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COMPACT SETS IN NON-METRIZABLE PRODUCT SPACES

BY

J. VAN DER SLOT

INTRODUCTION

Denote by s the countable infinite topological product of real lines. A result in infinite dimensional topology [1], [2] and [3] states that whenever C is a compact subset of s, then $s\backslash C$ is homeomorphic with s. In this note our aim is to show that for products of more than countable many real lines the situation is completely different; in fact we have the following result: Let C_1 and C_2 be compact subsets of an <u>uncountable</u> product P of real lines. Then the condition C_1 homeomorphic with C_2 is necessary and sufficient in order that the complements $P\backslash C_1$ and $P\backslash C_2$ are homeomorphic. We also study some generalizations of this result which might be of interest.

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P stands for some uncountable product of real lines. If $P = \Pi\{R_\alpha \mid \alpha \in A\}$ where $R_\alpha = R$ for $\alpha \in A$ then π_α denotes the natural projection of P onto R_α; if $p = (p_\alpha)_\alpha \in P$ then $\Sigma(p) \subset P$ is defined by $\Sigma(p) = \{x = (x_\alpha)_\alpha \in P | x_\alpha \neq p_\alpha$ for at most countably many $\alpha\}$.

The following lemma is due to Corson [4]. See also [5] page 98.

LEMMA

If f is a continuous map of $\Sigma(p)$ into $\mathbb{R}$ then f can be extended uniquely to a continuous function over P.

This lemma gives us the necessary machinery to prove the only if part of our statement. It gives us the following theorem:

THEOREM 1

<u>If</u> C_1 <u>and</u> C_2 <u>are compact subsets of</u> P <u>and</u> i <u>is a homeomorphism of</u> $P\backslash C_1$ <u>onto</u> $P\backslash C_2$ <u>then there is an autohomeomorphism</u> i^* <u>of</u> P <u>onto</u> P <u>which is an extension of</u> i <u>and which carries</u> C_1 <u>onto</u> C_2.

PROOF

First we notice that if E is an arbitrary compact set in P then for some $p = (p_\alpha)_\alpha \in P$ we have $E \subset P\backslash\Sigma(p)$. Indeed, for each α we have $\pi_\alpha E \neq \mathbb{R}_\alpha$, thus there is $p_\alpha \in \mathbb{R}_\alpha\backslash\pi_\alpha E$ for each α. If $p \in P$ is defined such that the α'th coordinate is p_α, then $E \subset P\backslash\Sigma(p)$.

Now, if i is a homeomorphism of $P\backslash C_1$ onto $P\backslash C_2$ then by lemma 1 the composition maps $i_\alpha = \pi_\alpha \circ i : P\backslash C_1 \to \mathbb{R}_\alpha$ can be extended to mappings $i^*_\alpha: P \to \mathbb{R}_\alpha$; thus $i^*: P \to P$ defined by $(i^*(x))_\alpha = i^*_\alpha(x)$ $(\alpha\in A)$ is a continuous extension of i taking C_1 into C_2. In the same way $j = i^{-1}: P\backslash C_2 \to P\backslash C_1$ can be extended to $j^*: P \to P$. Thus the mappings $j^* \circ i^*$ and $i^* \circ j^*$ are the identities on P, since both $P\backslash C_2$ and $P\backslash C_1$ are dense. Hence i^* satisfies our hypotheses.

REMARK

The lemma states that P is homeomorphic with the Hewitt-real-compactification of $\Sigma(p)$: $P = \upsilon(\Sigma(p))$ (see [5]). If $E \subset P$ is compact then $P = \upsilon(P\backslash E)$ i.e., E is the remainder of $P\backslash E$ in the Hewitt-real-compactification of $P\backslash E$.

The "if" part of the main result can now be stated as follows. For the proof we use the methods of Klee [7] and Anderson [1]. The present setting was suggested to me by A. Verbeek (oral communucation).

THEOREM 2

<u>If</u> C_1 <u>and</u> C_2 <u>are compact subsets of</u> P <u>and</u> h <u>is a homeomorphism of</u> C_1 <u>onto</u> C_2 <u>then</u> h <u>can be extended to an autohomeomorphism of</u> P <u>onto</u> P. <u>In particular it follows that</u> $P\backslash C_1$ <u>is homeomorphic with</u> $P\backslash C_2$.

PROOF

Let $P = \Pi\{P_\alpha \mid \alpha \in A\}$ where $P_\alpha = \Pi\{\mathbb{R}^\alpha_n \mid n = 1, 2, \ldots\}$ and $\mathbb{R}^\alpha_n = \mathbb{R}$ $(\alpha\in A, n=1,2,\ldots)$, and let $C = C_1 \cup C_2$. For each α there is an autohomeomorphism ϕ_α: $P \to P$ such that for some $n(\alpha) \in N$ $\pi_{n(\alpha)}(\phi_\alpha C)$ is a single element ([2] page 779, lemma 6.1.).

Now, let ϕ: $P \to P$ be defined by $(\phi(x))_\alpha = \phi_\alpha(x)$ $(\alpha\in A)$, then ϕ is a homeomorphism between P and the product of two copies P' and P'' of P such that $\pi_{P'}\phi(C)$ consists of a single element. Indeed, let $P' = \Pi\{\mathbb{R}^\alpha_{n(\alpha)} \mid \alpha \in A\}$ and P'' be the product of the factors $\mathbb{R}^\alpha_n$ for $n \neq n(\alpha)$ $(\alpha\in A)$. The proof of the theorem thus reduces to the following lemma:

LEMMA

Let p be a point of P; C_1 and C_2 compact subsets and h be a homeomorphism of C_1 onto C_2. Then h_1: $\{p\} \times C_1 \to \{p\} \times C_2$ defined by $h_1(p,x) = (p,h(x))$ can be extended to an autohomeomorphism of $P \times P$ onto $P \times P$.

PROOF

One can suppose that $p = 0 \in P$. Let h^*: $P \to P$ be a continuous extension of h and h^{-1*}: $P \to P$ be a continuous extension of h^{-1}: $C_2 \to C_1$. Such extensions exist because the compact sets C_1 and C_2 are C-embedded in P. For each $x = (x',x'') \in P \times P$ define

$$\psi_1(x',x'') = (x'+x'',x'')$$

$$\psi_2(x',x'') = (x',x''+h^*(x')-x')$$

$$\psi_3(x',x'') = (x'-h^{-1*}(x''),x'').$$

Here + and - stand for usual vector addition in the topological product P. Obviously ψ_1,ψ_2 and ψ_3 are autohomeomorphisms of $P \times P$ onto $P \times P$ and the composition $\psi = \psi_3 \circ \psi_2 \circ \psi_1$ satisfies the hypothesis of the lemma. Indeed, if $x'' \in C_1$ then
$\psi(0,x'') = (\psi_3\circ\psi_2)(x'',x'') = (x''-h^{-1}h(x''),h(x'')) = (0,h(x''))$. This completes the proof.

From Th. 1 and 2 we deduce:

COROLLARY

Two compact subsets of P are homeomorphic if and only if their complements in P are homeomorphic.

EXAMPLE

Let A be of cardinality of the continuum and $P = \Pi\{\mathbb{R}_\alpha \mid \alpha \in A\}$, $\mathbb{R}_\alpha = \mathbb{R}$ $(\alpha \in A)$. If $p \in P$ and C is an arc lying in P then the previous result implies $P\setminus\{p\} \not\cong P\setminus C$. We also have $P\setminus\{p\} \not\cong P$.

GENERALIZATIONS

Instead of considering products of real lines we can also consider uncountable products of intervals [0,1]. To obtain the corresponding theorem in this case, one has to introduce the concept of partial deficiency [3] and generalize it for uncountable products.

We say that a closed subset C of an uncountable product of $\underline{m}$ unit intervals I ($\underline{m}$ infinite cardinal) has a *complete partial deficiency* if $\overline{\pi_\alpha C}$ is contained in (0,1) for $\underline{m}$ indices α.

Now we have the following generalization:

THEOREM 3

If two closed subsets C_1 and C_2 of an uncountable product K of closed intervals are of complete partial deficiency, then C_1 and C_2 are homeomorphic if and only if $K\setminus C_1$ and $K\setminus C_2$ are homeomorphic.

PROOF

First, note that if C_1 and C_2 are completely partial deficient then there is an autohomeomorphism i of K onto itself such that $i(C_1 \cup C_2)$ is completely partial deficient. This can easily be deduced from the corresponding statement in the countable case: the union of two Z-sets in the Hilbert-cube is a Z-set (cf. [2]). Now the proof is reduced to the previous case except that in theorem 3 to prove necessity we have to use e.g. a piecewise linear homeomorphism rather than simple vector addition.

Theorem 3 also remains valid if we consider an uncountable product of <u>open</u> intervals and demand C_1 and C_2 to be <u>closed and C-embedded</u>. The condition C-embedded is essential to obtain the extensions h^* and h^{-1*} in Klee's lemma.

The second generalization consists in considering σ-compact subsets rather than compact sets. <u>Theorem 1 remains valid if we replace compact by σ-compact</u>.

Indeed, if $C = \cup\{C_i \mid i = 1, 2, \ldots\}$ where $C_i \subset P = \Pi\{\mathbb{R}_\alpha \mid \alpha \in A\}$ then write the index set A as a countable union of disjoint uncountable sets A_i, $i = 1, 2, \ldots$ and for each $\alpha \in A_i$ $i = 1, 2, \ldots$ define $p_\alpha \in \mathbb{R}_\alpha$ such that $p_\alpha \notin \pi_\alpha C_i$. The point $p \in P$ whose α'th coordinate is p_α $(\alpha \in A)$ satisfies the condition $C \subset P\backslash\Sigma(p)$.

The above argument implies that for two σ-compact C_1 and $C_2 \subset P$ the condition $P\backslash C_1$ homeomorphic with $P\backslash C_2$ necessarily implies that both or neither one of C_1 and C_2 must be closed and C-embedded. However, there exists a closed countable discrete subset of P which is not C-embedded in P (see [6]). <u>Thus there exist two closed countable discrete subspaces of P such that their complements in P are not homeomorphic</u>.

However, we do not know if for two <u>closed and C-embedded</u> σ-compact subsets C_1 and C_2 the condition $C_1 \simeq C_2$ necessarily implies $P\backslash C_1 \simeq P\backslash C_2$.

REFERENCES

[1] R.D. Anderson, Topological properties of the Hilbert cube and the infinite product of open intervals, Trans. Amer. Math. Soc., 126 (1967) pp. 200-206.

[2] R.D. Anderson, On topological infinite deficiency, Mich. Math. Journ. 14 (1967) pp. 365-383.

[3] R.D. Anderson and R.H. Bing, A completely elementary proof that Hilbert space is homeomorphic to the countable infinite product of real lines, Bull. Amer. Math. Soc. 74 (1968), 771-792.

[4] H.H. Corson, Normality in subsets of product spaces, Amer. Journ. of Math. 81 (1959) 785-796.

[5] R. Engelking, Outline of General Topology, North Holland Publishing Cie., Amsterdam, 1968.

[6] M. Hušek and J. van der Slot, Closed subsets of powers of natural numbers, in preparation as ZW-report, Math. Centre, Amsterdam.

[7] V.L. Klee, Some topological properties of convex sets, Trans. Amer. Math. Soc. 78 (1955) 30-45.