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Supplement to report ZW 1959-010, Solution of the Laplace inversion problem for a special function

by

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In report ZW 1959-010 the problem was how to find a function h(t), such that the given function

$$(1)\qquad f(p) = \int_0^\infty \frac{e^{-z\sqrt{x^2+a^2p^2}}\, J_0(\rho x)x\, dx}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} \qquad (p>0),$$

is the Laplace transform

$$(2)\qquad f(p) = p\int_0^\infty e^{-pt}\, h(t)dt$$

of h(t). In this report a different method for solving this problem will be given.

We again assume that ρ, z, a, b, c, d are positive constants and that $a \neq b$. We also put $\sqrt{\rho^2+z^2}=R$. Substituting $y=p^{-1}\sqrt{x^2+a^2p^2}$, we deduce from (1)

$$(3)\qquad f(p) = p\int_a^\infty \frac{e^{-zyp} J_0(p\rho\sqrt{y^2-a^2})y\, dy}{cy + d\sqrt{y^2+b^2-a^2}}.$$

By the well-known formula

$$(4)\qquad J_0(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{ixs}}{\sqrt{1-s^2}}\, ds,$$

we then have

$$(5)\qquad f(p) = p\int_a^\infty \frac{e^{-zyp} y\, dy}{cy+d\sqrt{y^2+b^2-a^2}}\, \frac{1}{\pi}\int_{-1}^{1} \frac{e^{isp\rho\sqrt{y^2-a^2}}}{\sqrt{1-s^2}}\, ds.$$

Replacing s by the new variable t

$$(6)\qquad t = zy - i\rho s\sqrt{y^2-a^2},$$

we obtain

$$(7) \qquad f(p) = \int_a^\infty dy \int_L \varphi(y,t)dt,$$

where $\varphi(y,t)$ is defined by

$$(8) \qquad \varphi(y,t) = \frac{p}{\pi i} \; \frac{y\, e^{-pt}}{\sqrt{\rho^2(y^2-a^2)+(zy-t)^2}\,(cy+d\sqrt{y^2+b^2-a^2})} .$$

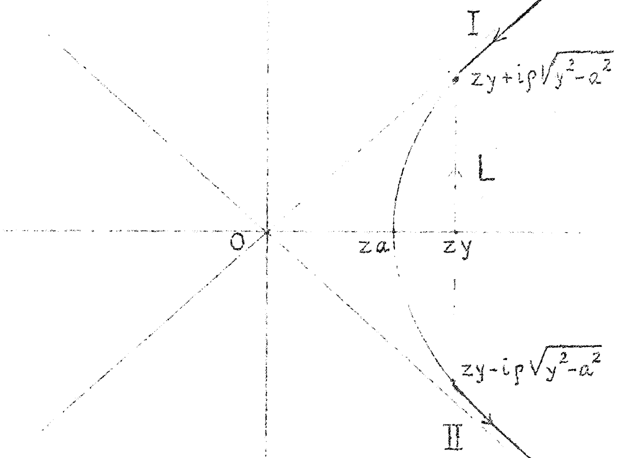

fig.1 The t-plane

The t-integration contour in (7) is a linesegment L connecting $zy-i\rho\sqrt{y^2-a^2}$ and $zy+i\rho\sqrt{y^2-a^2}$ (fig.1). If y varies from a to ∞, the points $zy-i\rho\sqrt{y^2-a^2}$ and $zy+i\rho\sqrt{y^2-a^2}$ describe a branch H of a hyperbola in the complex t-plane. If y is fixed, the function

$$\frac{e^{-pt}}{\sqrt{\rho^2(y^2-a^2)+(zy-t)^2}}$$

of t has no singularities in the region G to the right of H, and is $O(e^{-pt})$ if $t \to \infty$. We therefore have

$$(9) \qquad \int_L \varphi(y,t)dt = \int_I \varphi(y,t)dt + \int_{II} \varphi(y,t)dt,$$

where the sign of $\sqrt{\rho^2(y^2-a^2)+(zy-t)^2}$ has to be chosen in such a way that the square root is asymptotically equal to t if $t \to \infty$, $t \in G$. The contours I and II are parts of H as is shown in fig.1, and have parametric representations

$$(10) \qquad \begin{aligned} I &: \quad t=t_1(u) = zu+i\rho\sqrt{u^2-a^2}, \quad u \geq a; \\ II &: \quad t=t_2(v) = zv-i\rho\sqrt{v^2-a^2}, \quad v \geq a. \end{aligned}$$

From (7), (9) and (10) we deduce

$$f(p) = \int_a^\infty dy \int_I \varphi(y,t)dt + \int_a^\infty dy \int_{II} \varphi(y,t)dt =$$

$$(11) \qquad \int_a^\infty dy \int_\infty^y \varphi(y,t_1(u))t_1'(u)du + \int_a^\infty dy \int_y^\infty \varphi(u,t_2(v))t_2'(v)dv =$$

$$= -\int_a^\infty t_1'(u)du \int_a^u \varphi(y,t_1(u))dy + \int_a^\infty t_2'(v)dv \int_a^v \varphi(y,t_2(v))dy.$$

The integrations can be interchanged, as is justified in the following way. If $t_1 \in I$ we have

$$(12)\quad \left|\rho^2(y^2-a^2)+(zy-t_1)^2\right| = \left|t_1-(zy+i\rho\sqrt{y^2-a^2})\right|\left|t-(zy-i\rho\sqrt{y^2-a^2})\right| \geq$$
$$\left|zu-zy\right|\left|2\rho\sqrt{y^2-a^2}\right| .$$

We also have

$$(13)\quad \left|t_1'(u)\right| = \left|z+\frac{i\rho u}{\sqrt{u^2-a^2}}\right| \leq z + \frac{\rho y}{\sqrt{y^2-a^2}} .$$

From (12) and (13) it follows that

$$\psi(y) = \int_y^{\infty} \left|\varphi(y,t_1(u))\right| \left|t_1'(u)\right| du \leq \left|\frac{p}{\pi}\,\frac{y}{cy+d\sqrt{y^2+b^2-a^2}}\right| \frac{1}{\sqrt{2\rho z\sqrt{y^2-a^2}}} \cdot$$
$$\cdot\left(z+\frac{\rho y}{\sqrt{y^2-a^2}}\right)\int_y^{\infty}\frac{e^{-pzu}}{\sqrt{u-y}}\,du.$$

So there is a constant C (independent of y) with

$$\psi(y) \leq C\,\frac{y^2\,e^{-pzy}}{(cy+d\sqrt{y^2+b^2-a^2})(y^2-a^2)^{3/4}} ,$$

As $cy+d\sqrt{y^2+b^2-a^2} \geq db > 0$ if $y \geq a$, and since $a \neq 0$, $p > 0$, $z > 0$, we have

$$\int_a^{\infty}\psi(y)dy < \infty .$$

The integral over II can be handled with in the same way.

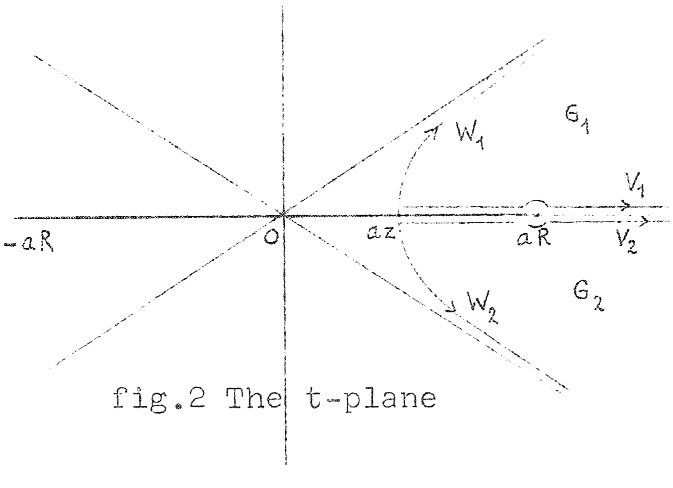

fig.2 The t-plane

If u ranges from a to ∞, $t_1(u)$ describes a contour W_1, which is the part of H above the real axis (fig.2). If $t \in W_1$, the corresponding value of u will be given by

$$(14) \qquad u(t) = \frac{tz - i\rho\sqrt{t^2-a^2R^2}}{R^2} .$$

From now on we cut the t-plane along the real axis from -aR to aR, taking $\sqrt{t^2-a^2R^2}$ positive if $t > aR$. Similarly, if v ranges from a to ∞, $t_2(v)$ describes a contour W_2, the part of H under the real axis, and now

$$(15) \qquad v(t) = \frac{tz + i\rho\sqrt{t^2-a^2R^2}}{R^2} \qquad (t \in W_2).$$

Hence (11) can be written

$$(16) \qquad f(p) = -\int_{W_1} dt \int_a^{u(t)} \varphi(y,t)dy + \int_{W_2} dt \int_a^{v(t)} \varphi(y,t)dy.$$

From now on y will also assume complex values. Let G_1 be the region bounded by W_1 and the part of the positive real axis from az to ∞. Let

$$(17) \qquad g(t) = \int_{W(t)} \varphi(y,t)dy,$$

first be defined as follows for $t \in G_1$.
G_1 is conformally mapped onto a region G_1' of the y-plane by $y=u(t)$ ((14)). G_1' is also bounded by the positive real axis, and a hyperbolic arc, which is the image of the part of the real axis $t > aR$ (fig.3).

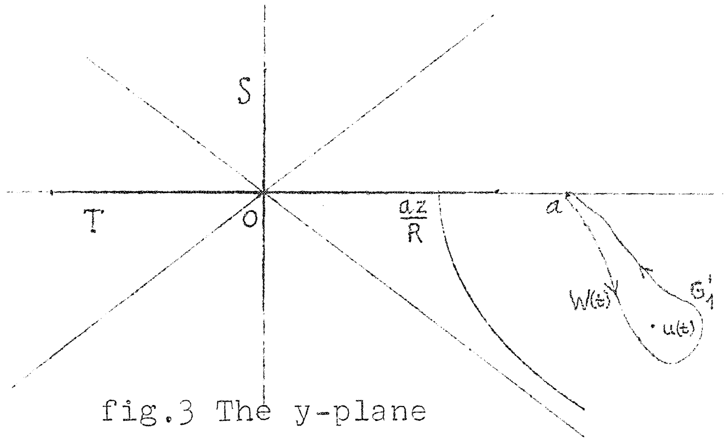

fig.3 The y-plane

$\sqrt{y^2+b^2-a^2}$ is defined in the following way.
I. If $a < b$, we cut the y-plane along the interval
$S : [-i\sqrt{b^2-a^2}, i\sqrt{b^2-a^2}]$
on the imaginary **axis**.
II. If $a > b$, the real axis is cut along the interval
$T : [-\sqrt{a^2-b^2}, \sqrt{a^2-b^2}]$.

In both cases the square root is positive for large positive values of y. W(t) is a simple curve in the y-plane. Starting in a, W(t) encircles u(t) in positive direction, ending in a again without leaving G_1'. Evidently, if t is fixed in G_1, only the root u(t) of $\rho^2(y^2-a^2)+(zy-t)^2=0$ is in G_1'.

On W(t) we define the function $\sqrt{\rho^2(y^2-a^2)+(zy-t)^2}$ by analytic continuation, taking the value t-za at the starting-point y=a of W(t). If W(t) satisfies the above conditions, the integral on the right of (17) is independent of W(t), and g(t) is uniquely defined on G_1. One can easily prove that g(t) is analytic on G_1. In fact, g(t) can be analytically continued to the boundary of G_1, the point t=Ra being excluded. If t is fixed and $t\neq Ra$, the conformal mapping y=u(t) can be extended across the cut (-aR,aR), and the roots of $\rho^2(y^2-a^2)+(zy-t)^2=0$ are separated. If u(t) is on the boundary of G_1', we can take a contour W(t), which leaves G_1' only in a small neighbourhood of u(t), but for the rest satisfies the above conditions. In case II it may occur that $u(t)\in T$; $\sqrt{y^2+b^2-a^2}$ then has to be continued analytically along W(t) across the cut T.

Finally we need an estimate of $|g(t)|$ if $t\in G_1$ and $t\to\infty$. It is not difficult to see that there exists a constant $k>0$ so that

$$(18)\qquad \left|\frac{y}{cy+d\sqrt{y^2+b^2-a^2}}\right|\leqslant k \qquad\qquad (y\in G_1').$$

We can deform W(t) into the line-segment

$$(19)\qquad y = a + (u(t)-a)s \qquad\qquad (0\leqslant s\leqslant 1).$$

Then, (17), (18) and (19),

$$(20)\qquad |g(t)|\leqslant \frac{2pk}{\pi}e^{-pRet}\int_0^1\frac{|u(t)-a|ds}{\sqrt{|a-u(t)||1-s||a(1-s)+u(t)(1+s)-\frac{2tz}{R^2}|}}\leqslant$$

$$\leqslant\frac{2pl}{\pi}e^{-pRet}\sqrt{\frac{2|t|}{R}+a},$$

if $|t|$ is sufficiently large (l is independent of t).

If u(t) is on the real axis and > a

$$g(t) = 2\int_a^{u(t)}\varphi(y,t)dy,$$

which integral occurs in (16). This can be proved by deforming W(t) into the interval [a,u(t)] . We therefore have

$$(21) \qquad \int_{W_1} dt \int_a^{u(t)} \varphi(y,t)dt = \tfrac{1}{2} \int_{W_1} g(t)dt.$$

Now, by (20) and since $g(t)$ is analytic in G_1 we can replace W_1 on the right of (21) by the contour V_1 of fig.2.
Hence

$$(22) \qquad \int_{W_1} dt \int_a^{u(t)} \varphi(y,t)dt = \int_{V_1} dt \int_a^{u(t)} \varphi(y,t)dt.$$

The second term on the right of (16) can be transformed in exactly the same way. Hence

$$(23) \quad f(p) = -\tfrac{1}{2} \int_{V_1} dt \int_a^{u(t)} \varphi(y,t)dy + \tfrac{1}{2} \int_{V_2} dt \int_a^{v(t)} \varphi(y,t)dy.$$

Adding the contours in the y-plane, we either obtain

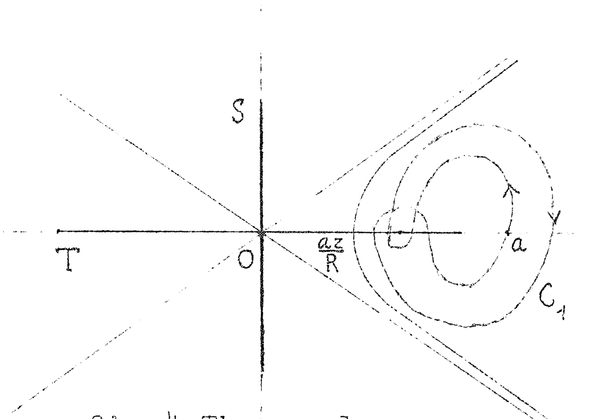

fig.4 The y-plane

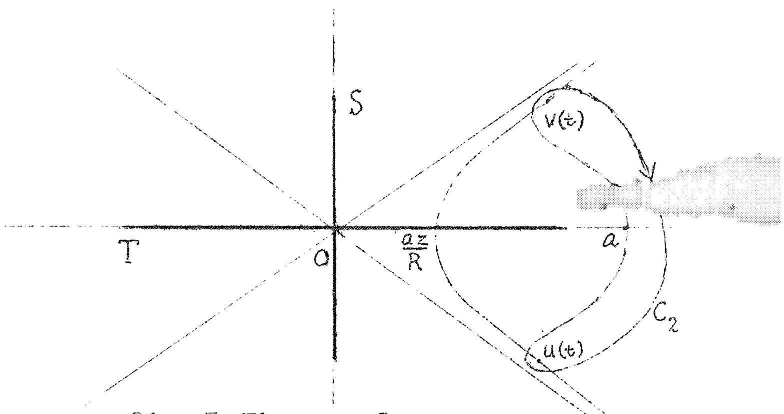

fig.5 The y-plane

a contour C_1 (fig.4) if $az < t < aR$, or a contour C_2 (fig.5) if $t > aR$. From (23) it is clear, that the function

$$(24) \qquad h(t) = \frac{1}{2\pi i} \int_{C_i} \frac{y\,dy}{(cy+d\sqrt{y^2+b^2-a^2})\sqrt{\rho^2(y^2-a^2)+(zy-t)^2}},$$

where $i=1$ if $az < t < aR$, $i=2$ if $t > aR$, satisfies (2). (Both roots in the denominator of the integral are > 0 if $y=a$). A further discussion of the function $h(t)$ can be found in § 5 of report ZW 1959-010, and need not be repeated here.