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DIMENSION MODULO A CLASS OF SPACES

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1. INTRODUCTION

We start our discussion with a problem which has been posed by de Groot in 1942.

A space is said to be semicompact if each point of the space has arbitrarily small neighborhoods with compact boundaries (the notion of semicompactness was first introduced by Zippin [26]).

In [15], de Groot proved that a separable metrizable space is semicompact if and only if it can be compactified by adding a set of dimension not exceeding zero.

The notions of compactness degree and deficiency of a space come up naturally. The compactness degree of a separable space X - cmp X - is defined in a similar way as the small inductive dimension, ind, but starting with the definition that cmp X = -1 if and only if X is compact. The compactness deficiency of a separable space X - def X - is defined by def X = min $\{\text{ind } Y\backslash X \mid Y$ is a metrizable compactification of $X\}$. Now the above characterization of semicompactness can be rephrased as follows:

Theorem. Let n = 0 or -1. For every separable metrizable space X, cmp X $\leq$ n if and only if def X $\leq$ n.

De Groot (l.c.) has posed the following problem:
Can the above theorem (or some modification of it) be proved for all n $\geq$ -1?
An affirmative answer to this question will give a nice and elegant "internal" characterization of the "external" property def X $\leq$ n.

The following questions are related to the problem above.

1. Wat are internal necessary and sufficient conditions on a separable metrizable space X so that def X $\leq$ n?
2. Is it possible to obtain a fruitful generalization of dimension theory by replacing the empty set in the definition of inductive dimension by members of some class of spaces?
3. What is the special role of the empty set in the theory of inductive dimension?

Since the original problem of de Groot is still unresolved in spite of considerable efforts ([16],[17]), the questions 1, 2 and 3 are of interest.

The internal characterizations of def X given so far (question 1) are quite complicated ([5], [24]).

The answer to question 2 is yes. In these notes an account will be given of what has been done so far in establishing the theory of dimension modulo a class of spaces.

From the results which have been obtained, the answer to question 3 will be clear. The role of the empty set is a minor one. Analogues of fundamental theorems like the sum and decomposition theorem can be proved for dimension modulo a class P under mild assumptions on P.

All new results discussed in these notes are due to joint research of Professor T. Nishiura and the author ([8], [9] and [10]).

CONVENTIONS AND NOTATIONS

If A is a subset of X, then B_X (A) (cl_X(A)) denotes the boundary (closure) of A in X. If the space X need not be emphasized the subscripts will be dropped.

All spaces under discussion are assumed to be heriditarily normal (and T_1). Beginning with section 6, all spaces are to be metrizable.

2. INDUCTIVE INVARIANTS

The most important dimension functions are the weak inductive dimension ind, the strong inductive dimension Ind and the covering dimension dim. We start with the generalization of the inductive dimension functions. We shall chiefly discuss the generalizations of the strong inductive dimension.

A class P of spaces is topologically closed if for every $X \in P$ the class P contains all spaces homeomorphic to X.

Definitions 1. Let P be a topologically closed class of spaces. The strong (weak) inductive dimension modulo P, denoted by P - Ind (P - ind), is defined for every space X as follows:

1. $\mathcal{P}$ - Ind X = $\mathcal{P}$ - ind X = -1 if and only X $\in \mathcal{P}$.
2. For each integer n $\geq$ 0, $\mathcal{P}$ - Ind $\leq$ n ($\mathcal{P}$ - ind X $\leq$ n) provided that each <u>non-empty</u> closed subset (each point) of X has arbitrarily small open neighborhoods U such that its boundary B(U) has $\mathcal{P}$ - Ind B(U) $\leq$ n-1 ($\mathcal{P}$ - ind B(U) $\leq$ n-1).

For each integer n $\geq$ 0, $\mathcal{P}$ - Ind X = n if $\mathcal{P}$ - Ind X $\leq$ n and $\mathcal{P}$ - Ind X $\nleq$ n-1. If $\mathcal{P}$ - Ind X $\nleq$ n for each n, then $\mathcal{P}$ - Ind X = ∞. Similarly, $\mathcal{P}$ - ind X = n, n = 0,1,2, ...,∞ are defined.

Clearly $\mathcal{P}$ - Ind and $\mathcal{P}$ - ind are topological invariants, since $\mathcal{P}$ is required to be topologically closed. Lelek, who first introduced the notion of inductive dimension modulo $\mathcal{P}$, used the name "inductive invariant" for invariants of this type [19].

By an easy inductive proof it is verified that $\mathcal{P}$ - ind $\leq$ $\mathcal{P}$ - Ind.

<u>Examples</u>.

1. {∅} - Ind = Ind, the strong inductive dimension.
   {∅} - ind = ind, the weak inductive dimension.
2. If X $\neq$ ∅ and $\mathcal{Q}$ - Ind X $\leq$ 0, then $\mathcal{Q} \neq$ ∅.
   It follows that ∅ - Ind X $\geq$ 1, whenever X $\neq$ ∅.
   So ∅ - Ind X = Ind X+1 for every space X.
   Similarly ∅ - ind X = ind X+1 for every space X.
3. Let $\mathcal{K}$ = {X|X is compact}.
   For every separable metrizable space X, $\mathcal{K}$ - ind X = cmp X (section 1).
   In section 5 we shall show $\mathcal{K}$ - Ind E^n = n, where E^n is the n-dimensional Euclidean space. From this it is clear that the condition $\mathcal{K}$ - Ind X $\leq$ n is <u>not</u> a necessary condition for def X $\leq$ n.
4. $\mathcal{P}$ = {X | all components of X are metric continua}.

<u>Theorem</u> ([20]). Let X be a compact Hausdorff space. If X is the continuous image of an ordered compactum, then $\mathcal{P}$ - Ind X $\leq$ 0.

5. Let $\mathcal{D}_m = \{X \mid \text{Ind } X \leq m\}$, $m = -1,0,1,2,\ldots$.
$\mathcal{D}_m$ - Ind X = sup{Ind X - (m+1), -1} as can be shown by a simple inductive proof.

The easy proofs of the following propositions are omitted.

Proposition 1. If $\mathcal{P} \subset \mathcal{Q}$, then $\mathcal{P}$ - Ind $\geq$ $\mathcal{Q}$ - Ind and $\mathcal{P}$ - ind $\geq$ $\mathcal{Q}$ - ind.

Proposition 2. For every space X and every class $\mathcal{P}$ we have
$\mathcal{P}$ - Ind X $\leq$ Ind X + 1 and $\mathcal{P}$ - ind X $\leq$ ind X + 1.
Moreover, if $\phi \in \mathcal{P}$, then $\mathcal{P}$ - Ind X $\leq$ Ind X and $\mathcal{P}$ - ind X $\leq$ ind X.

FURTHER CONVENTIONS.

All classes of spaces are assumed to be topologically closed.
In view of example 2 we shall assume that every class $\mathcal{P}$ is non-empty.
We shall also assume $\phi \in \mathcal{P}$. This assumption is made in order to avoid certain pathologies like the following:

If $\phi \notin \mathcal{P}$, then $\mathcal{P}$ - Ind X = 0 if and only if X = ϕ. Observe that $\phi \in \mathcal{P}$ follows from every monotonicity condition on $\mathcal{P}$ (section 4).

3. DEFICIENCY AND SURPLUS

First we formulate the extension problem. Let $\mathcal{P}$ be a class of spaces. We say that Y is a $\mathcal{P}$-hull of X if X $\subset$ Y and Y $\in \mathcal{P}$. The (strong) $\mathcal{P}$-deficiency of X is the infimum of the set of numbers
{Ind Y \ X | Y is a $\mathcal{P}$-hull of X}. This infimum will be denoted by
$\mathcal{P}$ - Def X.

EXTENSION PROBLEM. For what classes of spaces $\mathcal{P}$ is $\mathcal{P}$ - Ind = $\mathcal{P}$ - Def?

Dually, we can formulate the excision problem. If $\mathcal{P}$ is a class of spaces, we say that Y is a $\mathcal{P}$-kernel of X if Y $\subset$ X and Y $\in \mathcal{P}$. The (strong) $\mathcal{P}$-surplus of X is the minimum of the set of numbers
{Ind X \ Y | Y is a $\mathcal{P}$-kernel of X}. This minimum will be denoted by
$\mathcal{P}$ - Sur X.

EXCISION PROBLEM. For what classes of spaces $\mathcal{P}$ is $\mathcal{P}$ - Ind = $\mathcal{P}$ - Sur?

From the viewpoint of the theory of the inductive dimension modulo a class of spaces a theory of kernels and surplus is more natural than a theory of hulls and deficiency. As a matter of fact the theory of kernels is related to the theory of normal families (section 6).

Examples

1. $\{\phi\}$ - def X = -1 if and only if X = ϕ.
   $\{\phi\}$ - def X = ∞ if and only if X $\neq \phi$.
   $\{\phi\}$ - Sur = Ind.
2. Let $\mathcal{S} = \{X \mid X \text{ is } \sigma\text{-compact}\}$.
   Let $Z = B \times I^n$, where I is the unit interval and B is the space of the irrational numbers. Then
   $\mathcal{S}$ - Sur Z = $\mathcal{S}$ - Ind = n.
   $\mathcal{S}$ - Sur = n is proved as follows. If K is any $\mathcal{S}$-kernel of Z, then the natural projection of K into B is σ-compact. Since B is not σ-compact, there is a point $q \in B$ such that $\{q\} \times I^n \cap K = \phi$. So $Z \setminus K \supset \{q\} \times I^n$ and Ind $Z \setminus K = n$.
   From the results in section 8 it follows that $\mathcal{S}$ - Ind Z = $\mathcal{S}$ - Sur Z.
3. Let $\mathcal{K} = \{X \mid X \text{ is compact}\}$.
   R denotes the real line (with the usual topology).
   $\mathcal{K}$ - ind R = $\mathcal{K}$ - Def R = 0.
   $\mathcal{K}$ - Ind R = $\mathcal{K}$ - Sur R = 1.
   As for $\mathcal{K}$ - Ind R = 1, observe that the disjoint closed sets $F = \{n \mid n = 2,3,\dots\}$ and $G = \{n + \frac{1}{n} \mid n = 2,3,\dots\}$ cannot be separated by a compact set.

Note that by using other dimension functions, we can obtain several types of deficiency and surplus (see [9]).

4. DIMENSION MODULO A CLOSED MONOTONE CLASS

In our discussions we have to assume certain monotonicity conditions. A class $\mathcal{P}$ is said to be monotone if $X \in \mathcal{P}$, whenever $Y \in \mathcal{P}$ and $X \subset Y$. The other monotonicity conditions have the following form: If X is a certain type of subset of Y and $Y \in \mathcal{P}$, then $X \in \mathcal{P}$. The types we shall consider are closed, open, F_σ and G_δ. We will use these modifiers to express the type of monotonicity we wish to use. For example, closed monotone class $\mathcal{P}$.

In this section several properties of dimension functions modulo a closed monotone class are presented

Theorem 1. A class $\mathcal{P}$ of spaces if closed monotone if and only if for every space X for every closed subset F of X we have
$\mathcal{P}$ - Ind F $\leq$ $\mathcal{P}$ - Ind X.

Proof. Since the "if" part is obvious, we prove the "only if" part.
The proof is by induction on $\mathcal{P}$ - Ind X.
Suppose F is a closed subset of X. If $\mathcal{P}$ - Ind X = -1, then $X \in \mathcal{P}$, which implies $F \in \mathcal{P}$; hence $\mathcal{P}$ - Ind F = -1.
Assume the theorem for X with $\mathcal{P}$ - Ind X $\leq n - 1$.
Let G be a non-empty closed subset of F and U an open neighborhood of G in F. Then $X \setminus (F \setminus U)$ is an open neighborhood of G in X.
Since G is also closed in X and $\mathcal{P}$ - Ind X $\leq n$, there is a neighborhood V of G in X such that $G \subset V \subset X \setminus (F\setminus U)$ and $\mathcal{P}$ - Ind B(V) $\leq n - 1$. Then $W = V \cap F$ is a neighborhood of G in F, which satisfies $G \subset W \subset U$ and $B_F(W) \subset B_X(V)$.
Since $B_F(W)$ is a closed subset of $B_X(V)$, by the induction hypothesis $\mathcal{P}$ - Ind $B_F(W) \leq n - 1$. Thus we get $\mathcal{P}$ - Ind F $\leq n$.

Proposition 1. Let $n \geq 0$. Suppose $\mathcal{P}$ is closed monotone. Let A be a subset of X with $\mathcal{P}$ - Ind A $\leq n$.
Then for any disjoint closed subsets F and G of X there exists an open set U such that $F \subset U \subset X \setminus G$ and $\mathcal{P}$ - Ind$(B(U) \cap A) \leq n - 1$.

Proof. Since $\emptyset \in \mathcal{P}$, we may assume that F and G are disjoint non-empty closed subsets of X. By virtue of the normality of X there exist open sets V and W satisfying $F \subset V$, $G \subset W$ and $cl(V) \cap cl(W) = \emptyset$. Because $\mathcal{P}$ $\mathcal{P}$ - Ind $A \leq n$, there exists a neighborhood D of $cl(V) \cap A$ in A satisfying $cl(V) \cap A \subset D \subset A \setminus cl(W)$ and $\mathcal{P}$ - Ind $B_A(D) \leq n - 1$.
Observe that $B_A(D) = cl_A(D) \cap cl_A(A \setminus D)$.
Let $F_1 = F \cup (A \setminus cl_A(A \setminus D))$ and $G_1 = G \cup (A \setminus cl_A(D))$.
Neither of the disjoint sets F_1 and G_1 contains a cluster point of the other. By virtue of the heriditary normality there exists an open set U such that $F_1 \subset U$ and $cl_X U \cap G_1 = \emptyset$.
$B_X(U) = cl_X(U) \setminus U$ and $B_X(U) \cap A = B_A(D)$. The proposition follows.

By applying proposition 1 repeatedly we get

Propisition 2. Suppose $\mathcal{P}$ is closed monotone. Let $\mathcal{P}$ - Ind $X \leq n$. Then for every open collection $\{U_i \mid i = 1, \dots, n + 1\}$ and closed collection $\{F_i \mid i = 1, \dots, n + 1\}$ with $F_i \subset U_i$ there exists an open collection $\{V_i \mid i = 1, \dots, n + 1\}$ such that $F_i \subset V_i \subset cl(V_i) \subset U_i$ and $\cap \{B(V_i) \mid i = 1, \dots, n + 1\} \in \mathcal{P}$.

Proposition 3. Suppose $\mathcal{P}$ is closed monotone. Let A and B be subsets of a space X such that $X = A \cup B$.
If $\mathcal{P}$ - Ind $A \leq n$ and Ind $B \leq 0$, then $\mathcal{P}$ - Ind $X \leq n + 1$.

Proof. Let F be a non-empty closed subset of X and let V be a neighborhood of F in X. By virtue of proposition 1 there exists an open set U satisfying $F \subset U \subset V$ and Ind $(B(U) \cap B) \leq -1$.
Hence $B(U) \subset A$. By virtue of theorem 1 we have $\mathcal{P}$ - Ind $B(U) \leq n$.

Proposition 4. Suppose $\mathcal{P}$ is closed monotone. Then $\mathcal{P}$ - Ind $\leq$ $\mathcal{P}$ - Sur.

The proof of proposition 4 is similar to that of proposition 3, using induction on $\mathcal{P}$ - Sur. The theorem below can also be proved in a similar way, using double induction (cf. [1], p.28).

A class $\mathcal{P}$ is said to be additive if $Z \in \mathcal{P}$ whenever $Z = X \cup Y$, $X \in \mathcal{P}$ and $Y \in \mathcal{P}$.

Theorem 2. A closed monotone class P is additive if and only if

$$P - \text{Ind}\,(Y \cup Z) \le P - \text{Ind}\,Y + P - \text{Ind}\,Z + 1$$

for any subsets Y and Z of a space.
In particular, P - Ind X cannot be raised by adding a member of P to X provided that P is additive.

Proposition 5. Suppose P is closed monotone and open monotone.
Then $P - \text{Ind} \le P - \text{Def}$.

The proof is by induction on P - Def (see [6] for more details).

Counterexample.
The analogue of theorem 2 fails for P - ind as the following example shows.
Let X be the union of an open disc A in the plane and a boundary point b of A. Let K be the class of compact spaces.
K - ind A = 0, K - ind {b} = -1 and K - ind X = 1.

5. THE EILENBERG - BORSUK DUALITY THEOREM.

In this section we give an outline of the proof of the following theorem on extension of maps (i.e. continuous functions), which was first proved by Eilenberg in the compact case [13].

Theorem 1. Let $0 \le k \le n$. Suppose A is a closed set of a space X with Ind $X \setminus A \le n$. Then for each map $f: A \to S^k$, from A to the k-dimensional sphere, there exists a set $E \subset X \setminus A$, E closed in X with Ind $E < n - k$ such that f can be extended over $X \setminus E$.

The result of Eilenberg was first improved by Borsuk [12] (separable metrizable case) and later by Akasaki[11] (general metrizable case).
We shall indicate a simple and elegant proof for theorem 1 in the case that all spaces are heriditarily normal, employing inductive dimension modulo a class. For more details see [8].
Observe that the only information of theorem 1 is the upper bound for the dimension of E (since S^k is an absolute neighborhood retract!).

Usually (cf.[1] ,[3] and [4]), the special case k = n of theorem 1

is presented as a corollary of the following well-known result about mappings in spheres:
For every closed subset C of a heriditarily normal space X with Ind $X \leq n$ and for every map $f : C \to S^n$ there exists an extension of f over X.

It is surprising that this property remains valid and can be proved similarly if we pass from Ind to $\mathcal{P}$ - Ind.

Theorem 2. Suppose $\mathcal{P}$ is closed monotone. Let $\mathcal{P}$ - Ind $X \leq n$, $(n \geq 0)$. Then for every closed set C of X and every map f of C into S^n there exists a closed $\mathcal{P}$ - kernel E of X with $E \subset X \setminus C$ such that f can be extended over $X \setminus E$.

Outline of the proof of theorem 2. We shall regard S^n to be the boundary of I^{n+1}, where I^{n+1} is the $(n+1)$-dimensional cube in E^{n+1}; $I^{n+1} = \{(x_1, \ldots, x_{n+1}) \mid |x_i| \leq 1, i = 1, \ldots, n+1\}$. The point $(0, \ldots, 0)$ is denoted by q.

The proof has the same pattern as that of the above mentioned theorem about mappings in sphere (see[4], III 1.A and 2.A).
Let $f : C \to S^n$ be as mentioned. Since I^{n+1} is an absolute retract, f - considered as mapping into I^{n+1} - has an extension over X, which we denote again by f.
Let f(p) have the coordinates $f_1(p), \ldots, f_{n+1}(p)$.
Put $\varepsilon = \dfrac{1}{6\sqrt{n+1}}$ and $F_i = \{p \mid f_i(p) \geq \varepsilon\}$, $G_i = \{p \mid f_i(p) \leq -\varepsilon\}$, $i = 1, \ldots, n+1$.

By virtue of proposition 4.2 there exist open sets V_i, $i = 1, \ldots, n+1$, such that $F_i \subset V_i \subset cl(V_i) \subset X \setminus G_i$ and $E = \cap \{B(V_i) \mid i = 1, \ldots, n+1\}$ is a closed $\mathcal{P}$ - kernel of X. It is clear that

$$E \subset X \setminus \cup \{F_i \cup G_i \mid i = 1, \ldots, n+1\} \subset X \setminus f^{-1}(S^n) \subset X \setminus C.$$

Also $\cap \{B(V_i) \setminus E \mid i = 1, \ldots, n+1\} = \emptyset$. Since $X \setminus E$ is normal, in the subspace $X \setminus E$ there exists open sets W_i with
$cl_X(V_i) \setminus E \subset W_i \subset X \setminus G_i$ and
(*) $\cap \{W_i \setminus V_i \mid i = 1, \ldots, n+1\} = \emptyset$.
We construct maps $\phi_i : X \setminus E \to [-\varepsilon, \varepsilon]$ such that

$\phi_i^{-1}(0) \subset W_i \setminus V_i$, $\phi_i^{-1}(\varepsilon) = F_i$ and $\phi_i^{-1}(-\varepsilon) = G_i$.

Defining g_i by $\quad g_i(p) = f_i(p)$, $p \in F_i \cup G_i$

$$g_i(p) = \phi_i(p),\ p \in X \setminus (F_i \cup G_i \cup E)$$

we obtain maps of $X \setminus E$ into I.

Now $g : X \setminus E \to I^{n+1}$ defines by $g(p) = (g_1(p), \ldots, g_{n+1}(p))$ is continuous and $\rho(f(p), g(p)) < \frac{1}{3}$ for every $p \in X \setminus E$ (ρ denotes the metric of I^{n+1}). In view of (*) we have $q \notin g(X \setminus E)$.

In a standard way it can be proved that g can be required to be equal to f on $f^{-1}(S^n)$.

The compositon of g and the projection of $I^{n+1} \setminus \{q\}$ onto S^n from the origin is the desired extension of f.

Proof of theorem 1. Let $P = \{X \mid \text{Ind } X \leq n - k - 1\}$.

Then P - Ind $X \setminus A \leq k$ (example 2.5).

Let $f : A \to S^k$ as mentioned in the theorem. Since S^k is an absolute neighborhood retract, there is an open set U of X containing A and an extension g of f over U. We consider an open set V such that $A \subset V \subset cl(V) \subset U$. $X \setminus V$ is a closed subset of $X \setminus A$. By theorem 4.1, P - Ind $X \setminus V \leq k$. By virtue of theorem 2 the map $g \mid B_X(V) : B_X(V) \to S^k$ has an extension $k : (X \setminus V) \setminus E \to S^k$, where E is a closed P - kernel of X with $E \subset X \setminus cl_X(V)$.

Letting $\quad f^1(p) = g(p)$ for $x \in cl_X(V)$

$$h(p) \text{ for } x \in (X \setminus V) \setminus E$$

we obtain the deserved extension of f over $X \setminus E$.

Ind $E \leq n - k - 1 < n - k$.

Example. Let $K = \{X \mid X \text{ is compact}\}$.

We shall show K - Ind $E^n = n$, where E^n is the n-dimensional Euclidean space. In view of proposition 2.2. it is sufficient to prove K - Ind $E^n \geq n$. We shall derive a contradiction from the assumption K - Ind $E^n \leq n - 1$. Let $C_m = \{x \mid \rho(x, p_m) = 1\}$ where p_m is a point the distance of which to the origin q is 3m, m = 1, 2, ...

Let $C = \cup \{C_m \mid m = 1, 2, \ldots\}$. A map of the closed set C onto the standard sphere S^{n-1} is defined by sending each C_m isometrically onto S^{n-1}. From the assumption K - Ind $E^n \leq n - 1$, by virtue of theorem 2, it follows that there exists a compact set F in E^n with $F \subset E^n \setminus C$ such that f has a continuous extension over $X \setminus F$. Since a compact set is bounded, this results in retractions of the n - cell to its boundery.

6. NORMAL FAMILIES

FURTHER CONVENTION. From now on all spaces are assumed to be metrizable.

The theory of normal families is due to Hurewicz [18].
It greatly simplifies the deduction of the fundamental theorems of the inductive theory for separable spaces. The theory of normal families has been adapted for general metrizable spaces by Morita [21] in order to establish the theory of strong inductive dimension for (general) metrizable spaces.
In this section we give an outline of this theory (without proofs) and indicate its relation to the notion of surplus. In sections 7 and 8 this theory will be generalized (with proofs).

Let P be a class of spaces. We shall say that P is countably (locally finitely) closed additive if $X \in P$ whenever there is a countable (locally finite) closed cover F of X with $F \subset P$.

Definition 1. We shall say that a topologically closed class M of metrizable spaces is a normal family if the following conditions are satisfied:
N1. M is monotone,
N2. M is countably closed additive,
N3. M is locally finitely closed additive.

Examples of normal families.
1. $\{\emptyset\}$.
2. A space is said to be locally countable if each point has a countable neighborhoud. Let

$\mathcal{R} = \{X \mid X$ has a countable closed cover of locally countable sets$\}$.
$\mathcal{R}$ - Ind + 1 is known as the <u>rational dimension</u>.
$\mathcal{R}$ is a normal family.

<u>Definition 2</u>. For any topologically closed class of metrizable spaces $\mathcal{P}$ we define a <u>new class $\mathcal{P}'$ of metrizable spaces</u> as follows:
$X \in \mathcal{P}'$ if each non-empty closed subset F of X has arbitrarily small open neighborhoods U such that $B(U) \in \mathcal{P}$.
We agree that $\mathcal{P}^{(n+1)} = (\mathcal{P}^{(n)})'$ for $n = 0, 1, 2, \ldots$ and $\mathcal{P}^{(0)} = \mathcal{P}$.

<u>Example</u>. Let $\mathcal{P}$ be a class of spaces. Let $n \geq -1$.
$X \in \mathcal{P}^{(n+1)}$ if and only if $\mathcal{P}$ - Ind $X = n$.
(This is the reason why all dimension numbers should be raised by one).

Here we list the fundamental theorems of the theory of normal families.

<u>Theorem 1</u>. The class $\mathcal{D}_0 = \{X \mid \text{Ind } X \leq 0\}$ is a normal family.

<u>Theorem 2</u>. If $\mathcal{M}$ is a normal family, then $\mathcal{M}'$ is also a normal family.

<u>Theorem 3</u>. Let $\mathcal{M}$ be a normal family.
A space X belongs to $\mathcal{M}'$ if and only if there exist two subspaces Y and Z such that $X = Y \cup Z$, $Y \in \mathcal{M}$ and Ind $Z = 0$.

Observe that theorem 1 is a special case of theorem 2. However in the theory of normal families the proofs of theorems 1 and 2 are totally different and theorem 1 in combination with theorem 3 is used to prove theorem 2. In section 7 we shall show that the theorems can be rearranged in such a way that theorems similar to theorems 1 and 2 can be proved simultaneously.
Let us collect some consequences of theorems 1, 2 and 3.
$\mathcal{D}_k = \{X \mid \text{Ind } X \leq k\}$ is a normal family by virtue of theorems 1 and 2, $k = -1, 0, 1, 2, \ldots$. From this we get the following theorems.
<u>The subset theorem</u>. If Y is a subset of a space X and Ind $X \leq n$, then Ind $Y \leq n$.
<u>The countable sum theorem</u>. Let $\{F_i \mid i = 1, 2, \ldots\}$ be a countable closed cover of X such that Ind $F_i \leq n$ for $i = 1, 2, \ldots$. Then Ind $X \leq n$.

The locally finite sum theorem. Let $\{F_\gamma \mid \gamma \in \Gamma\}$ be a locally finite closed covering of X such that Ind $F_\gamma \leq n$ for each $\gamma \in \Gamma$. Then Ind $X \leq n$.

By applying theorem 3 repeatedly we get the following theorems.
The decomposition theorem. Ind $X \leq n$ if and only if X is the union of $n + 1$ subspaces of dimension ≤ 0.
The addition theorem. Let $X = Y \cup Z$. Then Ind $X \leq$ Ind Y + Ind $Z + 1$.
The excision theorem. Let M be a normal family. Then M - Ind = M - Sur.

Remarks 1. A generalization of the addition theorem has already been given in theorem 4.2.
2. Except for [10], the theory of normal families has never been employed for the study of dimension modulo a class. Normal families have been designed for a systematic deduction of the fundamental theorems of dimension theory. In view of the special role of zero-dimensional spaces (theorem 1 above) and because of the strength of condition N1, other applications of normal families can hardly be expected.

In sections 7 and 8 we shall show that by rearranging the theorems and by relaxing condition N1 (as has been done in [10]), a new theory with applications to dimension modulo a class can be obtained.

7. THE SUM THEOREMS

Definition 1. We shall say that a class N of metrizable spaces is a regular family if the following conditions are satisfied:
R1. N is closed monotone,
R2. N is countably closed additive.

Examples of regular families.
1. Every normal family is regular.
2. The class of all countable spaces is a regular family.
3. The class of all σ-compact spaces is a regular family.

Theorem 1. If N is a regular family, then N' is also a regular family.

Proof. Suppose N is a regular family. We shall show that N' is also regular in three steps.

A. N' is closed monotone.
This is proved in theorem 4.1.

B. N' is open monotone.
Proof (cf.[3] 11.3). Let $X \in N'$ and let V be an open subset of X. Let F be a closed subset of V and let U_0 be an open neighborhood of F in V. Write $V = \cup \{F_i \mid i = 1, 2, \ldots\}$ with F_i closed in X and $F_i \subset \text{Int } F_{i+1}$ (Int denotes the interior in X). Let W_i, $i = 1, 2, \ldots$, be open subsets of X such that
$F \subset cl_V W_{i+1} \subset W_i \subset U_o$ and $\cap\{W_i \mid i = 1, 2, \ldots\} = F$.
For each i, $F \cap F_i \subset W_i \cap \text{Int } F_{i+1} \subset F_{i+1}$.
By virtue of A we have $F_{i+1} \in N'$, $i = 1, 2, \ldots$.
From the definition of N' it follows that there exists an open set U_i with $F \cap F_i \subset U_i \subset W_i \cap \text{Int } F_{i+1}$ and $B_V(U_i) \in N$ $(i = 1, 2, \ldots)$.
Let $U = \cup \{U_i \mid i = 1, 2, \ldots\}$. Since the collection $\{U_i \mid i = 1, 2, \ldots\}$ is locally finite on $V \setminus F$, we have $B_V(U) \subset \cup\ B_V(U_i) \mid i = 1, 2, \ldots\}$.
Since N is regular, $B_V(U) \in N$. The inclusions $F \subset U \subset U_0$ are evident.
Thus $V \in N'$.

Remark. For later use we make the observation that $\{B_V(U_i) \mid i = 1, 2, \ldots\}$ is a locally finite cover of $\cup \{B_V(U_i) \mid i = 1, 2, \ldots\}$.

C. N' is countable closed additive.
Proof (cf.[3] 10.4). Let $X = \cup \{X_i \mid i = 1, 2, \ldots\}$ where X_i is closed in X and $X_i \in N'$.
Define $F_1 = X_1$ and $F_i = X_i \setminus \cup \{X_j \mid j = 1, \ldots, i - 1\}$, $i \geq 2$.
Then $X = \cup \{F_i \mid i = 1, 2, \ldots\}$ and the F_i are pairwise disjoint.
Moreover (1) $F_i \in N'$ (by virtue of B),
(2) $\cup \{F_j \mid j = 1, \ldots, i\}$ is closed in X for each i.
Now, let K_1 be a non-empty closed subset of X and U an open neighborhood of K_1 in X. Let $L_1 = X \setminus U$.
Let $G_1 = K_1 \cap F_1$ and $H_1 = L_1 \cap F_1$.

Since $F_1 \in N'$, there are a closed subset C_1 and open subsets U_1, V_1 of F_1 such that $G_1 \subset U_1$, $H_1 \subset V_1$ and $C_1 \in N$.
Now $K_1 \cup U_1$ and $L_1 \cup V_1$ are disjoint closed subsets of $X \setminus C_1$.
Let K_2 and L_2 be disjoint closed neighborhoods of $K_1 \cup U_1$ and $L_1 \cup V_1$ in $X \setminus C_1$.
It is to be observed that $cl_X(K_2) \cap cl_X(L_2) \subset C_1$.
Now let $G_2 = K_2 \cap F_2$ and $H_2 = L_2 \cap F_2$.
Since $F_2 \in N'$, there is a closed subset C_2 of F_2 and open subsets U_2, V_2 of F_2 such that $G_2 \subset U_2$, $H_2 \subset V_2$ and $C_2 \in N$.
Observe that $C_1 \cup C_2$ is closed in X (If $p \in F_1$ is an accumulation point of C_2, then $p \in C_1$). Also $C_1 \cup C_2 \in N$, because C_2 is an F_σ subset of $C_1 \cup C_2$ and N satisfies the conditions R1 and R2.
Inductively we define K_n, L_n and a set $C_n \in N$.
Finally, we let $K = \cup \{K_n \mid n = 1, 2, \ldots\}$, $L = \cup \{L_n \mid n = 1, 2, \ldots\}$ and $C = \cup \{C_n \mid n = 1, 2, \ldots\}$.
$K_1 \subset K \subset U$, K is open in X, and $B(K) \subset C$.
By virtue of conditions R1 and R2, $B(K) \in N$. Thus $X \in N'$.

A first consequence of theorem 1 is
The countable sum theorem. Suppose N is a regular family.
Let $\{F_i \mid i = 1, 2, \ldots\}$ be a countable closed covering of X such that N - Ind $F_i \leq n$ for $i = 1, 2, \ldots$, Then N - Ind $X \leq n$.

Definition 2. A regular family is said to be semi-normal if it is locally finitely closed additive.

Examples of semi-normal families.
1. The class $A(1)$ of all σ-locally compact spaces is a semi-normal family.
It is sufficient to check that this class is locally finitely closed additive. Let $\{X_\gamma \mid \gamma \in \Gamma\}$ be a locally finite closed cover of X with $X_\gamma \in A(1)$ for every $\gamma \in \Gamma$. Let $E_\gamma = \cup \{E_\gamma^i \mid i = 1, 2, \ldots\}$, where E_γ^i is locally compact. For every i, E_γ^i is open in $cl(E_\gamma^i)$, hence an F_σ in $cl(E_\gamma^i)$ and consequently E_γ^i is an F_σ in X.
Thus we may assume that $E_\gamma = \cup \{E_\gamma^i \mid i = 1, 2, \ldots$ with E_γ^i locally compact and closed in X. Then, for each i, the set $F_i = \cup \{E_\gamma^i \mid \gamma \in \Gamma\}$ is locally compact. Hence $X \in A(1)$.

As has been shown by Stone [25] the class $A(1)$ can be described as the class of all absolute F_σ spaces; a space X is said to be an absolute F_σ-space if, whenever X is a subspace of a metrizable space Y, X is an F_σ subset of Y (or, equivalently, X is a member of the additive Borel class 1 in Y).

2. The class $A(\alpha)$ of all absolute Borel sets of additive class α, where $2 \le \alpha < \Omega$ is a semi-normal family (see [2] for definitions). An elegant proof of the locally finite closed additivity is given in [14].

3. Every normal family is semi-normal.

Theorem 2. If N is a semi-normal family, then N' is also a semi-normal family.

Proof. Suppose N is semi-normal. Since any semi-normal family is regular, in view of theorem 1 we need only show that N' is locally finitely closed additive.

Let $X = \cup \{X_\gamma \mid \gamma \in \Gamma\}$ be a locally finite closed covering of X such that $X_\gamma \in N'$ for every $\gamma \in \Gamma$. In a standard fashion we can find a locally finite open cover $\{U_\delta \mid \delta \in \Delta\}$ and a closed cover $\{F_\delta \mid \delta \in \Delta\}$ of X such that for each δ, $F_\delta \subset U_\delta$ and $cl(U_\delta)$ meets at most finitely many members of the collection $\{X_\gamma \mid \gamma \in \Gamma\}$. Since N' is a regular family, $cl\ U_\gamma \in N'$ for each δ.

Let A be a closed subset of X and W and open neighborhood of A. $U_\delta \cap W$ is a neighborhood of $F_\delta \cap A$ in the subspace $cl\ U_\delta$.

Hence there exists an open subset V_δ of X such that

$F_\delta \cap A \subset V_\delta \subset U_\delta \cap W$ and $B(V_\delta) \in N$.

Since $\{B(V_\delta) \mid \delta \in \Delta\}$ is a locally finite closed cover of $\cup \{B(V_\delta) \mid \delta \in \Delta\}$, we have $\cup \{B(V_\delta) \mid \delta \in \Delta\} \in N$.

Let $V = \cup \{V_\delta \mid \delta \in \Delta\}$.Then $A \subset V \subset X \setminus B$ and, by theorem 1, $B(V) \in N$.

Hence $X \in N'$.

As a consequence of theorem 2 we get

The locally finite sum theorem. Suppose N is a semi-normal family. Let $\{F_\gamma \mid \gamma \in \Gamma\}$ be a locally finite closed covering of X such that $N - \text{Ind}\ F_\gamma \le n$ for each $\gamma \in \Gamma$. Then $N - \text{Ind}\ X \le n$.

Remark. The above theorem holds under weaker assumptions on N (see [10]).

8. THE EXCISION THEOREM.

Theorem 1. Let N be a semi-normal family.
A space X belongs to N' if and only if there exist two subspaces Y and Z such that $X = Y \cup Z$, $Y \in N$ and Ind $Z = 0$.

Proof. In view of proposition 4.2 it is sufficient to prove the " only if" part. Suppose $X \in N'$. In a standard fashion we can get a framework of a σ-locally finite basis of X i.e. an open collection $\{U_\gamma \mid \gamma \in \Gamma\}$ and a closed collection $\{F_\gamma \mid \gamma \in \Gamma\}$ such that

1. $\emptyset \neq F_\gamma \subset U_\gamma$ for every $\gamma \in \Gamma$, and

2. if $\{V_\gamma \mid \gamma \in \Gamma\}$ is an open collection such that $F_\gamma \subset V_\gamma \subset U_\gamma$, $\gamma \in \Gamma$, then it is a σ-locally finite basis of X.

For each $\gamma \in \Gamma$ there is an open W_γ satisfying

$F_\gamma \subset W_\gamma \subset U_\gamma$ and $B(W_\gamma) \in N$.
Let $Y = \cup \{B(W_\gamma) \mid \gamma \in \Gamma\}$ and $Z = X \setminus Y$.
Then $Y \in N$ in view of the condition R2 and the locally finite closed additivity of N. The subspace Z has a σ-locally finite open base $\mathcal{B}$ such that $B(V) = \emptyset$ for every $V \in \mathcal{B}$.

It remains to show Ind $Z \leq 0$. Let $\mathcal{B} = \cup \{\mathcal{B}_i \mid i = 1, 2, \ldots\}$ for locally finite open collections $\mathcal{B}_i$. Let F be a non-empty closed subset of Z and U a neighborhood of F in Z.
For every i let

$$U_i = Z \setminus \cup \{V \mid V \cap F = \emptyset,\ V \in \cup \{\mathcal{B}_j \mid j = 1, \ldots, i\}\}, \text{ and}$$

$$V_i = \cup \{V \mid V \subset U,\ V \in \cup \{\mathcal{B}_j \mid j = 1, \ldots, i\}\}.$$

Both U_i and V_i are open as well as closed.
Let $W_i = U_i \cap V_i$. Then $\{W_i \mid i = 1, 2, \ldots\}$ is locally finite on $Z \setminus F$.
It follows that $W = \cup \{W_i \mid i = 1, 2, \ldots\}$ satisfies
$F \subset W \subset U$ and $B(W) = \emptyset$. Hence Ind $Z \leq 0$.

Corollary. Suppose $\mathcal{M}$ is a semi-normal family. Let $n \geq 0$. $\mathcal{M}$-Ind $X \leq n$ if and only if there exists a σ-locally finite open base $\mathcal{B}$ for X such that $\mathcal{M}$-Ind $B(V) \leq n - 1$ for every $V \in \mathcal{B}$.

From the special case $n = 0$, $\mathcal{M} = \{\emptyset\}$ of this corollary it is easily deduced that $\mathcal{D}_0 = \{X \mid \text{Ind } X \leq 0\}$ is monotone. Combined with theorem 1, this result gives a proof of theorem 6.2. NOW, ALL THEOREMS OF SECTION 6 HAVE BEEN PROVED.

By applying theorem 1 repeatedly we get

THE EXCISION THEOREM. Suppose $\mathcal{M}$ is a semi-normal family. Then $\mathcal{M}$-Ind = $\mathcal{M}$-Sur.

An important consequence of the excision theorem is the following:

If $\mathcal{M}$ is a semi-normal family, then a theorem for $\mathcal{M}$-Ind gives a theorem for $\mathcal{M}$-Sur, and conversily. Thus the corollary above holds also for $\mathcal{M}$-Sur. The theorems for $\mathcal{P}$-Sur below, which easily follow from the definition of $\mathcal{P}$-Sur and the theorems of dimension theory in section 6, yield corresponding theorems for $\mathcal{P}$-Ind, whenever $\mathcal{P}$ is a semi-normal family.

Subset theorem. If $\mathcal{P}$ is monotone, then $\mathcal{P}$-Sur $Y \leq n$, whenever $Y \subset X$ and $\mathcal{P}$-Sur $X \leq n$.

Decomposition theorem. Suppose $\mathcal{P}$ is additive. Then $\mathcal{P}$-Sur $X \leq n$ if and only if $X = \cup \{X_i \mid i = 1, \ldots, n + 1\}$ with $\mathcal{P}$-Sur $X_i \leq 0$, $i = 1, \ldots, n + 1$.

The following proposition will give a partial answer to the extension problem.

Proposition 1. Suppose $\mathcal{P}$ is an additive family which contains the class of topologically complete spaces. Then $\mathcal{P}$-Def $\leq$ $\mathcal{P}$-Sur.

Proof. Suppose $\mathcal{P}$ - Sur $X \leq n$. Let $X \subset Y$ and Y topologically complete (so $Y \in \mathcal{P}$). There exists a $\mathcal{P}$ - kernel A of X such that Ind $X \setminus A \leq n$. By virtue of a theorem of Tumarkin ([4], theorem II.10) there exists a G_δ subset B of Y such that $X \setminus A \subset B$ and Ind $B \leq n$.
B is topologically complete, so $B \in \mathcal{P}$. The additivity of $\mathcal{P}$ gives $Z = A \cup B \in \mathcal{P}$. Ind $Z \setminus X \leq$ Ind $B \leq n$. Hence $\mathcal{P}$ - Def $X \leq n$.

Theorem 2. Let $2 \leq \alpha < \Omega$. Let $A(\alpha)$ be the class of all absolute Borel sets of additive class α. Then
$A(\alpha)$ - Ind = $A(\alpha)$ - Sur = $A(\alpha)$ - Def.

The following example shows that for the class $A(1)$ of all α-locally compact spaces the exquality $A(1)$ - Ind = $A(1)$ - Def does not hold (thus resolving a conjecture of Nagata in the negative [22]).

Example. Here we present an example of a separable space X with $A(1)$ - Ind X = 0 and $A(1)$ - Def X = 1.
Let I denote the unit interval, $Q = \{t \mid t \in I$ and t is rational$\}$ and $B = I \setminus Q$.
X is the subspace of $I \times I \times I$ given by
$X = (I \times I \times Q) \cup (B \times B \times B)$.
As is easily seen $A(1)$ - Ind X = $A(1)$ - Sur X = 0.
The proof of $A(1)$ - Def X = 1 makes use of the Baire category theorem ($B \times B \times B$ is topologically complete) and of the fact that X is not rimcompact (theorem in section 1) (see [9] for more details).

9. COVERING DIMENSION MODULO A CLASS

The most satisfactory answer to the extension problem is given through covering dimension modulo a class [10].

Let Y be a $\mathcal{P}$ - hull of a space X. Covering dimension applied to the remainder $Y \setminus X$ leads naturally to the concept of a $\mathcal{P}$ - border cover and the order of a $\mathcal{P}$ - border cover (cf.[24] and [6]). In this section we shall define the covering dimensions modulo a class $\mathcal{P}$ and summarize the main results on the relations between $\mathcal{P}$ - Sur, $\mathcal{P}$ - Ind and the covering dimensions.

We adopt the sntadard conventions for collections (e.g. [4] I.1). Let $\mathcal{U}$ be a collection in a topological space X and p a point of X. The order of $\mathcal{U}$ at p is the number of distinct members of $\mathcal{U}$ which contain p, and we denote it by $\mathrm{ord}_p\, \mathcal{U}$. The order of $\mathcal{U}$ is the supremum of $\{\mathrm{ord}_p\, \mathcal{U} \mid p \in X\}$.

Definition 1. Let $\mathcal{P}$ be a class of spaces. A $\mathcal{P}$ - border cover of a space X is an open collection $\mathcal{V}$ such that $(X \setminus \cup \mathcal{V}) \in \mathcal{P}$

The closed $\mathcal{P}$ - kernel $X \setminus \cup \mathcal{V}$ is called the <u>enclosure of $\mathcal{V}$</u>.
(Recall the convention $\emptyset \in \mathcal{P}$).

<u>Definition 2</u>. Let $\mathcal{P}$ be a class of spaces and X be a space.
$\mathcal{P}$ - dim X $\leq$ n if for any <u>finite</u> $\mathcal{P}$ - border cover $\mathcal{U}$ of X there exists a $\mathcal{P}$ - border cover $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}$ and order $\mathcal{V} \leq n + 1$.
$\mathcal{P}$ - Dim X $\leq$ n if for any $\mathcal{P}$ - border cover $\mathcal{U}$ of X there exists a $\mathcal{P}$ - border cover $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}$ and order $\mathcal{V} \leq n + 1$.
<u>$\mathcal{P}$ - dim</u> and <u>$\mathcal{P}$ - Dim</u> are called the <u>small</u> and <u>large covering</u> dimension <u>modulo $\mathcal{P}$</u> respectively.
It will be agreed that $\mathcal{P}$ - dim X = $\mathcal{P}$ - Dim X = 1 if and only if X $\in \mathcal{P}$.

When $\mathcal{P} = \{\emptyset\}$, we drop the prefix $\mathcal{P}$ and simply write dim.
As is well-known Dim = dim on the class of metrizable spaces.
An elementary proof of this equality for paracompact spaces, which is due to de Vries, is given in [4]. It is surprising that this proof can be adapted for dimensions modulo $\mathcal{P}$ under mild assumptions on $\mathcal{P}$ (theorem 1 below).

<u>Proposition 1</u>. $\mathcal{P}$ - dim and $\mathcal{P}$ - Dim are topological invariants.

<u>Proposition 2</u>. $\mathcal{P}$ - dim $\leq$ $\mathcal{P}$ - Dim $\leq$ dim.

<u>Problem</u>. Recall that if $\mathcal{P} \subset \mathcal{Q}$, then $\mathcal{Q}$ - Ind $\leq$ $\mathcal{P}$ - Ind (proposition 2.1).
A smilar result holds for Sur and Def. We conjecture that $\mathcal{P} \subset \mathcal{Q}$ <u>does not</u> imply $\mathcal{Q}$ - dim $\leq$ $\mathcal{P}$ - dim in general (a $\mathcal{Q}$ - border cover need not be a $\mathcal{P}$ - border cover!).

<u>Example</u>. Let $\mathcal{C}$ be the class of topologically complete spaces.
Let $Z = Q \times I^n$, when I denotes the unit interval and $Q = \{t \mid t \in I$ and t is rational$\}$.
We shall show $\mathcal{C}$ - dim Z = $\mathcal{C}$ - Dim Z = n. In view of proposition 2 above we need only prove $\mathcal{C}$ - dim Z $\geq$ n.
Since dim $I^n = n$, there is a finite open cover $\mathcal{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ of I^n such that for every open cover $\mathcal{V}$ of I^n with $\mathcal{V} < \mathcal{U}$ we have order $\mathcal{V} \geq n + 1$. $\mathcal{U}^* = \; Q \times U_\gamma \mid \gamma \in \Gamma\}$ is a cover of Z.
Let $\mathcal{W}$ be a border cover of Z with enclosure C such that $\mathcal{W} < \mathcal{U}^*$.

Since Q is not topologically complete, there is a $q \in Q$ with $\{q\} \times I^n \cap C = \emptyset$ (the natural projection of $Q \times I^n$ onto Q is a perfect map). It follows that the restruction of $\mathcal{W}$ to $\{q\} \times I^n$ has order $\geq n + 1$. Hence order $\mathcal{W} \geq n + 1$ and C - dim $Z \geq n$.

A class of spaces P is said to be <u>weakly additive</u> if $Z \in P$, whenever $Z = X \cup Y$ with X closed, $X \in P$ and $Y \in P$.

<u>Definition 3</u>. We shall say that a topologically closed class M of metrizable spaces is <u>a cosmic family</u> if the following conditions are satisfied:

C1. M is closed monotone,

C2. M is weakly additive,

C3. M is locally finitely closed additive.

Any semi-normal family is a cosmic family, and any cosmic family which is countably closed additive, is semi-normal. Every regular family satisfies conditions C1 and C2.

It should be observed that a <u>cosmic family is open monotone</u>. This is clear since any open set V in X can be written as $V = \cup \{H_k \mid k = 0, 1, \ldots\}$, where $H_k = \{x \mid \frac{1}{k+1} \leq \rho(x, X \setminus V) \leq \frac{1}{k}\}$. By virtue of C1 and C3 we have $V \in M$, whenever $X \in M$ and M is cosmic.

<u>Further examples</u> of cosmic families.

The classes $M(\alpha)$ of all absolute Borel sets of multiplicative class α, where $1 \leq \alpha < \Omega$, are cosmic families.

Observe that $M(1)$ is the class C of topologically complete spaces.

Now we mention without proofs some important results from [10] (or small modifications of them).

<u>Theorem 1</u>. Suppose M is a cosmic family. Then

<u>A.</u> M' is also a cosmic family (remark in the proof of theorem 7.1);

<u>B.</u> M - dim, M - Dim and M - Ind satisfy the locally finite sum theorem;

<u>C.</u> M - dim = M - Dim $\leq$ M - Ind;

<u>D.</u> M - dim $X \leq n$ if and only if for every closed M - kernel F of X and every closed subset C of $X \setminus F$ and every (continuous) map $f : C \to S^n$, there exists a closed M - kernel G of $X \setminus F$ such that

$G \subset X \setminus C$ and f can be extended over $X \setminus (F \cup G)$;

E. Let F be a closed M - kernel of X. Then M - Dim $X \setminus F$ = M - Dim X (Note the equality).

Theorem 2. Suppose R is a regular family. Then R - Dim = R - Sur.

10. DUALITY THEOREMS.

The duality of kernels and hulls (surplus and deficiency) becomes clear through covering dimension (cf. [10] section 7).

Definition 1. We shall say that two subsets X and Y of a space Z are complementary in Z if $Z = X \cup Y$ and $X \cap Y = \emptyset$.
Let P and Q be classes of spaces. A space Z is called ambiguous relative to P and Q provided that $X \in P$ if and only if $Y \in Q$, whenever X and Y are complementary in Z.

Observe that in view of our convention $\emptyset \in P$ for every P , each space Z which is ambiguous with respect to P and Q , belongs to $P \cap Q$.

Examples.
1. Let S and C be the classes of σ-compact spaces and topologically complete spaces respectively. Each compact space Z is ambiguous relative to S and C .
2. Let $A(\alpha)$ and $M(\alpha)$ be the classes of all absolute Borel sets of additive and multiplicative class α respectively, $2 \leq \alpha < \Omega$.
$A(\alpha) \cap M(\alpha)$ is the family of all spaces which are ambiguous relative to $A(\alpha)$ and $M(\alpha)$. Using the classical terminology [2], $A(\alpha) \cap M(\alpha)$ is the family of absolute ambiguous sets of class α.
Observe that both $A(\alpha)$ and $M(\alpha)$ are G_δ monotone as well as F_σ monotone ($\alpha \geq 2$). Moreover $M(1) \subset M(\alpha) \cap A(\alpha)$ for $\alpha \geq 2$.

Lemma 1. Suppose Z is ambiguous relative to P and Q .
Then Q - Def X $\leq$ P - Sur Y, whenever X and Y are complementary in Z.

The proof of lemma 1 is trivial.

Lemma 2. Suppose P is F_σ monotone. Then
P - Sur $\leq$ P - Def.

Proof. Let X be a space with $\mathcal{P}$ - Def X < ∞. Let Y be a $\mathcal{P}$ - hull of X with Ind Y \ X = $\mathcal{P}$ - Def X. By virtue of a theorem of Tumarkin ([4], theorem II.10) there exists a G_δ subset G of Y such that Y \ X ⊂ G and Ind G = Ind Y \ X.
Then F = Y \ G is an F_σ subset of Y contained in X.
Hence, F is a $\mathcal{P}$ - kernel of X. Ind X \ F ≤ Ind G = $\mathcal{P}$ - Def X.
So $\mathcal{P}$ - Sur X ≤ $\mathcal{P}$ - Def X.

Theorem 1. Suppose that both $\mathcal{P}$ and $\mathcal{Q}$ are F_σ monotone.
Suppose Z is ambiguous relative to $\mathcal{P}$ and $\mathcal{Q}$.
Then $\mathcal{P}$ - Def X = $\mathcal{P}$ - Sur X = $\mathcal{Q}$ - Def Y = $\mathcal{Q}$ - Sur Y.

Proof. $\mathcal{P}$ - Def X ≥ $\mathcal{P}$ - Sur X ≥ $\mathcal{Q}$ - Def Y ≥ $\mathcal{Q}$ - Sur Y ≥ $\mathcal{P}$ - Def X.

Theorem 2. Let $2 \le \alpha < \Omega$. Then
$A(\alpha)$ - Sur = $A(\alpha)$ - Def, $M(\alpha)$ - Sur = $M(\alpha)$ - Def and
$[M(\alpha) \cap A(\alpha)]$ - Sur = $[M(\alpha) \cap A(\alpha)]$ - Def.

Proof. Let $X \in A(\alpha)$. Let $X \subset Y \in M(1)$. Then X and Y \ X are complementary sets of Y and Y is ambiguous relative to $A(\alpha)$ and $M(\alpha)$.

Now we are going to prove similar results for covering dimensions modulo a class.
Convention. If $\mathcal{V} = \{V_\gamma \mid \gamma \in \Gamma\}$ is a collection in X and Y ⊂ X then $\mathcal{V} \mid Y$ - the restruction of $\mathcal{V}$ to Y - is the collection $\{V_\gamma \cap Y \mid \gamma \in \Gamma\}$.

Lemma 3. Let Y be a subspace of Z. For every open collection $\mathcal{U}$ of Y there exists an open collection $\mathcal{U}$ of X such that $\mathcal{V} \mid Y = \mathcal{U}$ and order $\mathcal{U}$ = order $\mathcal{V}$.

Proof. See [2], 15, XIII.

Lemma 4. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are closed monotone.
Let Z be ambiguous relative $\mathcal{P}$ and $\mathcal{Q}$ and let X and Y be complementary in Z. Let $\mathcal{U}$ be an open collection in Z.
Then $\mathcal{U} \mid X$ is a $\mathcal{P}$ - border cover of X if and only if $\mathcal{U} \mid Y$ is a $\mathcal{Q}$ - border cover of Y.

Proof. Let $W = \cup \mathcal{U}$. If $X \setminus W \in \mathcal{P}$, then $Z \setminus (X \setminus W) \in \mathcal{Q}$.
But $Z \setminus (X \setminus W) = Y \cup W$ and $Y \setminus W$ is a closed subset of $Y \cup W$. Hence $Y \setminus W \in \mathcal{Q}$, whenever $X \setminus W \in \mathcal{P}$. The lemma follows.

Theorem 2. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are closed monotone. Suppose Z is ambiguous relative to $\mathcal{P}$ and $\mathcal{Q}$.
Then $\mathcal{P}$ - Dim X = $\mathcal{Q}$ - Dim Y and $\mathcal{P}$ - dim X = $\mathcal{Q}$ - dim Y, whenever X and Y are complementary in Z.

Proof. Since the proofs for the small and large covering dimensions are very similar, we only prove the theorem for Dim.
Due to symmetry we need only prove $\mathcal{P}$ - Dim X $\leq$ $\mathcal{Q}$ - Dim Y, whenever X and Y are complementary in Z.
Suppose $\mathcal{Q}$ - Dim Y $\leq$ n. Let $\mathcal{U}$ be a $\mathcal{P}$ - border cover of X.
Let $\mathcal{U}^*$ be an open collection in Z with $\mathcal{U}^* \mid X = \mathcal{U}$. By virtue of lemma 4, $\mathcal{U}^* \mid Y$ is a $\mathcal{Q}$ - border cover of Y.
Since $\mathcal{Q}$ - Dim Y $\leq$ n, this $\mathcal{Q}$ - border cover has a $\mathcal{Q}$ - border cover refinement $\mathcal{V}$ of order $\leq n + 1$. Using lemma 3, we form an open collection $\mathcal{V}^*$ in Z with $\mathcal{V}^* \mid Y = \mathcal{V}$, order $\mathcal{V}^*$ = order $\mathcal{V}$ and $\mathcal{V}^* < \mathcal{U}^*$. Again by virtue of lemma 4, $\mathcal{V}^* \mid X$ is a $\mathcal{P}$ - border cover of X. $\mathcal{V}^* \mid X < \mathcal{U}$ and order $\mathcal{V}^* \mid X \leq n + 1$. So $\mathcal{P}$ - Dim X $\leq$ n.

Example. Let $Z = I^{n+1}$, where I denotes the unit interval.
Let $Q = \{t \mid t \in I \text{ and } t \text{ rational}\}$.
$X \subset Z$ is given by $X = Q \times I^n$. Let $Y = Z \setminus X$.
Z is ambiguous relative to $\mathcal{S}$ and $\mathcal{C}$ (the classes of all σ - compact and topologically complete spaces respectively). By virtue of theorem 2,

$\mathcal{C}$ - Dim X = $\mathcal{S}$ - Dim Y (= n).

$\mathcal{S}$ - Dim X = $\mathcal{C}$ - Dim Y (= -1).

11. EXTENSION THEOREM

In this section we shall give sufficient conditions for $\mathcal{P}$ - Dim = $\mathcal{P}$ - Def (cf. [10], section 6).
In view of proposition 4.5 ($\mathcal{P}$ - Ind $\leq$ $\mathcal{P}$ - Def) and theorem 9.1. C ($\mathcal{P}$ - Dim $\leq$ $\mathcal{P}$ - Ind) we have $\mathcal{P}$ - Dim $\leq$ $\mathcal{P}$ - Def for every cosmic family.
Here we give a direct proof of this.

Thoerem 1. Suppose $\mathcal{P}$ is a cosmic family. Then

$$\mathcal{P} - \mathrm{Dim} \leq \mathcal{P} - \mathrm{Def}.$$

Proof. Suppose $\mathcal{P}$ - Def $X \leq n$. Let F be a $\mathcal{P}$ - hull of X with Ind $F \setminus X \leq n$. Let $\mathcal{U}$ be a $\mathcal{P}$ - border cover of X with enclosure G (G is a closed $\mathcal{P}$ - kernel of X). Let $\mathcal{U}^*$ be an open collection in F with $\mathcal{U}^* \mid X = \mathcal{U}$. The set $W = \cup\, \mathcal{U}^*$ is an open subset of F and hence $W \in \mathcal{P}$. Since $G \cap W = \emptyset$, $Y = G \cup W$ is a $\mathcal{P}$ - hull of X, because $\mathcal{P}$ is weakly additive. Also, Ind $Y \setminus X \leq n$. As in the proof of theorem 10.2 we can obtain an open collection $\mathcal{V}$ in Y such that order $\mathcal{V} \leq n + 1$, $\mathcal{V} \mid X < \mathcal{U}$ and $\mathcal{V}$ is a cover of $Y \setminus X$.
$Y \setminus \cup\, \mathcal{V}$ is a closed subset of Y contained in X. It follows that $\mathcal{V} \mid X$ is a $\mathcal{P}$ - border cover of X of order $\leq n + 1$, which refines the given $\mathcal{P}$ - border cover $\mathcal{U}$.

The following definition is suggested by the excision theorem and the duality of surplus and deficiency.

Definition 1. A class $\mathcal{P}$ is said to be countably open multiplicative if for each $Y \in \mathcal{P}$ and each non-empty countable collection $\{X_i \mid i = 1, 2 \ldots\}$ of $\mathcal{P}$ - kernels of Y the intersection $X = \cap \{X_i \mid i = 1, 2, \ldots\}$ belongs to $\mathcal{P}$, whenever $X_j \setminus X$ is open in $Y \setminus X$ for $j = 1, 2, \ldots$.

It is clear that the notion of countably open multiplicatively and that of countable closed additivity are dual concepts. Observe that open monotonicity and countable open multiplicativity of a class imply G_δ monotonicity of the class.

Lemma 1. Suppose $\mathcal{P}$ is a cosmic family which is countably open multiplicative. Then $\mathcal{P}$ - Dim X = $\mathcal{P}$ - Def X provided X has a $\mathcal{P}$ - hull.

Proof. In view of theorem 1 we need only prove $\mathcal{P}$ - Def $\leq \mathcal{P}$ - Dim. Let Y be a $\mathcal{P}$ - hull of X and $\mathcal{P}$ - Dim $X = n$. We assume X is dense in Y. Let $\mathcal{U}_1 = \{S_1(x) \mid x \in Y \setminus X\}$, where $S_1(x) = \{y \mid \rho(x,y) < 1\}$. $\mathcal{U}_1' = \mathcal{U}_1 \mid X$ is a $\mathcal{P}$ - border cover of X. Let $\mathcal{V}_1$ be a border cover of X with enclosure F_1 such that $\mathcal{V}_1 < \mathcal{U}_1'$ and order $\mathcal{V}_1 \leq n + 1$. Let $\mathcal{W}_1$

be an open collection in Y such that $\mathcal{W}_1 \mid X = \mathcal{V}_1$ and order $\mathcal{W}_1$ = order $\mathcal{V}_1$. Suppose $\mathcal{W}_1, \ldots, \mathcal{W}_{k-1}$ have been defined.
Let $\mathcal{U}_k = \{S_{\frac{1}{k}}(x) \mid x \in Y \setminus X\}$. Let $\mathcal{V}_k$ be a border cover of X with inclosure F_k which refines $(\mathcal{U}_k \wedge \mathcal{W}_{k-1}) \mid X$ and has order $\leq n + 1$. Let $\mathcal{W}_k$ be an open collection such that $\mathcal{W}_k \mid X = \mathcal{V}_k$, $\mathcal{W}_k < \mathcal{W}_{k-1}$ and order $\mathcal{W}_k \leq n + 1$.
Weak additivity of $\mathcal{P}$ implies $X_k = F_k \cup (\cup \mathcal{W}_k) \in \mathcal{P}$.
Denote by Z the set $\cap \{X_k \mid k = 1, 2, \ldots\}$. Since $F_k \subset X \subset X_k$ for each K, $X \subset Z$ and $X_k \setminus Z = (\cup \mathcal{W}_k) \setminus Z$. Consequently, countable open multiplicitavily of $\mathcal{P}$ implies Z is a $\mathcal{P}$ - hull of X.

We shall show Ind $Z \setminus X \leq n$.
Let $\mathcal{W}_k' = \mathcal{W}_k \mid Z \setminus X$. Then $\{\mathcal{W}_k' \mid k = 1, 2, \ldots\}$ is a sequence of open coverings of $Z \setminus X$ such that

1. $\mathcal{W}_{k+1}' < \mathcal{W}_k'$, k = 1, 2, ... ;
2. order $\mathcal{W}_k' \leq n + 1$, k = 1, 2, ... ;
3. mesh $\mathcal{W}_k' \leq$ mesh $\mathcal{W}_k$ = mesh $\mathcal{V}_k \leq$ mesh $\mathcal{U}_k \leq \frac{2}{k}$.

By [4] theorem V.1 it follows that Ind $Z \setminus X \leq n$.
Hence $\mathcal{P}$ - Def $X \leq n = \mathcal{P}$ - Dim X.

<u>Definition 2</u>. A class $\mathcal{P}$ is said to be <u>universal</u> if every space X has a $\mathcal{P}$ - hull.

<u>Proposition 1</u>. Suppose $\mathcal{P}$ is a G_δ monotone class. Then $\mathcal{P}$ is universal if and only if $\mathcal{P}$ contains the class $\mathcal{C}$ of all topologically complete spaces.

<u>Proof</u>. Observe that a topologically complete space is an absolute G_δ.

<u>Example</u>. $\{\emptyset\}$ is <u>not</u> universal.

From lemma 1, theorem 1, proposition 4,5 and theorem 9.1 we get the following result.

<u>THE EXTENSION THEOREM</u>. Suppose $\mathcal{P}$ is a cosmic family which is countably open multiplicative and universal. Then

$\mathcal{P}$ - Ind = $\mathcal{P}$ - Dim = $\mathcal{P}$ - Def.

Examples. The conditions of the extension theorem are satisfied by the classes $M(\alpha)$ of all absolute Borel sets of multiplicative class α, $1 \leq \alpha < \Omega$. Observe that $M(1) = C$, the class of topologically complete spaces, is the smallest class which satisfies the conditions of the preceding theorem.

Many results on surplus and deficiency as well as the duality of these concepts can be summarized as follows ([10], theorem 7.5).

Thoerem 2. Suppose P and Q are F_σ monotone and weakly additive. Suppose Z is ambiguous relative to P and Q.
Then, if P is countably open multiplicative or if Q is countably closed additive,
P - Def X = P - Sur X = P - Dim X = Q - Dim Y = Q - Sur Y = Q - Def Y, whenever X and Y are complementary in Z.

Application. For $2 \leq \alpha < \Omega$,
$A(\alpha)$ - Dim = $A(\alpha)$ - Sur = $A(\alpha)$ - Def, and
$M(\alpha)$ - Dim = $M(\alpha)$ - Def = $M(\alpha)$ - Sur.

Example. Let C denote the class of topologically complete spaces. Let Z be the subspace of $I \times I$ given by $Z = I \times Q \cup Q \times I$ (I is the unit interval and Q is the set of rationals).
By virtue of the extension theorem, C - Ind Z = C - Dim Z = C Def Z = 0. By a simple application of the Baire category theorem, it can be shown that C - Sur Z = 1. So the complement of any C - kernel of Z has dimension one! None the less, every C - border cover of X has a C - border cover refinement of order one !

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