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On the differential operators of first order in tensor calculus

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by

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The most important property of any calculus dealing with geometric objects is that it contains one or more differential operators ledding from some well defined geometric object to other well defined geometric objects. Ricci calculus had from the beginning such an operator, the covariant differentiation. The Christoffel symbol $\{{}^{\kappa}_{\mu\lambda}\}$ was already introduced in 1869 [1]) and this symbol constituted as Ricci said in 1901 [2]), the "material instrument" necessary to build his method, but we owe to Ricci the idea [3]) to use this instrument for the construction of an invariant differential operator that can be applied to every tensor field and leads to the covariant derivative of the field.

Later on Levi Civita and the present author found independently about 35 years ago that the covariant differential could be interpreted geometrically in terms of a new kind of parallelism, and though this idea had a great influence on the development of modern differential geometry, it was by no means so important as Ricci's invention of covariant differentiation.

Now invariant differential operators were not entirely unknown at the time Ricci published his new calculus. From vector analysis we know the operations rotation and divergence that could easily be generalized for multivectors (= alternating quantities), in n dimensions:

1) a) Rot: $(p+1)\partial_{[\mu}w_{\lambda_1 \ldots \lambda_p]}$; w = p-vector

b) Div: $\partial_\mu \mathfrak{w}^{\mu\kappa_2 \ldots \kappa_q}$; $\mathfrak{w}$ = q-vector density of weight +1.

But there is a big difference between these operators and covariant differentiation:

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1) Christoffel, E.B.: Uber die Transformation des homogenen Differentialausdrücke zweiten Grades, Crelle's Journal 70, 46-70; gesammelte Abh. I, p. 352-377.

2) Ricci, G. and Levi Civita, T.: Méthodes de calcul différentiel absolu et leurs applications, Math. Ann. 54, 125-201. Reprint: collection de monographies etc. no. 5, Blanchard, Paris.

3) Ricci, G.: Sulle derivazione covariante ad una forma quadratica differenziale, Rend. Acc. Linc. (4) 3, 15-18, (1887).

2) a) $\nabla_\mu v^\kappa = \partial_\mu v^\kappa + \Gamma^\kappa_{\mu\lambda} v^\lambda$;

b) $\nabla_\mu w_\lambda = \partial_\mu w_\lambda - \Gamma^\kappa_{\mu\lambda} w_\kappa$;

$$\Gamma^\kappa_{\mu\lambda} \stackrel{\text{def}}{=} \{^{\,\kappa}_{\mu\lambda}\} = \tfrac{1}{2} g^{\kappa\varrho}(\partial_\mu g_{\lambda\varrho} + \partial_\lambda g_{\mu\varrho} - \partial_\varrho g_{\mu\lambda}).$$

Rot and Div exist in "empty" space, that means, we need only the fields operated on and nothing more. But covariant differentiation needs (in its first version) besides a field as v^κ or w_λ another field $g_{\lambda\kappa}$ from which the $\{^{\,\kappa}_{\mu\lambda}\}$ can be derived. In fact the covariant derivative of a field, for instance v^κ, may be considered as a differential concomitant of the two fields v^κ and $g_{\lambda\kappa}$. But this implies that covariant differentiation can be interpreted in two ways, first as an operator depending on $g_{\lambda\kappa}$ and working on v^κ and secondly as an operator depending on v^κ and working on $g_{\lambda\kappa}$.

In this way a differential concomitant of two quantities gives use to two differential operators and from this we see that asking for more differential concomitants and asking for more differential operators is essentially the same problem.

In 1931 [4]) Slebodzinski found a new differential operator depending on a contravariant vectorfield v^κ and appliable to all kinds of quantities (and geometric objects as was found later)

3) a) $\underset{v}{\pounds}\, u^\kappa = v^\mu \partial_\mu u^\kappa - u^\mu \partial_\mu v^\kappa$;

b) $\underset{v}{\pounds}\, w_\lambda = v^\mu \partial_\mu w_\lambda + w_\mu \partial_\lambda v^\mu$.

Also in this case the "material instrument" occurred already in publications of Lie but the interpretation as a differential operator that could be applied to all quantities was new. Van Dantzig [5]) called the new operator the Lie derivation. It was interpreted geometrically and applied to various problems of deformation by Van Kampen and the present author [6]) and it is now in generally use especially by english and japanese authors. Of course also this operator can be interpreted in two ways and there is a connexion between it and the covariant differentiation

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4) Slebodzinski, W.: Sur les équations de Hamilton, Bull. Acad. Roy. de Belgique (5) 17, 864-870.

5) Van Dantzig, D.: Zur allgemeinen projektiven Differentialgeometrie II, Proc. Kon. Ak. Amst. 35 (1932), 535-542.

6) Schouten, J.A. und Van Kampen, E.R.: Beiträge zur Theorie der Deformation, Warsz. Prac. Mat. Fiz. 41 (1933), 1-19.

4) $$\underset{v}{\mathcal{L}}\, g_{\mu\lambda} = 2\, \nabla_{(\mu} v_{\lambda)} .$$

In 1940 [7]) the present author succeeded in generalizing Lie's operator by forming a differential concomitant of two arbitrary <u>contravariant</u> quantities:

5) $$\sum_{i}^{0,\dots,a} \wp\, P^{\{\kappa_1 \dots \kappa_i |\mu| \kappa_{i+1} \dots \kappa_a}\, \partial_\mu Q^{\kappa_{a+1} \dots \kappa_{a+b+1}\}} - \sum_{j}^{0,\dots,b} \wp^{ab+a+b+j}\, Q^{\{\kappa_1 \dots \kappa_j |\lambda| \kappa_{j+1} \dots \kappa_b}\, \partial_\lambda P^{\kappa_{b+1} \dots \kappa_{a+b+1}\}} ,$$

where $\{\ \} = (\ ) + [\ ]$ and $\wp$ is the operator of an arbitrary <u>odd</u> permutation of the $a+b$ indices in $\{\ \}$. For instance $\wp\{\kappa\lambda\mu\} = \{\lambda\kappa\mu\}$ and $\wp^2\{\kappa\lambda\mu\} = \{\kappa\lambda\mu\}$.

This concomitant could be derived as follows. Let Λ be a so called collecting index standing for any number of upper and lower indices and let O_Λ be some differential operator whose working on the general quantities P and Q (indices suppressed) is known. Then we may <u>try</u> to find the working of O_Λ on the product PQ by using the rule of Leibnitz

6) $$O_\Lambda(PQ) = (O_\Lambda P)Q + P O_\Lambda Q .$$

But now it is by no means sure that the right hand side really can be expressed in terms of the product PQ and its derivatives. Here is an example. We know the Lie derivative of a tensor $p^{\kappa\lambda}$

7) $$\underset{v}{\mathcal{L}}\, p^{\kappa\lambda} = v^\mu \partial_\mu p^{\kappa\lambda} - p^{\mu\lambda} \partial_\mu v^\kappa - p^{\kappa\mu} \partial_\mu v^\lambda$$

and we may look upon $-\underset{v}{\mathcal{L}}\, p^{\kappa\lambda}$ also as the result of an operator O^κ working on v^λ:

8) $$O^\kappa v^\lambda = p^{\kappa\mu} \partial_\mu v^\lambda + p^{\mu\lambda} \partial_\mu v^\kappa - v^\mu \partial_\mu p^{\kappa\lambda} .$$

Now we try to use the rule of Leibnitz for the product $u^{\lambda_1\lambda_2} = v^{\lambda_1} w^{\lambda_2}$. That leads to

9) $$\begin{aligned} O^\kappa u^{\lambda_1\lambda_2} &= (O^\kappa v^{\lambda_1}) w^{\lambda_2} + v^{\lambda_1} O^\kappa w^{\lambda_2} = \\ &= p^{\kappa\mu} \partial_\mu u^{\lambda_1\lambda_2} - u^{\mu\lambda_2} \partial_\mu p^{\kappa\lambda_1} - u^{\lambda_1\mu} \partial_\mu p^{\kappa\lambda_2} + \\ &\quad + p^{\mu\lambda_1} \partial_\mu u^{\kappa\lambda_2} - p^{\mu\lambda_1} v^\kappa \partial_\mu w^{\lambda_2} + p^{\mu\lambda_2} v^{\lambda_1} \partial_\mu w^\kappa . \end{aligned}$$

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7) Schouten, J.A.: Uber Differentialkomitanten zweier kontravarianter Grössen, Proc. Kon. Ned. Akad. Amst. 43 (1940), 449-452.

The first four terms are expressed in terms of $p^{\kappa\mu}$, $u^{\lambda_1\lambda_2}$ and their derivatives but there are two disagreeing terms that do not contain $u^{\lambda_1\lambda_2}$ but only its factors. These disagreeing terms can be eliminated by taking $O^{\{\kappa_1}u^{\lambda_1\lambda_2\}}$ instead of $O^{\kappa_1}u^{\lambda_1\lambda_2}$ and this leads to the formula (5) for $a = b = 1$.

In 1918 [8]) E.Noether gave a general process for a given symmetric covariant tensor of valence >2 that leads to a complete set of differential concomitants but these concomitants do not depend on the coordinates ξ^κ only, as ordinary tensors do, but also on the differentials $d\xi^\kappa, d^2\xi^\kappa, \ldots$ as is usual in the geometrics of Finsler and Kawaguchi. But the results of E.Noether made it highly improbable that next to the covariant derivative and the Lie derivative other ordinary differential concomitants could be found.

So it was very astonishing that Nijenhuis found in 1951 [9]) a new concomitant of two mixed quantities of valence two $h_\lambda^{\cdot\kappa}$ and $l_\lambda^{\cdot\kappa}$:

10)
$$O_{[\mu}l_{\lambda]}^{\cdot\kappa} \overset{\text{def}}{=} h_{[\mu}^{\cdot\varrho}\partial_{|\varrho|}l_{\lambda]}^{\cdot\kappa} - l_\varrho^{\cdot\kappa}\partial_{[\mu}h_{\lambda]}^{\cdot\varrho} + l_{[\mu}^{\cdot\varrho}\partial_\varrho h_{\lambda]}^{\cdot\kappa} - h_\varrho^{\cdot\kappa}\partial_{[\mu}l_{\lambda]}^{\cdot\varrho}.$$

Mr. Tonolo [10]) had considered the necessary and sufficient conditions for the principal directions of a symmetric tensor in V_3 to be V_2-normal. He succeeded in finding conditions that did no longer contain the principal directions (as did all conditions formulated before) but only the tensor itself and its covariant derivatives. In dealing with this matter we found another more practical form of these conditions that could be generalized immediately for $n > 3$ [11]). Induced by this work Mr. Nijenhuis investigated the more general (non-metric) problem whether all pairs of eigendirections of a tensorfield $h_\lambda^{\cdot\kappa}$ could be X_2-building. This led him immediately to the new concomitant. In fact for $h = l$ and all eigenvalues different from each other this concomitant is zero if and only if the eigendirections have this special property.

The concomitant of Nijenhuis though entirely new, is so simple that there certainly must be more concomitants. Let us again try to use

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8) Noether, E.: Invarianten beliebiger Differentialausdrücke, Gött. Nachr. 1918, 1-8.

9) Nijenhuis, A.: X_{n-1}-forming sets of eigenvectors, Kon. Ned. Akad. Amst. 54 (1951) 200-212.

10) Tonolo, A.: Sopra una classe di deformazioni finite, Ann. Mat. Pura Appl. 4, 29 (1949), 29-53.

11) Schouten, J.A.: Sur les tenseurs de V_n aux directions principales V_{n-1}-normales, Coll. de géom. diff. Louvain 1951, 67-70.

the rule of Leibnitz and let us start with the negative Lie-derivative of $h_\lambda^{\cdot\kappa}$ looked upon as an operation working on v^κ:

11) $$O_\lambda v^\kappa = -\underset{v}{\mathcal{L}}\, h_\lambda^{\cdot\kappa} = h_\lambda^{\cdot\mu}\partial_\mu v^\kappa - h_\mu^{\cdot\kappa}\partial_\lambda v^\mu - v^\mu\partial_\mu h_\lambda^{\cdot\kappa}.$$

Then we try to write $(O_\lambda v^{\kappa_1})w^{\kappa_2} + v^{\kappa_1}O_\lambda w^{\kappa_2}$ in a form containing only $u^{\kappa_1\kappa_2} = v^{\kappa_1}w^{\kappa_2}$ and its derivatives. This gives six terms written down in the first column. Four of them have the form desired but the last two are disagreeing

12) $$\begin{array}{lll}
+h_\lambda^{\cdot\mu}\partial_\mu u^{\kappa_1\kappa_2} & +h_\lambda^{\cdot\mu}\Gamma^{\kappa_1}_{\mu\varrho}u^{\varrho\kappa_2} & +h_\lambda^{\cdot\mu}\Gamma^{\kappa_2}_{\mu\varrho}u^{\kappa_1\varrho} \\
-u^{\mu\kappa_2}\partial_\mu h_\lambda^{\cdot\kappa_1} & -h_\lambda^{\cdot\varrho}\Gamma^{\kappa_1}_{\mu\varrho}u^{\mu\kappa_2} & +h_\sigma^{\cdot\kappa_1}\Gamma^{\sigma}_{\mu\lambda}u^{\mu\kappa_2} \\
-u^{\kappa_1\mu}\partial_\mu h_\lambda^{\cdot\kappa_2} & -h_\lambda^{\cdot\varrho}\Gamma^{\kappa_2}_{\mu\varrho}u^{\kappa_1\mu} & +h_\sigma^{\cdot\kappa_2}\Gamma^{\sigma}_{\mu\lambda}u^{\kappa_1\mu} \\
-h_\mu^{\cdot\kappa_2}\partial_\lambda u^{\kappa_1\mu} & -h_\mu^{\cdot\kappa_2}\Gamma^{\kappa_1}_{\lambda\varrho}u^{\varrho\mu} & -h_\mu^{\cdot\kappa_2}\Gamma^{\mu}_{\lambda\varrho}u^{\kappa_1\varrho} \\
-w^{\kappa_2}h_\mu^{\cdot\kappa_1}\partial_\lambda v^\mu & -h_\mu^{\cdot\kappa_1}\Gamma^{\mu}_{\lambda\varrho}u^{\varrho\kappa_2} & \\
+w^{\mu}h_\mu^{\cdot\kappa_2}\partial_\lambda v^{\kappa_1} & +h_\mu^{\cdot\kappa_2}\Gamma^{\kappa_1}_{\lambda\varrho}u^{\varrho\mu} &
\end{array}$$

In order to get rid of them we write in the second column the defect, that is the term that should be added in order to obtain the covariant derivative with respect to some arbitrary symmetric connexion $\Gamma^\kappa_{\mu\lambda}$ instead of the ordinary derivative. Every defect contains $u^{\kappa_1\kappa_2}$ and the sum of all defects must be zero. Now we try to substitute the two disagreeing terms by a non-disagreeing term with the same defect. Let us take $u^{\varrho\kappa_2}\partial_\lambda h_\varrho^{\cdot\kappa_1}$. Then the defect is

13) $$h_\mu^{\cdot\varrho}\Gamma^{\kappa_1}_{\lambda\varrho}u^{\mu\kappa_2} - h_\mu^{\cdot\kappa_1}\Gamma^{\mu}_{\lambda\varrho}u^{\varrho\kappa_2}$$

hence the substitution is succesfull if

14) $$h_\mu^{\cdot\varrho}u^{\mu\sigma} = h_\mu^{\cdot\sigma}u^{\mu\varrho}.$$

But that means that two tensors $h_\lambda^{\cdot\kappa}$ and $u^{\kappa_1\kappa_2}$ have the differential comitant

15) $$\begin{array}{l}
h_\lambda^{\cdot\mu}\partial_\mu u^{\kappa_1\kappa_2} - u^{\mu\kappa_2}\partial_\mu h_\lambda^{\cdot\kappa_1} - u^{\kappa_1\mu}\partial_\mu h_\lambda^{\cdot\kappa_2} - \\
- h_\mu^{\cdot\kappa_2}\partial_\lambda u^{\kappa_1\mu} + u^{\varrho\kappa_2}\partial_\lambda h_\varrho^{\cdot\kappa_1}
\end{array}$$

provided that the condition (14) is satisfied.

We give two other examples:

$$\begin{aligned} &h_{\mu}^{\cdot\varrho}\partial_{\varrho}\mathcal{L}_{\lambda}^{\cdot\kappa\nu} - \mathcal{L}_{\varrho}^{\cdot\kappa\nu}\partial_{[\mu}h_{\lambda]}^{\cdot\varrho} + 2\mathcal{L}_{[\mu}^{\cdot\varrho[\nu}\partial_{|\varrho|}h_{\lambda}^{\cdot\kappa} + h_{\varrho}^{\cdot[\kappa}\partial_{[\mu}\mathcal{L}_{\lambda]}^{\cdot\nu]\varrho} - \\ &- \mathcal{L}_{[\lambda}^{\cdot[\nu|\varrho|}\partial_{\mu]}h_{\varrho}^{\cdot\kappa]} \quad ; \quad \text{for } \mathcal{L}_{\lambda}^{\cdot(\kappa\lambda)} = 0 \; ; \; \mathcal{L}_{\lambda}^{\cdot\varrho(\nu}h_{\varrho}^{\cdot\kappa)} = 0 \end{aligned} \tag{16}$$

and

$$\begin{aligned} &h_{[\mu}^{\cdot\varrho}\partial_{|\varrho|}\mathcal{L}_{\lambda\beta]}^{\cdot\cdot\kappa\alpha} - 2\mathcal{L}_{\varrho[\beta}^{\cdot\kappa\alpha}\partial_{\mu}h_{\lambda]}^{\cdot\varrho} + 2\mathcal{L}_{[\mu\beta}^{\cdot\cdot\varrho[\alpha}\partial_{|\varrho|}h_{\lambda]}^{\cdot\kappa]} - \\ &- h_{\varrho}^{\cdot[\kappa}\partial_{[\mu}\mathcal{L}_{\lambda\beta]}^{\cdot\cdot|\varrho|\alpha]} + \mathcal{L}_{[\lambda\beta}^{\cdot\cdot\varrho[\alpha}\partial_{\mu]}h_{\varrho}^{\cdot\kappa]} \; ; \; \text{for } \mathcal{L}_{\lambda\beta}^{\cdot\cdot(\kappa\alpha)} = 0 \; ; \; \mathcal{L}_{\lambda\beta}^{\cdot\cdot\varrho(\alpha}h_{\varrho}^{\cdot\kappa)} = 0, \end{aligned} \tag{17}$$

that can be derived in an analogous way. In all these cases some algebraic condition arises and the result is obtained by substituting the disagreeing terms by another term. It is not yet proved that such a substitution is always possible and there is not yet a general rule to find out the term to be introduced.

So the first problem is to find a general rule for the construction of all differential comitants of two or more tensorfields or tensordensityfields. But there is a still more general problem. Given any set of geometric objects it may be asked whether these objects have differential concomitants that are themselves geometric objects (not necessary quantities) with a given manner of transformation.