}{0}$ $\binom{0}{1}$ $\binom{0}{1}$ alone.

Case D

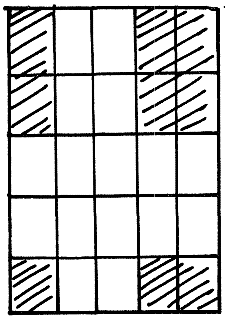

Diagram 4

$\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}$

Case D_1

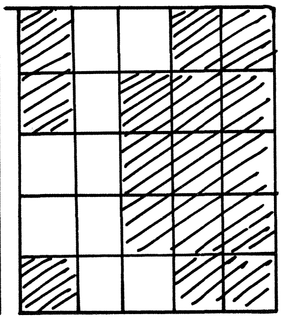

Diagram 5

$\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}\binom{1}{2}$

Case D_2

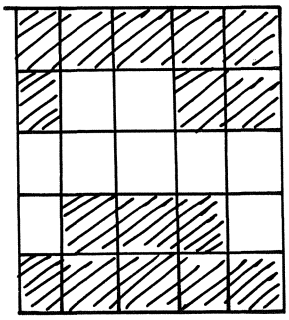

Diagram 6

$\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}\binom{2}{4}$

Case D_3

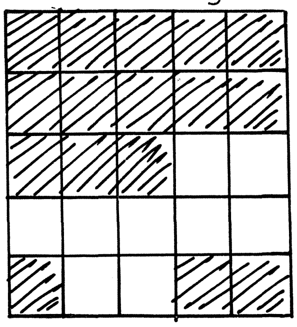

Diagram 7

$\binom{1}{0}\binom{1}{0}\binom{0}{1}\binom{0}{1}\binom{3}{1}$

Case D_1. The only "possible" elements left are $\binom{1}{1}$ $\binom{1}{2}$ $\binom{1}{3}$ $\binom{1}{4}$ and $\binom{2}{4}$. However the element $\binom{2}{4}$ excludes all the other elements for $\binom{2 + x}{4 + y}$ is forbidden for each possible $\binom{x}{y}$. Hence the four unknown elements must have the form $\binom{1}{a}$. But now some element of the form $\binom{3}{x + y + 2}$ is generated which is a forbidden element.

<u>Case D_2</u>. The only possible elements are $\binom{1}{2}$ $\binom{1}{3}$ $\binom{2}{2}$ $\binom{2}{3}$ $\binom{3}{2}$ $\binom{4}{2}$ and $\binom{4}{1}$.

The following combinations are forbidden because of the fact that they generate forbidden elements:

$\binom{4}{1}$ and $\binom{b}{3}$ for b = 1, 2:

$\binom{a}{2}$ and $\binom{b}{3}$ for any a, b

$\binom{a}{2}$ and $\binom{b}{2}$ for any a, b

$\binom{4}{1}$, $\binom{4}{1}$ and $\binom{a}{2}$ or $\binom{a}{3}$ for any a

$\binom{a}{3}$, $\binom{b}{3}$ and $\binom{c}{3}$ for any a, b, and c.

It is easy to see that it is impossible to choose four possible elements without creating a forbidden combination.

The proof of <u>case D_3</u> is analogous to that of <u>case D_1</u>.

Summarizing we see that in each case our argument has resulted either in some contradiction or in the existence of a zero-structure. Hence it is impossible to construct a primitive structure of length 9 over $C_5 \oplus C_5$. This proves $(C_5 \oplus C_5)$!

§12. <u>Proof of $(C_5 \oplus C_5)$!!!</u>

Let S be a structure of length 14 over $C_5 \oplus C_5$. We may assume that the structure contains no zero-substructure of elements taken from a single line as has been noticed in §10. Therefore we can assume that the structure contains at most four elements within a single line. Dividing 14 elements over six lines no single line containing 5 or more elements, we have two possibilities:

<u>Case A</u>. There are two lines each containing three or more elements.

<u>Case B</u>. There is one line containing four elements, all other lines containing two elements.

As has been noticed in §10 we may assume that elements contained in the same line are identical. After a suitable linear transform we may suppose that the structure contains a certain number of "known" elements. These known elements create a forbidden region in which each element possesses a critical number as has been described in §10. In the diagrams this number is marked down within the corresponding cell.

<u>Case A</u>. The structure consists of the elements $\binom{1}{0}$ $\binom{1}{0}$ $\binom{1}{0}$, $\binom{0}{1}$ $\binom{0}{1}$ $\binom{0}{1}$ and eight other elements. It will not be excluded that $\binom{1}{0}$ and $\binom{0}{1}$ appear each fourth times, and therefore we suppose only the existence of six other elements. The forbidden region for <u>case A</u> is shown in diagram 8.

4		1	2	3
3			1	2
2				1
5		2	3	4

Diagram 8

Possible choices for the six other elements are:

$\binom{1}{1}$, $\binom{1}{2}$ or $\binom{2}{1}$, $\binom{1}{3}$ or $\binom{3}{1}$, $\binom{2}{2}$, $\binom{2}{3}$ or $\binom{3}{2}$, and $\binom{1}{4}$ or $\binom{4}{1}$.

It is sufficient to prove the impossibility of chosing $\binom{1}{1}$, $\binom{2}{2}$, $\binom{1}{2}$ and $\binom{1}{3}$. Then the elements $\binom{2}{1}$ and $\binom{3}{1}$ are excluded for reasons of symmetry, and the remaining elements are exactly the elements from l_4 and it was excluded that six elements should be chosen from a single line.

<u>Case A_1</u>. The structure contains the elements $\binom{1}{0}$ $\binom{1}{0}$ $\binom{1}{0}$, $\binom{0}{1}$ $\binom{0}{1}$ $\binom{0}{1}$, and $\binom{1}{1}$. See diagram 9.

4	1	2	3	4
3		1	2	3
2			1	2
				1
5		2	3	4

Diagram 9.

Now the following choices are possible:

$\binom{1}{1}$ three other times.

$\binom{1}{2}$ or $\binom{2}{1}$ but these exclude each other

$\binom{1}{3}$ $\binom{2}{2}$ or $\binom{3}{1}$ but these exclude each other and all the other possible choices.

It is impossible to choose 5 elements. Therefore $\binom{1}{1}$ is ruled out. (In the diagrams 10, 11 and 12 related to the other subcases of <u>case A</u> the cell $\binom{1}{1}$ is marked black to indicate the impossibility of choosing this element.)

<u>Case A_2</u>. The structure contains $\binom{1}{0}$ $\binom{1}{0}$ $\binom{1}{0}$, $\binom{0}{1}$ $\binom{0}{1}$ $\binom{0}{1}$ and $\binom{1}{2}$. See diagram 10.

4		1	2	3
3	1	2	3	4
2		1	2	3
	▨		1	2
5		2	3	4

Diagram 10

4		1	2	3
3	2	3	4	2
2	1	2	3	1
	▨	1	2	
5		2	3	4

Diagram 11

4		1	2	3
3			1	2
2	1	2	3	4
	▨	1	2	3
5		2	3	4

Diagram 12

Now the possible choices are ($\binom{1}{1}$ already being excluded):

$\binom{1}{2}$ three other times possible

$\binom{2}{1}$ one time possible

$\binom{1}{4}$ one time possible but this choice excludes $\binom{2}{1}$.

It is impossible to choose 5 elements. Therefore $\binom{1}{2}$ (and also $\binom{2}{1}$) is ruled out.

<u>Case A_3</u>. The structure contains $\binom{1}{0}$ $\binom{1}{0}$ $\binom{1}{0}$, $\binom{0}{1}$ $\binom{0}{1}$ $\binom{0}{1}$ and $\binom{2}{2}$. See diagram 11.

The only possible choices are the elements $\binom{1}{4}$ and $\binom{4}{1}$. But these elements are contained within the line l_4 and it was excluded that we choose five elements from a single line. Therefore $\binom{2}{2}$ is ruled out.

<u>Case A_4</u>. The structure contains $\binom{1}{0}$ $\binom{1}{0}$ $\binom{1}{0}$, $\binom{0}{1}$ $\binom{0}{1}$ $\binom{0}{1}$ and $\binom{1}{3}$. See diagram 12.

The possible choices are:

$\binom{1}{3}$ three times

$\binom{1}{4}$ two times but then $\binom{1}{3}$ is excluded

$\binom{2}{3}$ one time but then $\binom{1}{3}$ is excluded.

It is impossible to choose 5 elements. Therefore $\binom{1}{3}$ (and also $\binom{3}{1}$) is ruled out.

This completes the proof of case A.

<u>Case B</u>. The structure consists of $\binom{1}{0}$ $\binom{1}{0}$ $\binom{1}{0}$ $\binom{1}{0}$, $\binom{0}{1}$ $\binom{0}{1}$ and two elements in each of the lines l_1, l_2, l_3 and l_4. The forbidden region "generated" by the elements $\binom{1}{0}$ $\binom{1}{0}$ $\binom{1}{0}$ $\binom{1}{0}$, $\binom{0}{1}$ and $\binom{0}{1}$ is shown in diagram 13. This diagram shows that the only possible choices of two elements within the line l_1 are:

<u>Case B_1</u>. The structure contains also $\binom{1}{1}$ and $\binom{1}{1}$ (diagram 14).

<u>Case B_2</u>. The structure contains also $\binom{1}{1}$ and $\binom{2}{2}$ (diagram 15).

<u>Case B_1</u>. It is impossible to choose two elements from l_4.

<u>Case B_2</u>. It is impossible to choose a single element from l_2.

This completes the proof of case B and therefore $(C_5 \oplus C_5)$!!! is proved.

4		1	2	3
3			1	2
5	1	2	3	4

Diagram 13

4	1	2	3	4
3	1	2	3	3
		1	2	2
			1	
5	1	2	3	4

Diagram 14

4	1	2	3	4
3	2	3	4	3
1	2	3	3	2
	1	2	2	
5	1	2	3	4

Diagram 15

§13. Proof of Theorem 3

The results from §8, 9, 10, 11, 12 imply that $(C_2 \oplus C_2)!!$, $(C_3 \oplus C_3)!!$ and $(C_5 \oplus C_5)!!$ are true. Therefore we can apply the induction procedure of §7. It proves that the strong conjecture holds for any group of the form $(C_k \oplus C_k)$ where $k = 2^n \cdot 3^m \cdot 5^l$.
It has been proved that $(C_k)!!$ is true for any k. The induction procedure of §7 now implies that $(G)!!$ is true for any G of the form $G = C_{d_1} \oplus C_{d_2}$ where $d_2 = 2^n \cdot 3^m \cdot 5^l$ and $d_1 = k \cdot d_2$. Then G! is also true for any such G. This completes the proof of Theorem 3.

It has been proved by P. Erdös, A. Ginzburg and A. Ziv, and independently by N.G. de Bruijn that any structure over C_n with length 2n-1 contains a zero-substructure of length = n. From this it follows that any structure of length 2n-1 over $C_n \oplus C_n$, such that all elements are contained within a coset a + H where H is a cyclic subgroup of order n, contains a zero-substructure of length n. See [6].
As has been noticed in the introduction it has been proved by H.B. Mann and J.E. Olson that any structure over $C_p \oplus C_p$ where p is some prime number consisting of different elements with length $\geq$ 2p-2 contains some zero-substructure. See [1].

§14. An equivalent formulation of $(C_p)^k!$

Let X be some finite dimensional vectorspace over C_p with dimension N. With respect to a given base $e_1, \ldots, e_N$ we define the unit-cell U as in §9.

Now we have the following equivalence:

Theorem 9. The following are equivalent:

a) $(C_p)^k!$ is true.

b) If $N \geq (p-1)k + 1$ and A is some N-k dimensional subspace of X then $A \cap U$ contains some element $t \neq 0$.

Proof: a) → b): Let A be some N-k dimensional subspace of X. Then A is determined by k linear functions on X:

$$A = \{x \in X \mid \phi_1(x) = \phi_2(x) = \ldots = \phi_k(v) = 0\}$$

Let $e_1, \ldots, e_N$ be some base with respect to which the unit-cell U is defined. Now we define the following structure S over $(C_p)^k$:

$$S = \{v_1, \ldots, v_N\} \quad \text{where} \quad v_j = \begin{pmatrix} \phi_1(e_j) \\ \phi_2(e_j) \\ \cdot \\ \cdot \\ \phi_k(e_j) \end{pmatrix}$$

If $N \geq (p-1)k + 1$ and $(C_p)^k!$ is true, then there exists a zero-substructure S'.

Now $\sum_{v_i \in S'} v_i = 0$ and therefore

$$\sum_{v_i \in S'} \phi_j(e_i) = 0 \quad \text{for } j = 1, 2, \ldots, k.$$

This implies: $\phi_j(\sum_{v_i \in S'} e_i) = 0$ for $j = 1, 2, \ldots, k$;

hence: $t = \sum_{v_i \in S'} e_i$ is an element of A. As S' is not empty we have $t \in A \cap U$ and $t \neq 0$. This completes the proof.

<u>b) → a)</u>: Let $S = \{v_1, v_2, \ldots, v_N\}$ be a structure over $(C_p)^k$ and suppose that $N \geq (p-1)k + 1$.

Now we consider the collection A of all solutions $(\lambda_1, \ldots, \lambda_N)$ of the equation:

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N = 0 \quad \text{with} \quad \lambda_i \in C_p.$$

This collection A is a linear subspace of the finite dimensional vectorspace $(C_p)^N$ of all N-tuples $(\lambda_1, \ldots, \lambda_N)$ with $\lambda_i \in C_p$, while the dimension of A is at least equal $N - k$.
Therefore A is at least equal $N - k$.
Therefore A contains some element $t \neq 0$ of the unit-cell U which is defined with respect to the canonical base of $(C_p)^N$. This means that there exists a solution of the equation $\lambda_1 v_1 + \ldots$ $\ldots + \lambda_N v_N = 0$ where $\lambda_i = 0$ or 1 for each j and not $\lambda_i = 0$ for all i. This proves the existence of a zero-substructure of S.

§15. Some applications for puzzelists and gamblers

I: If the conjecture $(C_n)^k!$ is valid we have the following corollary in the theory of finite undirected graphs:

Let G be a graph with k vertices and N edges. If $N \geq k(n-1) + 1$ then G contains a subgraph which is in each vertex of an order divisible by n.

Proof: We consider the $k \times N$ incidence matrix (a_{ij}) where $a_{ij} = 1$ if the i-th edge has the j-th vertex as an endpoint, $a_{ij} = 2$ if the i-th edge is a loop with the j-th vertex as endpoint, and $a_{ij} = 0$ else.

The N columns of this matrix can be considered as some structure over $(C_n)^k$. If $N \geq k(n-1) + 1$ there exists a zero-substructure. The edges corresponding to the columns that form this zero-structure form the demanded subgraph.

II: A gambler bets, after throwing 11 times with two discernable dices and marking the scores, that it is possible to elect a certain number of these throws such that the sums of the scores of both dices separately are both divisible by six.

It has been shown that $(C_6 \oplus C_6)!$ is valid and therefore he certainly wins. However if he plays the same game with three dices and 17 throws it is just a conjecture that he will never lose.

Literature

[1] H.B. Mann and J.E. Olson.
Sums of sets in the Elementary Abelian Group of Type (p,p).
Journal of combinatorial theory. Vol. 2. 275-284 (1967).

[2] P. Erdös and A. Rényi.
Probabilistic methods in group theory.
Journal d'Analyse Mathematique. Vol. 14. 127-138 (1965).

[3] P.C. Baayen, P. van Emde Boas and D. Kruyswijk.
A combinatorial problem on finite semigroups.
Report ZW 1965-006. Math. Centre, Amsterdam.

[4] P.C. Baayen and D. Kruyswijk.
A note on the multiplicative semigroup of the residue classes modulo n.
Report ZW 1965-007. Math. Centre, Amsterdam.

[5] P. van Emde Boas.
A combinatorial problem on the semigroup of all transformations of a finite set.
Report ZW 1965-009. Math. Centre, Amsterdam.

[6] P. Erdös, A. Ginzburg and A. Ziv.
Theorem in the additive number theory.
Bulletin of the research council of Israel. Vol. 10 F. 41-43 (1961).