STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

ZW 1967-10r

An Isomorphism principle in General Topology

J. de Groot.

Reprinted from:
Bull. Am. Mathem. Society, 73 (1967),
p 465-467.

MC

- 1967 -

AN ISOMORPHISM PRINCIPLE IN GENERAL TOPOLOGY

BY

J. DE GROOT

Reprinted from the
BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY
May, 1967, Vol. 73, No. 3
Pp. 465-467

AN ISOMORPHISM PRINCIPLE IN GENERAL TOPOLOGY

BY J. DE GROOT

Communicated by V. Klee, December 16, 1966

Introduction. To practically every topological space T of importance (including metrizable and locally compact Hausdorff spaces) one can let correspond (essentially by interchanging compact and closed sets) an "antispace" T^* which conversely determines T. If, for example, T is Hausdorff but not compact, T^* will be T_1, compact and superconnected (every open set is connected). One sacrifices the Hausdorff property but gains e.g. compactness. Furthermore the topology of T^* is weaker than that of T. This destroys the belief, generally held, that non-Hausdorff spaces are of minor or no importance. On the contrary, one could even say that they are "more elegant," since they perform the same job with a weaker topology.

Philosophically, the consequences seem to be of interest. If R denotes "time" (the real line), R^* has the same topology as R on every bounded closed interval. However R^* is compact. Time becomes unbounded but finite in the sense of compact. We have potential but no actual infinity.

A remark by J. M. Aarts (in our joint work on cocompactness) initiated this note.

Preliminaries. Let X be a set and $\{G\}$ a family of subsets G of X, closed under finite unions and arbitrary intersections. We do *not* assume the (usual) convention that X and $\emptyset$ are necessarily members of $\{G\}$. A pair $T_- = (X, \{G\})$ is called a (topological) *minusspace*, where $\{G\}$ indicates the family of closed sets of T_-. One can, of course, extend every T_- to a *topological* space T by adding X and $\emptyset$ as closed sets.

A subset S of T_- is called *squarecompact relative* to T_-, if for every family $\{C_\alpha\}$ of compact subsets C_α of T_-, for which $\{S \cap C_\alpha\}$ is centered (that is the intersection of finitely many $S \cap C_\alpha$ is nonempty), the intersection of all $S \cap C_\alpha$ is nonempty.

One can prove:

(i) The intersection of a compact and a squarecompact set is both compact and squarecompact.

(ii) The union of finitely many and the intersection of any number of squarecompact sets is squarecompact.

(iii) If in T_- every compact set is closed, then every closed set is squarecompact.

A topological space is called a *c-space* if the closed sets are exactly those sets for which the intersection with every compact closed set is compact. This notion is different from the well-known notion of a k-space (compactly generated space). However, every k-space is a c-space and for those spaces in which compact sets are closed, both notions coincide.

The following notions are equivalent for a topological space T:

(a) It is a c-space.

(b) Compact sets are closed and squarecompact sets are closed.

(c) Firstly, the intersection of a compact and a squarecompact set is closed; secondly, a set G is closed in T iff $G\cap C$ is closed for all compact sets C of T.

(d) Firstly, the intersection of two compact sets is compact; Secondly, G is closed, iff $G\cap C$ is compact for all compact C.

Which spaces are c-spaces?

(iv) In every locally compact topological space (that is, every point has arbitrarily small compact neighborhoods) or in every space satisfying the first axiom of countability the following properties are equivalent:

(a) Every compact set is closed,
(b) The space is a c-space,
(c) The space is Hausdorff.

Antispaces. Let X be a set, $T_-=(X, \{G\}, \{C\})$, $T_-^*=(X,\{C\}, \{G\})$ two minusspaces over X, where $\{G\}$ denotes the family of all closed sets G and $\{C\}$ the family of all compact sets C in T_-, while $\{C\}$ are the *closed* sets of T_-^* and $\{G\}$ the *compact* sets of T_-^*. So the identity map of X onto itself maps the closed (compact) sets of X onto the compact (closed) sets of T_-^*. Such a pair is called an antipair and T_- and T_-^* are called *antispaces* (of each other). A space T_- is an antispace, if there exists a T_-^* as indicated. Observe that antispaces determine each other. If in T_- the compact sets coincide with the closed sets, T_- and T_-^* coincide as e.g. is the case for compact Hausdorff spaces.

EXAMPLE. If X is any set, and T the discrete space over X, then T^* is determined by the cofinite topology on X.

THEOREM. *A minusspace is an antispace, iff the closed sets coincide with the squarecompact sets. The topological antispaces T are exactly the c-spaces.*

Observe that **the** *compact* anti(minus)spaces T^* pair off with the *topological* antispaces T. A minusspace is a compact antispace, iff every closed set is compact and the squarecompact sets coincide with the closed sets. Notice that according to (iv), most spaces of importance in mathematics are c-spaces.

A category of antispaces T and onto continuous mappings f corresponds to the category of spaces T^* and onto mappings f^* (f^* defined by the requirement, that the inverse of any compact image set is compact). This sets up an isomorphism as mentioned in the title.

UNIVERSITY OF AMSTERDAM AND
UNIVERSITY OF FLORIDA