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A GENERAL FIXED POINT THEOREM

by

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A general fixed point theorem

In general topology there exist some fixed point theorems for contracting mappings. Their common characteristic is that they guarantee uniqueness of the fixed point by means of a principle of contraction relative to the metric of the space. In this paper we shall describe yet another fixed point theorem of the same sort, orginating from a problem in differential equations, and we shall give a generalisation to uniform spaces. First we shall state two well-known contraction theorems and we shall then prove the main theorem in a metric space and a uniform space separately. We shall conclude by showing that the two well-known theorems are contained in the main theorem, although we are well aware that they are easier to prove by other methods.

Theorem 1. (BANACH) Let (X,ρ) be a complete metric space and let ϕ be a mapping from X into X such that there exists a positive real number α less than 1 with the property that $\rho(\phi(x), \phi(y)) \leq \alpha\, \rho(x,y)$ for all x and y in X, then X contains one and only one point x_ϕ for which $\phi(x_\phi) = x_\phi$ holds.

Theorem 2. Let (X,ρ) be a metric space and let ϕ be a mapping from X into itself such that $\overline{\phi(X)}$ is compact and $\rho(\phi(x),\phi(y)) < \rho(x,y)$ for all $x, y \in X$, then X contains one and only one fixed point relative to the mapping ϕ.

Convention. If X is a space and ϕ is a mapping from X into X then $\phi^0(x)$ is the identity on X and $\phi^n(x) = \phi(\phi^{n-1}(x))$ for every natural number n and every $x \in X$. Clearly ϕ^n can be considered as a mapping from X into X.

Theorem 3. Let (X,ρ) be a metric space, and let ϕ be a continuous function from X into X which satisfies the following properties:

(i) $\exists\, x_0 \in X$ such that $\{\phi^n(x_0)\}_{n=1}^{\infty}$ is conditionally compact in X.

(ii) $\forall x \in X;\ \forall y \in X$ we have $\lim_{n\to\infty} \rho(\phi^n(x),\phi^n(y)) = 0$.

Then the space X contains exactly one fixed point relative to the transformation .

Proof. Since $\{\phi^n(x_0)\}_{n=1}^{\infty}$ is conditionally compact in (X,ρ) there exists an infinite subset M of the natural numbers such that $\{\phi^m(x_0) \mid m \in M\}$ is convergent. Let $\hat{x}_0$ be its limit. From the continuity of ϕ it follows that $\{\phi^{m+1}(x_0) \mid m \in M\}$ is convergent with limit $\phi(\hat{x}_0)$.
Choose an $\varepsilon > 0$.
From condition (ii)it follows that $\exists N_0$ such that for every natural number $n > N_0$ we have

$$\rho(\phi^n(x_0),\phi^{n+1}(x_0)) < \frac{\varepsilon}{3}.$$

Furthermore there exists an N_1 such that

$$\forall m \in M;\ m > N_1 \text{ we have } \rho(\phi^m(x_0),\hat{x}_0) < \frac{\varepsilon}{3}$$

and

$$\forall m \in M;\ m > N_1 \text{ we have } \rho(\phi^{m+1}(x_0),\phi(\hat{x}_0)) < \frac{\varepsilon}{3}.$$

Since M is infinite we conclude that $\rho(\hat{x}_0,\phi(\hat{x}_0)) < \varepsilon$ for every positive number ε and therefore $\hat{x}_0$ has to be a fixed point of ϕ.
Suppose $\hat{y}_0$ is another fixed point, then $\lim_{n\to\infty} \rho(\phi^n(\hat{x}_0),\phi(\hat{y}_0)) = 0$. Since $\hat{x}_0 = \phi(\hat{x}_0) = \phi^n(\hat{x}_0)$ and $\hat{y}_0 = \phi(\hat{y}_0) = \phi^n(\hat{y}_0)$ for all $n \in \mathbb{N}$ we have $\hat{x}_0 = \hat{y}_0$.
Therefore $\hat{x}_0$ is the unique fixed point of the function ϕ.

Theorem 4. Let X be a Tychonoff space and let ϕ be a continuous mapping from X into X. If there exists a uniform structure $\mathcal{H}$ on X such that

$\forall x \in X,\ \forall y \in X,\ \forall H \in \mathcal{H},\ \exists N_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ with $n > N_0$ we have $(\phi^n(x),\phi^n(y)) \in H$,

then the following conditions are equivalent:

(i) $\exists x_0 \in X$ and an infinite subset M of $\mathbb{N}$ such that $\{\phi^m(x_0) \mid m \in M\}$ is a convergent sequence.

(ii) The space X contains exactly one fixed point $\hat{x}_\phi$ relative to ϕ.

(iii) For every $x \in X$ the sequence $\{\phi^n(x) \mid n \in \mathbb{N}\}$ converges.

Proof (i) $\Rightarrow$ (ii). Let $\hat{x}$ be the limit of $\{\phi^m(x_0) \mid m \in M\}$. From the continuity it follows that $\{\phi^{m+1}(x_0) \mid m \in M\}$ converges to $\phi(\hat{x})$.
Let H be an arbitrary diagonal neighbourhood in $\mathcal{H}$. Then there exists a $K \in \mathcal{H}$ such that $K = K^{-1}$ and $K \times K \times K \subset H$.
There exists an $N_0 \in \mathbb{N}$ such that

(a) $(\phi^m(x_0), \phi^{m+1}(x_0)) \in K$ for every $m \in \mathbb{N}$; $m \geq N_0$.

(b) $(\phi^m(x_0), \hat{x}) \in K$ for every $m \in M$; $m \geq N_0$.

(c) $(\phi^{m+1}(x_0), \phi(\hat{x})) \in K$ for every $m \in M$; $m \geq N_0$.

Since M is infinite we can choose m sufficiently large in M and we conclude that

$$(\hat{x}, \phi(\hat{x})) \in K \times K \times K \subset H.$$

We conclude that $(\hat{x}, \phi(\hat{x})) \in \bigcap \{H \mid H \in \mathcal{H}\}$ and since X is a Tychonoff space it follows that $\hat{x} = \phi(\hat{x})$.
Suppose that $\hat{y}$ is another fixed point of ϕ, then

$$\forall H \in \mathcal{H};\ \exists N \in \mathbb{N};\ \forall n > N \text{ we have } (\phi^n(\hat{y}), \phi^n(\hat{x})) = (\hat{y}, \hat{x}) \in H.$$

and hence $(\hat{y}, \hat{x}) \in \bigcap \{H \mid H \in \mathcal{H}\}$. This implies that $\hat{y} = \hat{x}$. Therefore $\hat{x}$ is the unique fixed point of ϕ in X.

(ii) $\Rightarrow$ (iii). Let x_0 be the fixed point of ϕ in X and let x be an arbitrary member of X. Let U be a neighbourhood of x_0 then U contains a neighbourhood of x_0 of the form: $\{y \mid (y, x_0) \in H\}$ for some $H \in \mathcal{H}$. By definition there exists an $N_0 \in \mathbb{N}$ such that for every $n > N_0$ we have $(\phi^n(x), \phi^n(x_0)) \in H$; hence $(\phi^n(x), x_0) \in H$. Therefore $\phi^n(x)$ is eventually in every neighbourhood of x_0, i.e. $\phi^n(x)$ converges to x_0.

(iii) $\Rightarrow$ (i). Obvious.

Proof of theorem 1. Let x and y be two arbitrary points of X. Then $\rho(\phi^{n+1}(x),\phi^{n+1}(y)) \leq \alpha . \rho(\phi^{n}(x),\phi^{n}(y)) \leq \alpha^{n+1} . \rho(x,y)$. Since $\alpha < 1$ we have $\lim_{n\to\infty} \rho(\phi^{n}(x),\phi^{n}(y)) = 0$.

Moreover, for every $x \in X$ we have $\rho(\phi^{n}(x),x) \leq \sum_{i=1}^{n} \rho(\phi^{i}(x),\phi^{i-1}(x)) \leq$

$\leq \sum_{i=1}^{n} \alpha^{i-1} . \rho(\phi(x),x) \leq \frac{1}{1-\alpha} \rho(\phi(x),x)$.

Therefore, for every k and $l \in \mathbb{N}$; $k \geq l$ we have $\rho(\phi^{k}(x),\phi^{l}(x)) \leq$

$\leq \alpha^{l} \rho(\phi^{k-l}(x),x) \leq \frac{\alpha^{l}}{1-\alpha} . \rho(\phi(x),x)$.

This implies that $\{\phi^{n}(x) \mid n \in \mathbb{N}\}$ is a Cauchy sequence and from the completeness of X it follows that its limit exists. We conclude that $\{\phi^{n}(x) \mid n \in \mathbb{N}\}$ is conditionally compact, and theorem 1 follows from theorem 3.

Proof of theorem 2. The proof given here is by no means the most efficient proof of this theorem, but it merely is included to show that this theorem may be taken as a consequence of theorem 3.

First, since $\phi[X]$ is conditionally compact, the set $\{\phi^{n}(x) \mid n \in \mathbb{N}\}$ has to be conditionally compact too; this implies condition (i).

In order to prove that $\forall x, \forall y, \lim \rho(\phi^{n}(x),\phi^{n}(y)) = 0$ we shall prove that the diameter $D(\phi^{n}[X]) \overset{def}{=} \sup \{\rho(a,b) \mid a, b \in \phi^{n}[X]\}$ tends to 0 for $n \to \infty$.

Suppose that $\lim_{n\to\infty} D(\phi^{n}[X]) \neq 0$ then also $\lim_{n\to\infty} D(\phi^{n}[\overline{\phi[X]}]) \neq 0$. Define $X_n = \phi^{n}[\overline{\phi[X]}]$, then X_n is closed and compact for every natural number n. Obviously $X_{n+1} \subset X_n$ and since X_n is compact we can choose in every X_n a x_n and a y_n such that $D(X_n) = \rho(x_n,y_n)$.

Clearly $D(X_m) \leq D(X_n)$ if $m > n$. The sets $\{x_n\}$ and $\{y_n\}$ are subsets of $\overline{\phi[X]}$ and hence relatively compact. Therefore there exists a subset M of $\mathbb{N}$ such that both sequences $\{x_m \mid m \in M\}$ and $\{y_m \mid m \in M\}$ converge to x_0 and y_0 respectively. Since x_m and y_m are in X_n for all $m \geq n$ and since X_n is closed it follows that x_0 and y_0 are in X_n and hence x_0 and y_0 are in $\bigcap_{n=1}^{\infty} X_n$.

$$\rho(x_0,y_0) \geq \liminf_{m\in M;\, m\to\infty} D[X_m] = \lim_{n\to\infty} D[X_n] = \lim_{n\to\infty} \rho(x_n,y_n).$$

Therefore $\rho(x_0,y_0) = D(\bigcap_{n=1}^{\infty} X_n)$ which is non-zero by assumption. $\{y_0\}$ is closed in X and hence $\phi^{-1}(y_0)$ is a closed subset of X. $y_0 \in X_{n+1}$ for every n and hence $\phi^{-1}(y_0) \cap X_n \neq \emptyset$. $\{\phi^{-1}(y_0) \cap X_n\}_{n=1}^{\infty}$ is a closed nest in the compact set X_1 and so $\bigcap_{n=1}^{\infty} X_n \cap \phi^{-1}(y_0) \neq \emptyset$. Let y_1 be a member of this set.

Similarly we can find a point $x_1 \in \bigcap_{n=1}^{\infty} X_n \cap \phi^{-1}(x_0)$.

Clearly, $\rho(x_1,y_1) \leq D(\bigcap_{n=1}^{\infty} X_n)$; $\phi(x_1) = x_0$ and $\phi(y_1) = y_0$.

Therefore $\rho(x_1,y_1) > \rho(x_0,y_0) = D(\bigcap_{n=1}^{\infty} X_n) \geq \rho(x_1,y_1)$.

We conclude that the assumption $\lim_{n\to\infty} D(\phi^n[X]) \neq 0$ leads to a contradiction and thus for every x and y of X and for every $\varepsilon > 0$ there exists an N_0 such that $\rho(\phi^n(x),\phi^n(y)) < \varepsilon$ for every $n > N_0$; and also the second condition of theorem 3 is proved.

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