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ORDERABLE SPACES
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A CHARACTERIZATION OF CONNECTED (WEAKLY) ORDERABLE SPACES

BY

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Introduction

A topological space $X$ is said to be (weakly) orderable if there exists
a total order $<$ on $X$ such that all open intervals

$$(a, b) = \{x \in X \mid a < x < b\}$$

are open in $X$.

In the following $X$ will denote a connected $T_1$-space.

$X$ is said to have the property

$$(B')$$

iff for each $p \in X$, $X \setminus p$ has at most two components.

$$(B'0)$$

iff for each $p \in X$, all components of $X \setminus p$ are open.

Clearly $(B')$ implies $(B'0)$.

It is well known (see e.g. [2] and [3]) that a connected topological
space $X$ is orderable iff among every three points of $X$ there is exactly
one which separates the other two.

Here we prove:

Theorem

If the connected $T_1$-space $X$ satisfies the following three conditions:

(i) among any three points of $X$ there is at least one which lies
    in a connected set separating the other two;

(ii) among any three points of $X$ there is at most one which lies
    in an open connected set which separates the other two points;

(iii) $(B'0)$,

then $X$ is orderable, (and conversely, an orderable space certainly
satisfies $(i)$ - $(iii)$).
Corollary 1

If among three points of \( X \) there is exactly one which has an open connected neighbourhood which separates the other two then \( X \) is orderable.

Corollary 2

If among any three points of \( X \) there is exactly one which lies in a connected set that separates the other two, then \( X \) is orderable.

We use the following notation:

\[
A = B + C \\
\quad b \quad c
\]

means that \( A \) is the topological sum of its subsets \( B \) and \( C \), \( b \in B \) and \( c \in C \).

Proof of the corollaries

In both cases it is sufficient to prove that \( X \) satisfies \( B' \) since \( B' \) implies \( B'0 \). Suppose \( X \setminus p = A + B + C \).

\[
\quad a \quad b \quad c
\]

Then \( A \cup \{p\} \), \( B \cup \{p\} \) and \( C \cup \{p\} \) are connected sets containing a resp. b resp. c and separating b and c resp. a and c resp. a and b which gives a contradiction in case 2. In case 1 let \( U_{ab} \) be an open connected neighbourhood of \( p \) which does not contain \( a \) and \( b \), then \( C \cup U_{ab} \) is an open connected neighbourhood of \( c \) separating \( a \) and \( b \). Likewise \( B \cup U_{ac} \) and \( A \cup U_{bc} \) are open connected neighbourhoods of \( b \) resp. \( a \) separating \( a \) and \( c \) resp. \( b \) and \( c \). Contradiction.

Proof of the theorem

1. \( X \) satisfies \( B' \).

For, suppose \( X \setminus p = A + B + C \); \( \tilde{A} \setminus a = A_1 + A_2 \);

\[
\quad a \quad b \quad c \quad p
\]

\( \tilde{B} \setminus b = B_1 + B_2 \); \( \tilde{C} \setminus c = C_1 + C_2 \) (where \( A_2, B_2, C_2 \) may be empty).
Since the components of $X \setminus a$ are open, the components of $\overline{A} \setminus a$ are open in $\overline{A}$, hence we may assume that $A_1$, $B_1$ and $C_1$ are connected.

Now $A \cup B_1 \cup C_1$ is an open connected neighbourhood of $a$ separating $b$ and $c$, and $B \cup A_1 \cup C_1$ is an open connected neighbourhood of $b$ separating $a$ and $c$, a contradiction.

2. The complement of an open connected set has at most two components.

For, suppose $X \setminus C = A_1 + A_2 + A_3$, where $C$ is open and connected.

\[ p_1, p_2, p_3 \]

Then $C \cup A_i$ is an open connected neighbourhood of $p_i$ separating $p_j$ and $p_k$ ($i = 1, 2, 3; i \neq j \neq k \neq i$).

Contradiction.

3. If $X$ contains at least two cut points, $X$ is orderable. For, let $p$ and $q$ be two cut points; $X \setminus p = A_p + B_p, X \setminus q = A_q + B_q$.

\[ a_p q_p p q \]

Then $A_p \subset A_q$, and so $a \in A_q$ and $b \in B_p$.

\[ Y = X \setminus (A_p \cup B_p) = A_q \setminus A_p \]

is closed and connected;

\[ Y' = Y \setminus \{ p, q \} \]

Let $A_p \setminus a = E_a + F_a$ and $B_q \setminus b = E_b + F_b$ where $E_a$ and $E_b$ are connected and $F_a$ and $F_b$ may be empty. Then $Y \cup E_a \cup E_b$ is connected and open.

3A. $Y$ can have no endpoints other than $p$ and $q$.

For if $Y \setminus r$ is connected ($r \neq p, q$), $(Y \setminus r) \cup E_a \cup E_b$ is open and connected with complement $\{ r \} + (F_a \cup \{ a \}) + (F_b \cup \{ b \})$ which contradicts 2.

3B. $Y$ satisfies $B'$.

For if $Y \setminus r = A + B + C$ or $Y \setminus r = A + B + C$ then

\[ p q p q \]

\[ X \setminus r = (A \cup A_p) + (B \cup B_q) + C \] or $X \setminus r = (A \cup A_p \cup A_q) + B + C$

which contradicts 1.

3C. Each point $r$ of $Y'$ separates $p$ and $q$.

For if $Y \setminus r = A + B$ then $A$ and $B$ are connected by 3B, so

\[ p q \]
A \cup E_a \cup E_b is open and connected and has complement
\((F_a \cup \{a\}) + (F_b \cup \{b\}) + (B \cup \{r\})\), which contradicts 2.

3D. Y is orderable.
For Y is connected and \(Y = p + E(p,q) + q\) (see Whyburn, [4]).

3E. X is orderable:
Let Z be the collection of all cut points of X.
Then first of all Z is orderable. For, if \(z \in A_p\) let
\(X \setminus z = A_z + B_z\) and if \(z \in B_p\) let \(X \setminus z = A_z + B_z'.\)

Then \(y < z \iff \overline{A_y} \subset A_z\) defines an order on Z, compatible with the
topology. Now \(X = (\bigcap_{z \in Z} A_z) \cup Z \cup (\bigcup_{z \in Z} B_z)\) since \(x \in A_r \cap B_s\) for
some \(r, s \in Z\) implies that \(x\) separates \(r\) and \(s\) (3C) and therefore
that \(x \in Z\).
Suppose \(\bigcap_{z \in Z} A_z\) contains two distinct points \(e, f\).

Then \(\overline{A_p} \setminus e\) is connected, hence \(A_q \setminus e\) is an open connected neigh-
bourhood of \(f\) separating \(e\) and \(q\).
Likewise \(A_q \setminus f\) is an open connected neighbourhood of \(e\) separating
\(f\) and \(q\). Contradiction.
Therefore both \(\bigcap_{z \in Z} A_z\) and \(\bigcap_{z \in Z} B_z\) can contain at most one point,
and we can extend the order on Z in the obvious way to an order
on X.
This proves 3.

4. X cannot contain exactly one cut point.
Let \(p\) be the only cut point of X, and \(X \setminus p = A + B\).
\(P_1, P_2, P_3\)

Then \(A \setminus p_1\) and \(A \setminus p_2\) and \(B \setminus p_3\) are connected and consequently
\((3 \setminus p_3) \cup (p) \cup (A \setminus p_1)\) is an open connected neighbourhood of
\(p_j\) separating \(p_i\) and \(p_j\) (\(i, j = 1, 2; i \neq j\)).
Contradiction.
5. X contains at least one cut point.
   For suppose no point of X is a cut point. We consider two cases.

5A. Suppose \( X \setminus \{p, q\} \) is disconnected for all \( p, q \in X \).
   Now \( X \setminus \{p, q\} = A + B \) and \( E = B \cup \{p, q\} \) is connected. So \( r \) cannot
   lie in a connected set separating \( p \) and \( q \). Since \( p, q, r \) are arbitrary we arrive at a contradiction.

5B. Let \( X \setminus \{p, q\} \) be connected for some fixed pair of distinct points \( p, q \).
   Now \( X \setminus \{p, q\} \) is an open connected neighbourhood of \( r \) separating \( p \)
   and \( q \) for each point \( r \in X \setminus \{p, q\} \).
   Therefore \( q \) cannot have an open connected neighbourhood separating \( p \) and some other point \( r \).
   Hence \( X \setminus \{p, r\} \) cannot be connected for \( r \neq q \).
   Thus \( X \setminus p \) has at most one end point.
   Choose \( r_1, r_2 \) different from \( p, q \).
   Let \( X \setminus \{p, r_1\} = A_1 + A_2 \) and \( X \setminus \{p, r_2\} = B_1 + B_2 \).
   Then \( \overline{A_1} = A_1 \cup \{p, r_1\} \) and \( \overline{B_1} = B_1 \cup \{p, r_2\} \) so \( r_2 \) and \( r_1 \) cannot lie
   in a connected set separating \( r_1 \) and \( p \) resp. \( r_2 \) and \( p \). Therefore \( p \)
   must lie in a connected set separating \( r_1 \) and \( r_2 \) and \( X \setminus \{r_1, r_2\} \)
   must be connected. (Otherwise we would have \( X \setminus \{r_1, r_2\} = A + B \)
   and \( E = B \cup \{r_1, r_2\} \) which gives a contradiction.)
   But now in \( X \setminus r_1 \) all points except possibly \( p \) and \( q \) are end points,
   which is impossible by the above argument.
   Contradiction.

Finally 3, 4 and 5 together prove the theorem.

**Remark**

The condition B'0 in the theorem is needed to ensure the existence of
sufficiently many open connected sets without which the second condition
(among three points there is at most one which lies in an open connected
set that separates the other two points) would be useless.
For example, if a $V_1$-space is defined as a connected $T_1$-space satisfying the following property: "each connected subset has at most one end point" then we have for $V_1$-spaces:

1. Among every three points there is at least one which lies in a connected set that separates the other two points.
2. The complement of an open connected set is connected. Therefore no open connected set separates any two points.
3. For all but at most one point $p \in X \setminus X$ does have infinitely many open components and exactly one closed component.

(see [1]).
References

[1] Brouwer, A.E. On connected spaces in which each connected subset has at most one endpoint, Rapport nr. 22, Vrije Universiteit, Amsterdam (1971).

