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On a theorem in the theory of binary relations

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by G. Fodor Szeged.

This paper was inspired by a theorem of Lazar<sup>1)</sup> which we shall state below.

Let  $M$  be a set of positive measure. To every  $x \in M$  we adjoin a set of elements  $y \neq x$  of  $M$  which may have the cardinal number of the continuum. This means that we define a function  $y = \varphi(x)$  of  $x \in M$ , where  $y \in M$ , which may assume continuously many values for every  $x$ , whereas the equation  $x = \varphi(x)$  cannot occur. If neither of the two equations  $y = \varphi(x)$  and  $x = \varphi(y)$  holds, the two elements  $x$  and  $y$  are called independent.

Let  $\{\varphi(x)\}$  denote the set of values  $\varphi(x)$  for a given  $x$ , <sup>assume</sup> moreover that  $x$  is not a point of accumulation of  $\{\varphi(x)\}$  and that  $\{\varphi(x)\}$  is of measure zero.

**Theorem.** We can find a set of positive exterior measure of elements of  $M$  that any pair of its elements are independent.

**Proof.** As the set  $\{\varphi(x)\}$  does not contain  $x$  and as  $x$  is no point of accumulation, its complement  $C\{\varphi(x)\}$  with respect to  $M$  contains  $x$  and an interval  $I_x$  surrounding  $x$ . Now ~~for every~~  $x$  let us choose in  $I_x$  a closed segment  $S(x)$  with rational endpoints. Hence all values of  $\varphi(x)$  are situated outside  $S(x)$ .

Now the segments  $S(x)$  form an enumerable system  $S_1, S_2, \dots$ . To every  $S_n$  there belongs at least one  $x \in M$ , such that  $S(x) = S_n$ . Let  $N_n$  denote the set of all  $x \in M$  with  $S(x) = S_n$ . We assert that there is at least one segment  $S_n$  for which  $N_n$  has positive exterior measure. For suppose that for every  $S_n$  the set  $N_n$  would be of measure zero. Then, as  $M \subset N_1 + N_2 + \dots$ , the well-known theorem that the sum of enumerably many sets of measure zero is a set of measure zero, we must conclude that  $M$  was a set of measure zero.

The elements  $x \in N_n$  ~~are evidently independent as the adjoined values~~ all belong to  $S_n$ ; they are evidently independent as the adjoined values are all outside the segment  $S_n$ .

Lazar's theorem<sup>2)</sup> states only that we can find a set with the power of the continuum under the same assumptions.

1) Compositio Mathematica 3 (1936), p. 304.

2) Evidently Lazar uses the word "condensation-point", where only a point of accumulation is meant.