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Nowhere differentiable continuous functions

with an extended list of references

by

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§0. Introduction

Nowhere differentiable continuous functions have been discovered, independently, by WEIERSTRASS [8, 96], CELERIER [15], and BOLZANO [42, 69]. This discovery in the nineteenth century created a new subject in mathematics, which occupies mathematicians up to now.
In the Mathematical Reviews of the last years one easily finds in each volume a couple of references to papers treating this subject.
In general these papers contain a new elementary examply of a non-differentiable function or a more elegant proof that an earlier constructed function is non-differentiable.

I became interested in this problem while trying to find an elementary proof for the non-differentiability of the function $f(x) = \sum_{n=0}^{\infty} 2^{-n}\sin(2^n\pi x)$.
This function is of the same type as CELERIER's function
$f(x) = \sum_{n=0}^{\infty} a^{-n} \sin(a^n \pi x)$. In his original example CELERIER takes for a an even integer > 1000 and his proof fails for a = 2.
The non differentiability of this function can be deducted from a theorem of HARDY [36] but this proof cannot be considered to be elementary.

After finding the proof which is described in §1, I looked for similar methods in the litterature. None of the papers I was able to study gives the non-differentiability of this special function f(x) as an easy corollary with exception of HARDY's above mentioned paper.
However, none of the principles on which the argument is based, can be considered to be new or unknown. The result thus demonstrates that by simply considering "more" terms in a development classical methods can lead to better results.

In §2 I present a list of the references I was able to study. I also mention references I found in other publications, the original papers of which were not available in Amsterdam. This list of references is preceded by a survey of the history of the problem and a description of the methods used in solving it. Information of this kind is rarely given but can be found in the papers 36, 40, 46 and 56.

§1. An elementary proof for the non-differentiability of $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \sin(2^n \pi x)$

Lemma 1: Let $f(x)$ be a real-valued function on the real line $\mathbb{R}$ and let $x_0 \in \mathbb{R}$. Then f has a finite derivative $f'(x_0)$ in $x_0 \in \mathbb{R}$ iff the following condition holds:

for any $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$x_0 - \delta \leq a_1, a_2 \leq x_0 \leq b_1, b_2 \leq x_0 + \delta, \ a_1 \neq b_1, \ a_2 \neq b_2$$

implies

$$\left| \frac{f(b_1) - f(a_1)}{b_1 - a_1} - \frac{f(b_2) - f(a_2)}{b_2 - a_2} \right| < \varepsilon$$

Proof: Suppose first that the condition is fulfilled. It then follows that the difference-quotient

$$\frac{f(x) - f(x_0)}{x - x_0}$$

satisfies a Cauchy condition for $x \to x_0$, hence it has a finite limit for $x \to x_0$, and f is differentiable in x_0 with a finite derivative.

Reversily if f has a finite derivative $f'(x_0)$ we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0) \quad \text{for } x \to x_0$$

This implies:

$$\lim_{a,b \to x_0} \frac{f(b) - f(a)}{b - a} = f'(x_0)$$

provided that x_0 is in between a and b! The proof of this fact is completely elementary

The condition given by Lemma 1 is used by many authors in this field of mathematics. It is generally supposed to be not commonly known. It appears however already in 1882 in the works of T.J. STIELTJES [87].

Lemma 2: Let $g(x)$ be the function

$$g(x) = \sin 2\pi x + \sin \pi x + (2-\sqrt{2}) \sin(\pi x/2)$$

and let x_0 be some real number.
Then there exists a real number y such that

$$y \leq x \leq y + 1 \quad \text{and} \quad |g(y)| > \frac{\pi^2}{16} + 0{,}05$$

Proof: The function $g(x)$ is tabulated in Table 1 for $x = 0$ (1/8) up to 4. $g(x)$ is periodic modulo 4 and it is therefore sufficient to find points $y_0,\ldots,y_N \in [0,4]$ so that

$$y_i - y_{i-1} < 1 \qquad \text{for } i = 1,\ldots,N$$

and

$$y_0 + 4 - y_N < 1$$

where we have

$$|g(y_i)| > 0{,}7 > \pi^2/16 + 0{,}05$$

We now take $\{y_0,\ldots,y_N\} = \{1/8, 4/8, 9/8, 13/8, 19/8, 23/8, 28/8\}$

y	g(y)	y	g(y)	y	g(y)	y	g(y)
0	+ 0,0000	8/8	+ 0,5858	16/8	+ 0,0000	24/8	- 0,5858
1/8	+ 1,2041	9/8	+ 0,8990	17/8	+ 0,9755	15/8	- 0,2501
2/8	+ 1,9313	10/8	+ 0,8341	18/8	+ 1,4829	26/8	- 0,2483
3/8	+ 1,9564	11/8	+ 0,2703	19/8	+ 1,3055	27/8	- 0,7038
4/8	+ 1,4142	12/8	- 0,5858	20/8	+ 0,5858	28/8	- 1,4142
5/8	+ 0,7038	13/8	- 1,3055	21/8	- 0,2703	29/8	- 1,9564
6/8	+ 0,2483	14/8	- 1,4829	22/8	- 0,8341	30/8	- 1,9313
7/8	+ 0,2501	15/8	- 0,9755	23/8	- 0,8990	31/8	- 1,2041
8/8	+ 0,5858	16/8	+ 0,0000	24/8	- 0,5858	32/8	+ 0,0000

Table I. Values of $g(x)$ in 4 decimals.

Lemma 3. Let f be a real valued function and put

$$A(x,h) = \frac{1}{h}\,(f(x) + f(x+h) - 2f(x+h/2))$$

If a positive number $M \geq 0$ exists so that for each $x_0 \in \mathbb{R}$ and $\delta > 0$ there exists a number h with $0 < h < \delta$, and a point $x \in \mathbb{R}$ so that $x \leq x_0 \leq x + h$ and

$$|A(x,h)| \geq M$$

then f has nowhere a finite derivative.

Proof: We have

$$A(x,h) = \frac{f(x+h) - f(x)}{f} - \frac{f(x+h/2) - f(x)}{h/2} =$$

$$= \frac{f(x+h) - f(x+h/2)}{h/2} - \frac{f(x+h) - f(x)}{h} .$$

It follows that $A(x,h)$ is the difference between the increment-ratios of f on the intervals $[x,x+h]$ and $[x,x+h/2]$ resp. $[x+h/2,x+h]$ and $[x,x+h]$. For each point x_0 between x and x+h there are two intervals $[a_1\; b_1]$ and $[a_2\; b_2]$ so that

$$x \leq a_1,\; a_2 \leq x_0 \leq b_1,\; b_2 \leq x + h\;,\; a_1 \neq b_1,\;\; a_2 \neq b_2$$

and

$$\left| \frac{f(b_1) - f(a_1)}{b_1 - a_1} - \frac{f(b_2) - f(a_2)}{b_2 - a_2} \right| = |A(x,h)| .$$

Now let $x_0 \in \mathbb{R}$ and $\delta > 0$. Let M, h and x be chosen as above. Then we have

$$x_0 - \delta < x \leq x_0 \leq x + h < x_0 + \delta \quad \text{and}$$

$$|A(x,h)| > M .$$

It follows that there exists a_1, a_2, b_1 and b_2

$$x_0 - \delta \leq a_1,\; a_2 \leq x_0 \leq b_1,\; b_2 \leq x_0 + \delta\;,\; a_1 \neq b_1,\; a_2 \neq b_2$$

and

$$\left| \frac{f(b_1) - f(a_1)}{b_1 - a_1} - \frac{f(b_2) - f(a_2)}{b_2 - a_2} \right| = |A(x,h)| > M$$

By Lemma 1 this is impossible if f has a finite derivative in x_0. It follows that f has nowhere a finite derivative.

Theorem: The function $f(x)= \sum_{n=0}^{\infty} \frac{1}{2^n} \sin(2^n \pi x)$ has nowhere a finite derivative.

Proof: Let $A(x,h) = \frac{1}{h} (f(x) + f(x+h) - 2f(x+h/2))$

Then $A(x,h) = \sum_{n=0}^{\infty} A_n(x,h)$ where

$$A_n(x,h) = \frac{1}{2^n} \cdot \frac{1}{h} (\sin(2^n \pi x)+\sin(2^n\pi(x+h))-2\sin(2^n\pi(x+h/2)))$$

$$= \frac{1}{2^n} \cdot \frac{1}{h} \cdot 2(\sin(2^n\pi(x+h/2)).\left[\cos(2^n\pi h/2)-1\right])$$

For $h = 2^{-k}$ this gives

$$A_n(x,2^{-k}) = -2^{-(n-k-1)}.\sin(2^n\pi(x+2^{-k-1})).(1-\cos(2^{n-k-1}\pi)) .$$

It follows that $A_n(x,2^{-k}) = 0$ for $k \geq n+2$.

For $n \leq k - 2$ we have

$$|A_n(x,2^{-k})| \leq 2^{-(n-k-1)} |\sin(2^n\pi(x+2^{-k-1}))||1-\cos(2^{n-k-1})| \leq$$

$$\leq 2^{-(n-k-1)} . 1 . \frac{1}{2}(2^{n-k-1}\pi)^2 =$$

$$= \pi^2 . 2^{n-k-2} .$$

Therefore

$$\left|\sum_{n=0}^{k-2} A_n(x,2^{-k})\right| \leq \sum_{n=0}^{k-2} |A_n(x,2^{-k})| \leq \sum_{h=0}^{k-2} \pi^2 . 2^{n-k-2} =$$

$$= \frac{\pi^2}{8} \sum_{n=1}^{k-1} 2^{-n} < \frac{\pi^2}{8} .$$

For the remaining terms we have:

$$A_{k+1}(x,2^{-k}) + A_k(x,2^{-k}) + A_{k-1}(x,2^{-k}) =$$

$$= -(2\sin(2^{k+1}\pi(x+2^{-k-1})) + 2\sin(2^k\pi(x+2^{-k-1})) +$$

$$+ 2(2-\sqrt{2})\sin(2^{k-1}\pi(x+2^{-k-1}))) =$$

$$= -2\ g(2^k(x+2^{-k-1})) = -2\ g(2^k x + 1/2)$$

$g(y)$ is defined as in Lemma 2.

Putting $R(x,k) = \sum_{n=0}^{k-2} A_n(x,2^{-k})$ we conclude

$$A(x,2^{-k}) = -2.g(2^k x + 1/2) + R(x,k)$$

$$\text{with } |R(x,k)| < \frac{\pi^2}{8}.$$

Now let $x_0 \in \mathbb{R}$ and $\delta > 0$. Take $M = 0{,}1$. Let $k \in \mathbb{N}$ be chosen so that $2^{-k} < \delta$. By Lemma 2 there exists a point $y \in \mathbb{R}$ so that

$$y \leq 2^k\ x_0 + 1/2 \leq y + 1 \quad \text{and} \quad |g(y)| > \frac{\pi^2}{16} + 0{,}05$$

Putting $x = 2^{-k}(y-1/2)$ we have

$$x \leq x_0 \leq x + 2^{-k} \quad \text{and also}$$

$$|A(x,2^{-k})| \geq |-2\ g(y)| - |R(x,k)| >$$

$$> 2(\pi^2/16 + 0{,}05) - \pi^2/8 = 0{,}1$$

We conclude that the condition in Lemma 2 is fulfilled, hence f has nowhere a finite derivative.

The above proof is restricted to finite derivatives. It is known that $f(x)$ has an infinite derivative at a dense point set in R.

§2. The history of the non-differentiable continuous function

Before the first examples of non-differentiable continuous functions were given, differentiability was believed to be a consequence of continuity, although a number of isolated singularities were possible. There are authors known to have tried to prove rigidly the truth of this conviction. For example AMPÈRE [2] has given a proof for the differentiability of a continuous function using the (also erroneous) "fact" that for such a function the domain can be decomposed into a collection of intervals, on which the function is monotone.
WEIERSTRASS' function is the first example that has been published. P. du BOIS-REYMOND [8] describes in 1875 this function which Weierstrass has send to him in a letter. The following comment on this "horrible" example illustrates the mathematical feeling of the time (see [8]):
"Noch manches Rätsel scheint mir die Metaphysik der Weierstrassen Funktionen zu bergen, und ich kann mich des Gedankens nicht erwehren dass hier tieferes Eindringen schliesslich vor eine Grenze unseres Intellects führen wird, ähnlich der in der Mechaniek durch die Begriffe Kraft und Materie Gezogenen. Diese Funktionen scheinen mir um es kurz zu sagen, räumliche Trennungen zu setzen nicht wie die Rationalzahlen in unbegrenzt Kleinen, sondern in unendlich Kleinen. Doch es ist hier nicht der Ort, auf so kontroverse Fragen näher einzugehen".
CELERIER's example was first published after his death in 1890 [15]. He may have discovered it already in 1830 but the exact date is unknown. BOLZANO describes in an unpublished paper in 1834 an example of a continuous function where in between any two points where the function has no derivative, there is another such point. The function he describes has however no finite derivative anywhere. A description of the original manuscript is given in [42]. For a proof of the non-differentiability see [69].

RIEMANN is known to have claimed that the function $f(x) = \sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^2}$ is a non-differentiable function but no proof of his is known. (see the remark in [8]). This example is much more difficult then the others.

HARDY proves that f has no finite derivative for all irrational values of x [36]. His proof is based on earlier results of Littlewood and himself established "by reasoning of a highly trancendental character". To my knowledge up to now no elementary proof for Riemann's function exists.

In 1900 E.H. MOORE [58] proved that the coordinate-functions of the space-filling curves constructed by PEANO [65] and HILBERT [38] are also examples of nowhere differentiable functions. Research in this field has been carried out also by A.N. SING [76,80].

The presented examples of non-differentiable continuous functions can be devided in different types.

<u>Type I</u>: These are functions of the type

$$f(x) = \sum_{n=0}^{\infty} a_n \, \phi(b_n x)$$

where $\{a_n\}$ is a sequence so that $\sum_{n=0}^{\infty} |a_n| < \infty$, $\{b_n\}$ is a rapidly increasing sequence of integers and ϕ is a periodical function. For ϕ is mostly chosen one of the following functions:

$\sin x$, $\cos x$, $|\sin x|$, $\Delta(x)$ (= the distance of x to the nearest integer), or $2\Delta(x/2 + 1/4) - 1/2$ (which is a "piecewise linear sine-function").

This class contains as a special case the functions $f(x) = \sum_{n=0}^{\infty} a^n \, \phi(b^n x)$ with integral $b > 1$ and $ab \geq 1$. WEIERSTRASS' and CELERIER's example are of this type.

Type II: These functions are constructed geometrically as the limit function f of a uniform convergent sequence of piecewise linear functions f_k. Sometimes f_{k+1} is derived from f_k by applying a fixed transformation on the maximal linear subsegments of (the graph of) f_k. Such a linear subsegment of f_k is transformed into the union of a finite number of subsegments of f_{k+1} having "large" differences in their increment-ratio's. In general the functions f_k are defined in such a way that $f_j(x) = f_k(x)$ for $j > k$ if x is one of the "corners" of f_k. Thus we have for all "corners" x of f_k equality $f(x) = f_k(x)$. BOLZANO's function is a typical example of this type. The theory of these constructions has been developed for example by K. KNOPP [47].

Type III: These functions are defined by considering the integers x_i in the expansion of x in some base $b \geq 2$: $x = \sum_{i \in N}^{-\infty} x_i b^i$, and by defining f(x) in terms of the integers x_i.

A well-known example of a function of this type is the function $f(x) = \sum_{n=1}^{\infty} 2^{-n} \Delta(2^n x)$. This function is in fact a representative for each of the three types. Its definition as a type III function is the following:

Let $x = n + \sum_{n=1}^{\infty} 2^{-n_i} \qquad n \in \mathbb{N},\ n_i \in \mathbb{N},\ n_i > n_{i-1}$

then $f(x) = \sum_i (n_i - 2(i-1)).2^{-n_i}$.

There are proofs for the existence of non-differentiable continuous functions by topological argument. These proofs are applications of the Baire category theorem on the space of all continuous function on a closed interval. This approach into functional analysis has been developped after 1920 (BANACH).

Another development has been the construction of even "worse" examples of functions. Weierstrass' function has nowhere an infinite derivative but it has a left or a right derivative $\pm \infty$ at a dense subset. It was an open problem up to 1925 whether there existed a continuous function having nowhere a left or a right derivative. The first example of a function of this class has been given by BESICOVITCH [5]. E.D. PEPPER [66] provided a simplified description of Besicovitch function while

A.P. MORSE [59] presented an analytical definition.
All examples of this class are much more complicated then the "simple" non-differentiable functions. It has been proved that this class of Besicovitch functions is much coarser then the class of non-differentiable functions. The Besicovitch functions form a first category set while the class of all non-differentiable functions is a residual set of a nowhere dense set, and therefore a second category set. See for example S. SAKS [72].

The behaviour of the four Dini-derivatives* of a continuous function has equally become a field of research (DENJOY, GARG). Also the point-sets $f^{-1}(a) \subset \mathbb{R}$ have been studied for non-differentiable f.

The methods used to prove non-differentiability for some function f depend on the type of the example f.

If f is a first type function $f = \sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} a_n \phi(b_n x)$ one usually calculates the difference quotient $\frac{1}{h}(f(x+h) - f(x))$ for suitable values of h-mostly $\pm 1/b_k$ and $\pm 1/2b_k$; one then has an expansion

$$\frac{1}{h}(f(x+h) - f(x)) = \sum_{n=0}^{\infty} \frac{1}{h}(f_n(x+h) - f_n(x)).$$

For a suitably chosen $h = h_{1,k}$ and $h = h_{2,k}$ the terms with $n \geq k+1$ vanish in this expansion while the sum $\sum_{n=0}^{k-1} \frac{1}{h}(f_n(x+h) - f_n(x))$ is uniformly bounded by some constant M. Further we have for the k-th term, for some $\delta > 0$

$$\left|\frac{1}{h_{1,k}}(f_k(x+h_{1,k}) - f_k(x)) - \frac{1}{h_{2,k}}(f_k(x+h_{2,k}) - f_k(x))\right| > M + \delta.$$

As $h_{i,k} \to 0$ it follows that $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is undefined.

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* These are the functions: $D^+(x) = \limsup_{h>0} \frac{1}{h}(f(x+h) - f(x))$, $D^-(x) = \limsup_{h<0} \frac{1}{h}(f(x+h) - f(x))$, $D_+(x) = \liminf_{n>0} \frac{1}{h}(f(x+h) - f(x))$ and $D_-(x) = \liminf_{h<0} \frac{1}{h}(f(x+h) - f(x))$, which are always defined.

For second type functions the non-differentiability is generally proved by means of the criterion given in Lemma 1. In formulating this criterion some authors forget the crucial condition $x - \delta \leq a_1$, $a_2 \leq x \leq b_1$, $b_2 \leq x + \delta$ which is necessary to make the Lemma true (see for example [13]). The criterion together with a warning that this condition is crucial is present in the works of T.J. STIELTJES [87].

In the proof of the non-differentiability of a second type function one takes for a_1, a_2 and b_1, b_2 values of x where for some $k \in \mathbb{N}$ the function f_k has corners (and thus generally $f(a_i) = f_k(a_i)$ and $f(b_i) = f_k(b_i)$). By construction the difference of the increment ratios $\frac{f(b_i) - f(a_i)}{b_i - a_i}$ does not tend to zero and from there follows the non-differentiability.

If f is a third type function one generally takes $h = b^{-N}$. This way the expansions of x and x + h become equal, with the exception of the N-th place. Exact calculation of the difference quotient $\frac{1}{h}(f(x+h) - f(x))$ becomes possible and the result is found to be diverging.

Functions of the type $f(x) = \sum_{n=1}^{\infty} \frac{1}{a^n} \Delta(a^n x)$ are the most generally given examples. As mentioned before they belong to each of the three types. For a = 10 this function is generally attributed to VAN DER WAERDEN [95]. The elegant proof he describes in this paper is in fact the solution of HEYTING and BUZEMAN which he received upon presenting this function as a problem in the journal WISKUNDIGE OPGAVEN MET DE OPLOSSINGEN in 1930 [37]. This proof however fails for $a < 4$. The example $f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \Delta(2^n x)$ is given already in [90] and [49].
Weierstrass gives in his original example $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x)$ the condition $ab > 1 + \frac{3}{2}\pi$. Many authors have tried to exchange this artificial condition by the more natural one $ab \geq 1$. HARDY [36] succeeded in proving that the condition $ab \geq 1$, $a,b \in \mathbb{R}$ was sufficient to exclude a finite derivative in any point for both the functions $\sum_{n=0}^{\infty} a^n \sin(b^n x)$ and $\sum_{n=0}^{\infty} a^n \cos(b^n x)$. An infinite derivative becomes impossible if $ab > 1$.

This result is more general then any of the results which have been proved by elementary methods. All elementary proofs restrict themselves to integral b.

In the list of references the papers marked T are unknown to me, as they were not available in Amsterdam. As far as these papers are mentioned in "Mathematical Review", volume and page are indicated.
The papers marked F contain functional-analytical treatment while the papers treating more generalised derivatives are marked D. Papers marked L contain an interesting list of references or other background information. A Roman figure in between parentheses indicates the type of a given example.

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