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CARDINAL FUNCTIONS ON TOPOLOGICAL GROUPS

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Cardinal Functions on Topological Groups

by

J. de Vries

In these notes we make some remarks about cardinal functions on topological groups. In Section 2 we introduce a new cardinal function on the class of all topological spaces, which function coincides with the Lindelöf degree on all paracompact spaces. In Section 3 some applications are made to topological groups, in particular to locally compact Hausdorff groups. In the appendix we describe the weight, the local weight and the density of the space $C(X)$ of all bounded real-valued continuous functions on a locally compact Hausdorff space X in terms of the weight and the Lindelöf degree of X; here $C(X)$ is endowed with the compact-open topology. Section 1 is devoted to the definition of the several cardinal functions which we shall consider. In addition, some trivial remarks are made there about the behaviour of these functions on topological groups.

1. Conventions and preliminaries

In these notes a cardinal function is a function which assigns to every topological space a cardinal number in such a way that equal cardinal numbers are assigned to homeomorphic spaces.
Since our motivation lies in applications to topological groups, and many of the theorems about cardinal functions require the T_0- separation axiom for the topological spaces involved, we shall restrict our attention from the outset to Hausdorff spaces (observe that any topological group which satisfies the T_0 - separation axiom is a Hausdorff space). In these notes every topological space is supposed to be a Hausdorff space.
The interior (closure) of a subset A of a topological space X is denoted by int (A) ($\bar{A}$, respectively). The symbol "$\subset$" is used in the strict sense. The formula "$A \subset B$ or $A = B$" is abbreviated "$A \subseteq B$".
The cardinality of a set A is denoted by $|A|$; in particular, $\aleph_0 := |\mathbb{N}|$.
As usual, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of the real numbers, the integers

and the non - negative integers, respectively.

Now we shall list some cardinal functions. Let X be a topological space with topology $\mathcal{T}$. Define

the <u>weight</u> of X by

$$w(X) := \min \{|\mathcal{B}| \mid \mathcal{B} \text{ is an open base for } X\};$$

the <u>local weight</u> of X by

$$\chi(X) := \sup_{x \in X} [\min\{|\mathcal{V}| \mid \mathcal{V} \text{ is a local base at } x\}];$$

the <u>density</u> of X by

$$d(X) := \min \{|A| \mid A \subseteq X \ \& \ \bar{A} = X\};$$

the <u>cellularity</u> (or <u>Souslin</u>) <u>number</u> of X by

$$c(X) := \sup \{|G| \mid G \subseteq \mathcal{T} \ \& \ G \text{ disjoint}\};$$

the <u>Lindelöf degree</u> of X by

$$L(X) := \min \{\kappa \mid \text{each open covering of } X \text{ has a subcovering of cardinality } \kappa\}.$$

For a systematical treatment of these cardinal functions we refer to [6]. In that monograph also other cardinal functions are considered and the reader may find there many references to the literature.

The following relations are well-known:

(1.1) $c(X) \leq d(X) \leq w(X) \leq 2^{|X|}$ *);

(1.2) $L(X) \leq w(X)$;

(1.3) $d(X)\ \chi(X) \leq w(X)$.

(inequalities (1.1) and (1.3) are quite trivial, and (1.2) follows

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*) It is known that in locally compact Hausdorff spaces X even $w(X) \leq |X|$; cf. [3], Theorem 3.6.9. See also [6], 2.2.

from [3], Theorem 1.1.6). Our next observation concerns the Lindelöf degree in locally compact spaces:

Let X be a locally compact space. For any infinite cardinal number κ, the following conditions are equivalent:

(i) $L(X) \leq \kappa$;

(ii) X has a covering of cardinality κ, consisting of relatively compact, open sets;

(iii) X has a covering of cardinality κ, consisting of compact sets.

For any completely regular space X we may define the <u>uniform weight</u> of X by:

$$u(X) := \min \{ |\mathcal{U}| \mid \mathcal{U} \text{ is a base for a uniformity, compatible with } \mathcal{T} \}.$$

Obviously, we have

(1.4) $\chi(X) \leq u(X)$,

(1.5) $w(X) = d(X).u(X)$,

and in [6], p. 36 a proof may be found of the fact that

(1.6) $w(X) = L(X).\ u(X)$.

A <u>topological group</u> is a group G endowed with a Hausdorff topology such that the function

$$p: (s, t) \longmapsto st^{-1} : G \times G \to G$$

is continuous. It is well-known that a topological group G is completely regular, and that a locally compact topological group is paracompact, hence normal (cf. [5], 8.4 and 8.13).

The <u>left uniformity</u> for a topological group G is the uniformity which has as a base the family of sets

$$\{(s,t) \mid (s,t) \in G \times G \ \& \ s^{-1}t \in U\},$$

where U runs through a base for the neighbourhood system of the identity e of G. It follows immediately that $u(G) \leq \chi(G)$ (indeed, the left uniformity is compatible with the topology of G), so that by (1.4):

(1.7) $u(G) = \chi(G)$.

Hence in any topological group G we have

(1.8) $w(G) = d(G).\ \chi(G)$,

(1.9) $w(G) = L(G).\ \chi(G)$.

Consequently, for any topological group G,

$$\chi(G) < L(G) \Longrightarrow L(G) = w(G) = d(G).$$

However, in any <u>locally compact</u> group G the inequality

(1.10) $L(G) \leq d(G)$

holds, because G may be covered by d(G) translates of a compact symmetrical neighbourhood of the identity of G.

Consequently, for a locally compact group G there are four possibilities:

A. $\chi(G) < d(G) = L(G) = w(G)$,

B. $\chi(G) = d(G) = L(G) = w(G)$,

$$C \begin{cases} 1. & L(G) < d(G) = \chi(G) = w(G), \\ 2. & L(G) < d(G) < \chi(G) = w(G). \end{cases}$$

Notice that in all finite and all infinite discrete groups we have A. In non-metrizable, σ-compact groups, C.1 or C.2 holds.
We shall give now some non-discrete, non-compact examples of all possibilities.

EXAMPLES.

1°. $G = \mathbb{R} \times \mathbb{R}_d$, where $\mathbb{R}_d$ denotes the additive group of the reals, endowed with the discrete topology. Then

$$\chi(G) = \chi(\mathbb{R}) = \aleph_0 < 2^{\aleph_0} = |\mathbb{R}_d| \leq L(G).$$

In this example $\mathbb{R}_d$ may be replaced by any discrete group of cardinality greater than $\aleph_0$.

2°. $G = \mathbb{R}$. Now we have $\chi(G) = L(G) = \aleph_0$.

3°. Let κ be any infinite cardinal number, and let G_0 be a product of κ copies of the compact circle group $\mathbb{T}(:= \{ \lambda \mid \lambda$ a complex number & $|\lambda| = 1 \}$, this subset of the space of complex numbers being endowed with the usual topology and complex mulitiplication).
Since G_0 is compact, $L(G_0) \leqq \aleph_0$. In addition,

$$\chi(G_0) = \kappa \text{ and } d(G_0) = \log \kappa,$$

where $\log \kappa := \min \{\beta \mid 2^\beta \geq \kappa\}$ (cf.[6], 4.3 and 4.5).

If κ is a strong limit, i.e.

$$\forall\beta : \beta < \kappa \Longrightarrow 2^{\beta} < \kappa,$$

then it is clear that $\log\kappa = \kappa$; if, in addition, $\kappa > \aleph_0$, then we have a group G_0 satisfying C.1.

If $\kappa = 2^{\beta}$ for some $\beta \geq 2^{\aleph_0}$, then

$$\aleph_0 < \log\kappa \leq \beta < \kappa,$$

so that G_0 satisfies C.2.

Notice that these examples are compact; for non-compact examples, simply replace G_0 by $G := G_0 \times \mathbb{Z}$. Then all examples remain locally compact and non-discrete.

2. The function $X \longmapsto \eta(X)$

In this section we introduce a new cardinal function. Its most interesting property seems to be that it equals the Lindelöf degree on every paracompact space; in that case we have, in fact, to do with some new properties of the function $X \longmapsto L(X)$ on the class of all paracompact spaces.

DEFINITION. Let X be a topological space with topology $\mathcal{T}$, and let

$$\eta(X) := \sup\{|\mathcal{W}| \mid \mathcal{W} \subseteq \mathcal{T}\setminus\{\phi\},\ \mathcal{W} \text{ is locally finite and } \mathcal{W} \text{ is disjoint}\}.$$

A locally finite family of non-empty open subsets of X which are pairwise disjoint is called a defining family for η on X.

REMARK. For every topological space X the cardinal number $\eta(X)$ is well-defined, since each defining family for η on X has cardinality less than or equal to c(X). In particular,

(2.1) $\eta(X) \leq c(X) \leq d(X) \leq w(X)$.

Our first intention is to give another characterization of $\eta(X)$ for spaces X such that $\eta(X) \geq \aleph_0$. Before doing this, we shall prove that $\eta(X) < \aleph_0$ if and only if X is a finite, discrete space.

PROPOSITION 2.1. *For every discrete space* X *we have*

$$\eta(X) = L(X) = |X|.$$

In addition, for every topological space X *the inequality*

(2.2) $\eta(X) \leq L(X)$

holds.

PROOF. The first statement is trivial. To prove inequality (2.2), consider a defining family $\mathcal{W}$ for η on X. If $\bigcup\mathcal{W} = X$, we have an open covering of X which has no proper subcovering, so that $L(X) \geq |\mathcal{W}|$. In the other case, fix for every $W \in \mathcal{W}$ an element $t_W \in W$. Since X is a Hausdorff space and $\mathcal{W}$ is locally finite, $T := \{t_W | W \in \mathcal{W}\}$ is a closed subset of X. Now $\mathcal{W} \cup \{X \setminus T\}$ is an open covering of X which has no proper subcovering, so that $L(X) \geq |\mathcal{W}| + 1$. We have proved, that every defining family $\mathcal{W}$ for η on X satisfies $|\mathcal{W}| \leq L(X)$, so that, indeed, $\eta(X) \leq L(X)$.

PROPOSITION 2.2. *The following conditions are equivalent in any topological space* X:

(i) $|X| < \aleph_0$ (*in which case* X *is discrete*),

(ii) $L(X) < \aleph_0$,

(iii) $\eta(X) < \aleph_0$.

In that case, $\eta(X) = L(X) = |X|$.

PROOF. Only (iii) $\Longrightarrow$ (i) needs a proof.
Suppose $\eta(X)$ is finite; then there is a defining family $\mathcal{W}$ for η on X with exactly $\eta(X)$ members. Now $\mathcal{W}$ is a (finite) family of one-point sets, otherwise some $W \in \mathcal{W}$ would contain two disjoint open sets W_1 and W_2, and $[\mathcal{W}\setminus\{W\}] \cup \{W_1, W_2\}$ would be a defining family for η on X with $\eta(X) + 1 > \eta(X)$ members. In addition, $\bigcup\mathcal{W} = X$, otherwise the open set $X\setminus\bigcup\mathcal{W}$ could be joined to $\mathcal{W}$ in order to get a contradiction with the definition of $\eta(X)$. Consequently, X is discrete and finite.

COROLLARY 2.1. *Let X be a non-finite space which is either separable or a Lindelöf space.*
Then $\eta(X) = \aleph_0$.

PROOF. Use Proposition 2.2 and the inequalities (2.1) and (2.2).

LEMMA 2.1. *Let X be a topological space and let* $\mathcal{W}$ *be a locally finite family of non-empty open sets in X. Suppose* $|\mathcal{W}| \geq \aleph_0$. Then there is a defining family $\mathcal{W}_0$ *for* η *on* X *such that* $|\mathcal{W}_0| = |\mathcal{W}|$.
In addition, $\mathcal{W}_0$ *may assumed to be a refinement of* $\mathcal{W}$.

PROOF. Let Φ be the set of all defining families $\mathcal{V}$ for η on X which satisfy the following conditions:

(i) $\mathcal{V}$ is a refinement of $\mathcal{W}$, i.e. each $V \in \mathcal{V}$ is contained in some $W \in \mathcal{W}$,

(ii) each $V \in \mathcal{V}$ meets only finitely many members of $\mathcal{W}$,

(iii) each $W \in \mathcal{W}$ contains at most one $V \in \mathcal{V}$.

To show that $\Phi \neq \phi$, select for some $W \in \mathcal{W}$ an element $t \in W$ and a neighbourhood V of t which meets only finitely many members of $\mathcal{W}$. Then $\{V \cap W\} \in \Phi$.
Define a partial ordering of Φ by

$$\mathcal{V}_1 \leq \mathcal{V}_2 \Longleftrightarrow \mathcal{V}_1 \subseteq \mathcal{V}_2.$$

Since any disjoint family $\mathcal{V}$ of non-empty open subsets of X which satisfies conditions (i), (ii) and (iii) above is locally finite, hence a member of Φ, Φ is inductively ordered by this partial ordering. Consequently, by Zorn's Lemma, there is a maximal element $\mathcal{W}_0 \in \Phi$. We claim, that $|\mathcal{W}| \leq \aleph_0 . |\mathcal{W}_0|$.
Suppose the contrary, i.e. $\aleph_0 . |\mathcal{W}_0| < |\mathcal{W}|$. Since $\mathcal{W}_0$ satisfies condition (ii) above, the collection

$$\mathcal{W}_1 := \{W \mid W \in \mathcal{W} \ \& \ \exists \ V \in \mathcal{W}_0 : V \cap W \neq \phi\}$$

has cardinality

$$|\mathcal{W}_1| \leq \aleph_0 . |\mathcal{W}_0| < |\mathcal{W}|,$$

so that $\mathcal{W} \backslash \mathcal{W}_1 \neq \phi$; let $W_0 \in \mathcal{W} \backslash \mathcal{W}_1$. Now it is clear that $\mathcal{W}_0 \cup \{W_0\} \in \Phi$, which contradicts the maximality of $\mathcal{W}_0$. Hence $\aleph_0 . |\mathcal{W}_0| \geq |\mathcal{W}|$.
A similar argument shows that $\mathcal{W}_0$ cannot be finite (because $\mathcal{W}$ is infinite), so that $\aleph_0 . |\mathcal{W}_0| = |\mathcal{W}_0|$. It follows, that $|\mathcal{W}_0| \geq |\mathcal{W}|$.
However, $|\mathcal{W}_0| \leq |\mathcal{W}|$ by condition (iii), hence $|\mathcal{W}_0| = |\mathcal{W}|$.

PROPOSITION 2.3. *Let X be an infinite topological space. Then, if $\mathcal{T}$ denotes the topology of X,*

(2.3) $\quad \eta(X) = \sup \{ |\mathcal{W}| \mid \mathcal{W} \subseteq \mathcal{T} \backslash \{\phi\} \ \& \ \mathcal{W} \text{ is locally finite}\}.$

PROOF. Denote the expression in the right hand side of (2.3) by η^*. It is trivial that $\eta(X) \leq \eta^*$.
Since X is infinite, both $\eta(X)$ and η^* are infinite.
Consequently, it is sufficient to prove that any locally finite family $\mathcal{W}$ of non-empty open sets gives rise to such a family $\mathcal{W}_0$ which is, in addition, disjoint and which satisfies the inequality $|\mathcal{W}| \leq \aleph_0 \, |\mathcal{W}_0|$. Indeed, then we have $|\mathcal{W}| \leq \aleph_0 \cdot \eta(X) = \eta(X)$, hence $\eta^* \leq \eta(X)$. Now, if $\mathcal{W}$ is finite, the existence of such a family $\mathcal{W}_0$ is trivial, and if $\mathcal{W}$ is infinite, there is such a $\mathcal{W}_0$ by the preceding Lemma.

REMARK. For finite spaces X with at least two points we have

$$\eta^* = 2^{|X|} - 1 \neq |X| = \eta(X).$$

Consequently, for such spaces Proposition 2.3 is false.

THEOREM 2.1. *For every paracompact space X the equality* $\eta(X) = L(X)$ *holds.*

PROOF. For finite spaces the equality is trivial (cf. Proposition 2.2), so we may assume that X is infinite.
Let A be any open covering of X. Because X is paracompact there is a locally finite open covering A_0 of X such that A_0 is a refinement of A. It follows from Proposition 2.3 that $|A_0| \leq \eta(X)$. Now it is easy to see that A has a subcovering of cardinality less than or equal to $|A_0| \leq \eta(X)$.
This proves that $L(X) \leq \eta(X)$. Since the inequality $\eta(X) \leq L(X)$ is generally true (cf. Proposition 2.1), the theorem is proved.

COROLLARY 2.2. *A paracompact space X is a Lindelöf space if (and only*

<u>if</u>) <u>each locally finite family of non-empty open sets is at most countable</u>.

<u>COROLLARY</u> 2.3 <u>In any metrizable space</u> X <u>we have</u>

$$\eta(X) = L(X) = c(X) = d(X) = w(X).$$

<u>PROOF.</u> For finite X everything is clear. For infinite X, the first equality follows from the fact that X is paracompact; the other equalities may be derived from the formulas (1.1), (1.5) and (1.6), using the fact that $u(X) = \aleph_0$.

<u>EXAMPLES.</u>

1^0. Let X be a separable normal space which is not paracompact (cf. [3], Exercixe 5.1.F). Since X cannot be finite, it follows from Corollary 2.1 that $\eta(X) = \aleph_0$. Since X is not a Lindelöf space (otherwise X would be paracompact), we have $L(X) > \eta(X) = \aleph_0$.

2^0. Let X be the set of all ordinal numbers less than the first uncountable ordinal, endowed with the topology which has all order-intervals $(\alpha, \beta]$ as a subbase.
Then X is pseudocompact, i.e. every continuous real valued function on X is bounded (cf. [3], Example 3.5.1: every real valued continuous function on X is eventually constant). By a well-known characterization of pseudocompactness, every defining family for η on X is finite)[1], so that $\eta(X) \leq \aleph_0$.
On the other hand, $L(X) = |X| > \aleph_0$.

)[1] See, for example, part of the proof of Proposition 2.10 below.

3^0. Let $X = A \cup B$ with $A \cap B = \phi$, A and B open in X, A an infinite space which is not paracompact (e.g. the space of the preceding example) and B a discrete space with $|B| = L(A)$. Since $L(X) = L(A) + L(B) = L(A)$, and $\eta(X) \geq |B| = L(A)$, it follows from Proposition 2.1 that $\eta(X) = L(X)$. However, X is not paracompact, because its closed subspace A is not paracompact. This example shows that the converse of Theorem 2.1 is not generally true, not even if X is assumed to be locally compact (this in contradistinction to locally compact topological groups, which are always paracompact).

Cf. also Corollary 2.5 below.

We may consider Theorem 2.1 as a statement about the Lindelöf degree on the class of all paracompact spaces:

Whenever X is a paracompact space,

$$L(X) = \sup \{ |\mathcal{W}| \mid \mathcal{W} \text{ is a locally finite, disjoint family of non-empty open sets} \}.$$

On the other hand, Theorem 2.1 may be regarded as a characterization of $\eta(X)$ for paracompact X. Now paracompactness is important in analysis because of its relation with the concept of a partition of unity.

Recall that a family $\{f_s \mid s \in S\}$ of continuous functions on an arbitrary space X with values in the segment $[0,1]$ is called a <u>partition of unity</u> provided that $\sum_{s \in S} f_s(x) = 1$

for every $x \in X$.

From the proof of Lemma 2 to Theorem 5.1.3 in[3] it follows immediately that for any partition of unity $\{f_s \mid s \in S\}$ on an arbitrary infinite space X the inequality $|S| \leq \eta(X)$ holds (use Proposition 2.3). Conversely, let X be a completely regular space and $\mathcal{W}$ a defining family for η on X. For every $W \in \mathcal{W}$ there is a continuous function $f_W : X \longrightarrow [0,1]$ such that $f_W(x) = 1$ for some $x \in W$ and $f_W(y) = 0$ for every $y \in X\backslash W$. Because $\mathcal{W}$ is locally finite, the function g: $X \longrightarrow [0,1]$ defined by

$$g(x) := \sum \{f_W(x) \mid W \in \mathcal{W}\} \qquad (x \in X)$$

is continuous. Now $\{f_W \mid W \in \mathcal{W}\} \cup \{1-g\}$ is a partition of unity of cardinality $\geq |\mathcal{W}|$.
This proves:

PROPOSITION 2.4. *Let X be an infinite completely regular space. Then*

$$(2.4) \qquad \eta(X) = \sup \{\kappa \mid \exists \text{ partition of unity on X with } \kappa \text{ members}\}.$$

REMARK. Let X be a topological space and let f_n be the function which is defined by $f_n(x) = 2^{-n}$ $(x \in X)$. Then $\{f_n \mid n \in \mathbb{N}\}$ is a partition of unity on X. This shows that Proposition 2.4 does not hold for finite spaces X: the supremum in the right hand member of (2.4) equals $\aleph_0 \cdot \eta(X)$.

Recall that a closed subset A of a topological space is said to be regularly closed whenever $A = \overline{\text{int}(A)}$. It is easy to see that A is regularly closed if and only if $A = \overline{U}$ for some open set U.

PROPOSITION 2.5. *Let F be a regularly closed subset of a topological space X. Then we have* $\eta(F) \leq \eta(X)$.

PROOF. Let $\{W_s \mid s \in S\}$ be a defining family for η on F.

For each $s \in S$, select $V_s \subseteq X$ such that V_s is open in X and $W_s = V_s \cap F$. Let U be the interior of F as a subset of X, so that $F = \bar{U}$. For each $s \in S$, we have $V_s \cap \bar{U} = W_s \neq \phi$, hence $V_s \cap U \neq \phi$. Consequently, $\{V_s \cap U \mid s \in S\}$ is a disjoint family of open subsets of X; it is even a defining family for η on X, i.e. it is locally finite. Indeed, for any $x \in X \backslash F$ the open set $X \backslash F$ is a neighbourhood of x which meets no $V_s \cap U$. In addition, each $x \in F$ has a neighbourhood O in X such that $O \cap F$ meets W_s for only a finite number of elements $s \in S$, so that O meets only finitely many members of $\{V_s \cap U \mid s \in S\}$. It follows, that $|S| \leq \eta(X)$.
Because this holds for any defining family for η on F, the proof is complete.

<u>REMARK.</u> In Proposition 2.5 we cannot replace F by an arbitrary open subset or a closed subset of X.
We give two examples:

1^o. Let $X = \{(x,y) \mid x \in \mathbb{R} \;\&\; y \in \mathbb{R} \;\&\; y \geq 0\}$, endowed with the topology defined by the following neighbourhood bases of its points:

> for $y > 0$, (x,y) has a local base consisting of open discs in the plane with centre in (x,y) and radius $< y$;
>
> for $y = 0$, $(x,0)$ has a local base consisting of sets $C \cup \{(x,0)\}$, where C is an open disc in the plane with centre (x, z), $z > 0$, and radius z.

Clearly X is separable, so that $\eta(X) \leq \aleph_0$. However, $F := \{(x,0) \mid x \in \mathbb{R}\}$ is a closed subset of X, which has the discrete topology, so that $\eta(F) = |\mathbb{R}| > \aleph_0$.

2^o. Let F be a locally compact topological space and let X be the Čech - Stone compactification of F.
Then F is open in X. Now $\eta(X) \leq \aleph_0$ because X is compact, but $\eta(F)$ may be arbitrarily large (e.g. F discrete and $|F| > \aleph_0$).

Notice that the first example concerns a non-paracompact space X (indeed, $L(X) = |\mathbb{R}| > \aleph_0 = \eta(X)$), even a non-normal space. For paracompact spaces we have

PROPOSITION 2.6. Let X be a paracompact space and let A be an F_σ set in X. Then $\eta(A) \leq \eta(X)$.

PROOF. Since the assertion is trivial for finite spaces, we may assume that X is infinite. For a closed subset B of any space Y the inequality $L(B) \leq L(Y)$ holds, hence for the union A of countably many closed subsets of X we have $L(A) \leq \aleph_0 \cdot L(X) = L(X)$. But $L(X) = \eta(X)$ and $\eta(A) = L(A)$, because X and A are paracompact, so that, indeed, $\eta(A) \leq \eta(X)$.

The second example preceding Proposition 2.6 shows that in the following result the inequality may be strict:

PROPOSITION 2.7. Let U be an open subset of a topological space and assume that U is dense in X.
Then $\eta(U) \geq \eta(X)$.

PROOF. Straightforward.

The cardinal functions η and L behave similarly under continuous mappings. Let X and Y be topological spaces and let f be a continuous mapping of X onto Y. Since $f^{-1}(\mathcal{W})$ is a defining family for η on X (an open covering of X) whenever $\mathcal{W}$ is a defining family for η on Y (an open covering of Y, respectively) it is clear that

$$\eta(Y) \leq \eta(X); \; L(Y) \leq L(X) \tag{2.5}$$

It is easily shown by examples that these inequalities may be strict, independent of each other:

if X and Y are discrete and $|Y| \lneq |X|$, then both inequalities in (2.5) are strict (f any mapping of X onto Y); if X_0 is a space such that $\eta(X_0) < L(X_0)$, if X is the disjoint union of $\eta(X_0)$ copies of X_0, if Y is a discrete space of cardinality $|Y| = \eta(X_0)$, and f is the obvious continuous mapping of X onto Y, then $\eta(Y) = \eta(X)$ and $L(Y) < L(X)$; if Y is a space with $\eta(Y) < L(Y) = |Y|$ (cf. Example 2 after Theorem 2.1), X is a discrete space with $|X| = |Y|$ and f is any function of X onto Y, then $\eta(Y) < \eta(X)$ and $L(Y) = L(X)$.

Recall that a continuous function $f : X \to Y$ is said to be a *perfect mapping* if f is closed and if, for every $y \in Y$, $f^{-1}(y)$ is compact.

PROPOSITION 2.8. *Let f be a perfect mapping of the topological space X onto the space Y. Then*

(2.6) $$L(Y) \leq L(X) \leq \aleph_0 . L(Y).$$

If, in addition, either f is open or X is paracompact, then

(2.7) $$\eta(Y) \leq \eta(X) \leq \aleph_0 . \eta(Y).$$

PROOF. The proof that $L(X) \leq \aleph_0 . L(Y)$ is a straightforward modification of the proof that a space is compact if one of its perfect images is compact (cf. [3], Problem 3Y and 5C for further references).

Now suppose X is paracompact. By a well-known theorem of Michael, Y is paracompact as well, so that in this case (2.7) is a direct consequence of (2.6) and Theorem 2.1.

Finally suppose f is an open mapping. If Y is finite, then X is compact, so that (2.7) is trivial (in this case, f is open !). If Y is infinite, then (2.7) follows from Proposition 2.3, once we have shown that $f(\mathcal{W})$ is locally finite for any locally finite family $\mathcal{W}$ of subsets of X. Let $\mathcal{W}$ be such a family and let $y \in Y$.

The compact set $f^{-1}(y)$ may be covered with a finite number of open sets, each of which meets only finitely many members of $\mathcal{W}$. Let U denote the union of these open sets and let $V := Y \setminus f(X \setminus U)$. Then V is an open neighbourhood of y (f is a closed mapping) and V meets only finitely many members of $f(\mathcal{W})$ ($V \cap f(W) \neq \phi$ implies that $U \cap W \neq \phi$). This completes the proof.

About the behaviour of the functions η and L under the formation of topological products no more can be said, in general, then what follows from (2.5) : if $X = \prod_{\alpha \in A} X_\alpha$, where X_α is a topological space for each $\alpha \in A$, then

(2.8) $$\sup_{\alpha \in A} \eta(X_\alpha) \leq \eta(X); \ \sup_{\alpha \in A} L(X_\alpha) \leq L(X) .$$

Infinite products of two-point spaces show that the first inequality may be strict, and the product of the Sorgenfrey-space (i.e. $\mathbb{R}$ with the half-open interval topology) with itself shows that the second inequality may be strict. However, if X is infinite and locally compact, we have $L(X) = \sup L(X_\alpha)$ (use the result on p. 3).
Finally, we consider topological direct sums. We start with a general result:

<u>PROPOSITION</u> 2.9 <u>Let</u> X <u>be an infinite topological space and let</u> $\mathcal{F}$ <u>be a locally finite covering of</u> X <u>consisting of non-empty regularly closed sets</u>.
<u>Then</u>

(2.9) $$\eta(X) = |\mathcal{F}| . \sup_{F \in \mathcal{F}} \eta(F).$$

PROOF. It is clear that $\{\mathrm{int}\,(F) \mid F \in \mathcal{F}\}$ is a locally finite family of non-empty open sets. So by Proposition 2.3 we have $|\mathcal{F}| \leq \eta(X)$. In addition, for every $F \in \mathcal{F}$, $\eta(F) \leq \eta(X)$ by Proposition 2.5. Since $\eta(X)$ is infinite this proves that

$$\eta(X) \geq |\mathcal{F}| . \sup_{F\in\mathcal{F}} \eta(F).$$

To prove the converse inequality, consider an arbitrary defining family $\mathcal{W}$ for η on X. For each $F \in \mathcal{F}$, define

$$\mathcal{W}_F := \{W \cap \mathrm{int}(F) \mid W \in \mathcal{W} \ \&\ W \cap \mathrm{int}(F) \neq \phi\}.$$

Then $\mathcal{W}_F = \phi$ or $\mathcal{W}_F$ is a locally finite, disjoint family of non-empty open subsets of F. In each case we have $|\mathcal{W}_F| \leq \eta(F)$.

Now let $\mathcal{W}^1 := \bigcup\{\mathcal{W}_F \mid F \in \mathcal{F}\}$; then $|\mathcal{W}^1| \geq |\mathcal{W}|$.

Indeed, $\mathcal{F}$ is a covering of X, so that for each $W \in \mathcal{W}$ there is an $F \in \mathcal{F}$ such that $W \cap F \neq \phi$. However, $F = \overline{\mathrm{int}(F)}$, so that $W \cap \mathrm{int}(F) \neq \phi$. Consequently, each member of the disjoint family $\mathcal{W}$ contains at least one element of $\mathcal{W}^1$, so $|\mathcal{W}^1| \geq |\mathcal{W}|$.
Combining our results, we get

$$|\mathcal{W}| \leq |\mathcal{W}^1| \leq |\mathcal{F}| . \sup_{F\in\mathcal{F}} |\mathcal{W}_F| \leq |\mathcal{F}| . \sup_{F\in\mathcal{F}} \eta(F).$$

Since this holds for any defining family $\mathcal{W}$ for η on X, it follows that

$$\eta(X) \leq |\mathcal{F}| . \sup_{F\in\mathcal{F}} \eta(F), \tag{2.10}$$

and the proof is finished.

REMARK. In the second half of the proof we did not use the fact that

X is infinite, so that (2.10) holds for finite spaces as well. It is easy to see that (2.9) is false, in general, for finite spaces, unless $\mathcal{F}$ is disjoint.

COROLLARY 2.4. Let $X = \bigcup \{X_\alpha \mid \alpha \in A\}$, each X_α open in X and the X_α's pairwise disjoint
Then

$$\eta(X) = |A| . \sup_{\alpha \in A} \eta(X_\alpha) \tag{2.11}$$

In particular, if, for each $\alpha \in A$, $\eta(X_\alpha) \leq |A|$, then $\eta(X) = |A|$ and $\{X_\alpha \mid \alpha \in A\}$ is a defining family for η on X of cardinality $\eta(X)$.

PROOF. For finite spaces X the result is trivial, and for infinite spaces X the result follows immediately from Proposition 2.9, since each X_α is open and closed in X, hence regularly closed.

REMARK. Similar methods show that in the situation of Corollary 2.4 we have

$$L(X) = |A| . \sup_{\alpha \in A} L(X_\alpha) \tag{2.12}$$

Consequently, if for each $\alpha \in A$ we have $L(X_\alpha) \leq |A|$, then $L(X) = |A|$. In that case we have also $\eta(X_\alpha) \leq |A|$ for each $\alpha \in A$ (cf. Proposition 2.1), hence $\eta(X) = L(X) = |A|$.

COROLLARY 2.5. Let X be the disjoint union of κ mutually disjoint open sets, each of which is a Lindelöf space. If κ is an infinite cardinal number, then we have $\eta(X) = L(X) = \kappa$.

PROOF. Cf. the preceding Remark.

The next lemma is well-known (cf. [4], 15Q). We insert a proof of it, because in the sequel we need some parts of this proof.

LEMMA. 2.2. *Let X be a completely regular space. The following conditions are equivalent:*

(i) *X is pseudocompact, i.e. every real-valued continuous function on X is bounded,*

(ii) *X is totally bounded in any of its admissible uniformities.*

PROOF. (i) $\Longrightarrow$ (ii). Assume that $\mathcal{U}$ is an admissible uniformity of X such that X is not totally bounded with respect to $\mathcal{U}$. Then there are an $\alpha \in \mathcal{U}$ and a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X such that, for every $k \in \mathbb{N}$, $x_{k+1} \notin \cup\{\alpha(x_i) \mid 1 \leq i \leq k\}$. Now let $\beta \in \mathcal{U}$, $\beta^{-1} = \beta$ and $\beta^4 \subseteq \alpha$.

$\{\beta^2(x_k) \mid k \in \mathbb{N}\}$ is a disjoint family, and, consequently,

$\{\beta(x_k) \mid k \in \mathbb{N}\}$ is a locally finite, disjoint family of subsets of X, each of which has a non-empty interior (hence $\{\text{int}\,(\beta(x_k)) \mid k \in \mathbb{N}\}$ is a defining family for η on X).

By complete regularity, for every $k \in \mathbb{N}$ there is a continuous function $f_k : X \to [0,k]$ such that $f_k(x_k) = k$ and $f_k(y) = 0$ for every $y \in X\backslash\text{int}\,\beta(x_k)$. Now the function $f : x \longmapsto \sum\{f_k(x) \mid k \in \mathbb{N}\}$ is well-defined and continuous (for $\{\beta(x_k) \mid k \in \mathbb{N}\}$ is locally finite). However, f is not bounded, so that X is not pseudocompact.

(ii) $\Longrightarrow$ (i). Let $\mathcal{U}_0$ denote the weakest uniformity such that every real-valued continuous function is uniformly continuous. Because X is completely regular, $\mathcal{U}_0$ is admissible, so that X is totally bounded with respect to $\mathcal{U}_0$. Now it is easy to see that each continuous real-valued function on X is bounded.

PROPOSITION 2.10 Let X be a locally compact paracompact space. Then $\eta(X) = L(X)$. In addition, there is a defining family for η on X of cardinality $\eta(X)$ if and only if X is either finite or non-compact.

PROOF. It follows from Theorem 2.1 that $\eta(X) = L(X)$.
For the second statement we consider two cases.
First assume X is not σ-compact. Then it follows from [2], Theorem I. 9.5, that X is a disjoint union of σ-compact subspaces X_α, each of which is open in X. Since there must be uncountably many of these X_α, it follows from Corollary 2.4 that X has a defining family for η on X of cardinality $\eta(X)$.
Now assume X is σ-compact, but non-finite. Then $\eta(X) = \aleph_0$, and we have to prove that there exists a countable defining family for η on X if and only if X is non-compact. In the case that X is not pseudocompact the existence of such a family follows immediately from the preceding Lemma and its proof (the first half of (i) $\Longrightarrow$ (ii)). Consequently, the proof is finished as soon as we have proved that a pseudocompact, paracompact locally compact space X is compact. The proof is almost trivial: since X is locally compact and paracompact, X may be covered by a locally finite family of relatively compact, open sets. Since X is pseudocompact this family must be finite, otherwise we could construct a disjoint sequence of non-empty open sets which is locally finite (cf. also Lemma 2.1); then the second part of the proof of (i) $\Longrightarrow$ (ii) in the preceding Lemma would show that X were not pseudocompact. Now X is covered by a finite collection of

relatively compact sets, hence X is compact.
Conversely, if X is compact, each defining family for η on X is finite. Indeed, X may be covered by a finite number of open sets each of which meets finitely many members of such a family. Hence X cannot have a defining family for η on X of cardinality $\eta(X)$, unless X is finite.

In the preceding results we have already given a partial answer to the question what classes of topological spaces X admit a defining family for η of cardinality $\eta(X)$. It is already shown that the class of all infinite compact spaces must be excluded (Proposition 2.10). On the other hand, if $\eta(X)$ is a successor cardinal, i.e. if

$$\sup \{\lambda \mid \lambda < \eta(X)\} < \eta(X),$$

then X has a defining family for η of cardinality $\eta(X)$.
In addition to these trivial facts we have

PROPOSITION 2.11. *An infinite topological space X admits a defining family for η of cardinality $\eta(X)$ if and only if there is a locally finite covering $\mathcal{F}$ of X, consisting of non-empty regularly closed sets with pairwise disjoint interiors such that, for every $F \in \mathcal{F}$, $\eta(F) \leq |\mathcal{F}|$.*

PROOF. Assume that $\mathcal{W}$ is a defining family for η on X such that $|\mathcal{W}| = \eta(X)$. If $\cup\{\overline{W} \mid W \in \mathcal{W}\} = X$, let $\mathcal{F} := \{\overline{W} \mid W \in \mathcal{W}\}$; in the other case, let $\mathcal{F} := \{\overline{X\setminus\overline{\cup\mathcal{W}}}\} \cup \{\overline{W} \mid W \in \mathcal{W}\}$)*. In both cases $\mathcal{F}$ is a covering of X, consisting of non-empty regularly closed sets. In addition, $\{\overline{W} \mid W \in \mathcal{W}\}$ is locally finite, and, consequently, $\mathcal{F}$ is locally finite as well. That $\{\text{int}(F) \mid F \in \mathcal{F}\}$ is a disjoint family follows from the

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)* It should be observed, that $\overline{\cup\mathcal{W}} = \cup\{\overline{W} \mid W \in \mathcal{W}\}$.

fact that for open subsets U and V of X always

$$U \cap V = \phi \Longrightarrow \mathrm{int}(\overline{U}) \cap \mathrm{int}(\overline{V}) = \phi .$$

Finally, for every $F \in \mathcal{F}$, $\eta(F) \leq \eta(X) = |\mathcal{F}|$ by Proposition 2.5. The "if" part of our Proposition is an immediate consequence of Proposition 2.9.

3. <u>The Lindelöf degree on locally compact groups</u>

It is well-known that every locally compact (Hausdorff !) topological group G is a disjoint union of pairwise disjoint, open subsets, which are σ-compact. Indeed, let U be a symmetrical, compact neighbourhood of the identity of G and let $G_0 := \cup\{U^n \mid n \in \mathbb{N}\}$. Then G_0 is an open subgroup of G and G_0 is σ-compact. Now G is the union of the distinct left cosets of G_0, which are pairwise disjoint, open in G and σ-compact. These cosets are, in addition, homeomorphic with G_0. It follows, that

$$\eta(G) = [G : G_0]\cdot\eta(G_0);\ L(G) = [G : G_0].\ L(G_0)$$

(cf. (2.11) and (2.12)). Here $[G : G_0]$ is the index of G_0 in G, that is the cardinality of the set of all left cosets of G_0 in G. Consequently, if G itself is not σ-compact, we have

(3.1) $$\eta(G) = L(G) = [G : G_0],$$

and the family of left cosets of G_0 is a defining family for η on G. In the other case, we have $\eta(G) = L(G) = |G|$ whenever G is finite and $\eta(G) = L(G) = \aleph_0$ whenever G is infinite and σ-compact. In the latter case, G has a defining family for η if and only of G is non-compact. One may form such a family by translating a sufficiently small compact neighbourhood of the identity of G over a suitable sequence of points in G (cf. the corresponding part of the proof of Proposition 2.10, that is, in fact, the first part of the proof of (i) $\Longrightarrow$ (ii) in

Lemma 2.2).
These facts for a locally compact group G might also be derived as direct consequences of Proposition 2.10, because any locally compact group G is paracompact (since G is a disjoint union of open, σ-compact subsets, as we have seen above).
Resuming, we have

PROPOSITION 3.1. <u>In a non-compact, locally compact topological group G the Lindelöf degree satisfies the equality</u>

$$L(G) := \max\{ |\mathcal{W}| \mid \mathcal{W} \text{ is a disjoint, locally finite family of non-empty open sets in } G\}.$$

In this context it is tempting to mention a result due to O.T. Alas [1]. Recall that the Haar measure μ on a locally compact, non-discrete group G satisfies the conditions

(M_1) $\forall\, x \in G\colon \mu(\{x\}) = 0$

(M_2) $\forall\, A \subseteq G\colon A$ is a Borelset $\Longrightarrow \mu(A) = \inf\{\mu(U) \mid A \subseteq U \;\&\; U \text{ open}\}$

(M_3) $\forall\, A \subseteq G$, A a Borelset: A is open or A has a σ-finite measure $\Longrightarrow \mu(A) = \sup\{\mu(K) \mid K \subseteq A \;\&\; K \text{ compact}\}$.

In general (M_3) is not true for arbitrary closed subsets of G, and the usual counter example is a closed subset of G which is a local null-set but not a null set (cf. [5], 11.33). Notice that any discrete closed subset A of G is locally null: for each compact $K \subseteq G$, $K \cap A$ is finite, hence $\mu(K \cap A) = 0$ by condition (M_1) above. In particular,

$$(3.2) \qquad \sup\{\mu(K) \mid K \subseteq A \;\&\; K \text{ compact}\} = 0.$$

Now the problem is to find such a set A with $\mu(A) > 0$;
in view of condition (M_3) such a set A cannot have a (σ-)finite measure, so that $\mu(A) = \infty$. The result of O.T. Alas, referred to above, may now be formulated in the following way:

PROPOSITION 3.2. *Let G be a non-discrete locally compact group. The following conditions are equivalent:*

(i) *G contains a discrete closed subset A such that* $\mu(A) = \infty$;

(ii) $L(G) > \aleph_0$;

(iii) *G is not* σ-*compact*.

PROOF (ii) $\Longleftrightarrow$ (iii). Trivial.

(ii) $\Longrightarrow$ (i). If (ii) is satisfied, there is an uncountable defining family $\mathcal{W}$ for η on G (Proposition 3.1). For each $W \in \mathcal{W}$, let $t_W \in W$, and let $A := \{t_W \mid W \in \mathcal{W}\}$. Then A is a discrete subset of G, and A is closed because $\mathcal{W}$ is locally finite.
Suppose for some open $V \supseteq A$ we have $\mu(V) < \infty$. Because $\mathcal{W}$ is a disjoint family of μ-measurable sets, it follows that for any $k \in \mathbb{N}$ only finitely many $W \in \mathcal{W}$ satisfy the inequality $\mu(V \cap W) > \frac{1}{k}$. Consequently, $\mu(V\cap W) > 0$ for at most countably many $W \in \mathcal{W}$.
Since any open subset U of G is non-empty if and only if $\mu(U) > 0$, it follows that $V \cap W \neq \emptyset$ for at most countably many $W \in \mathcal{W}$. This contradicts the fact that $t_W \in V \cap W$ for every $W \in \mathcal{W}$ and that $|\mathcal{W}| > \aleph_0$.
Consequently, $\mu(V) = \infty$ for every open $V \supseteq A$, so that $\mu(A) = \infty$ by (M_2).

(i) $\Longrightarrow$ (iii). Suppose G is σ-compact. Then each closed subset of G is a Borel set with a σ-finite measure. In particular, for any discrete closed subset A of G the equality $\mu(A) = \sup\{\mu(K) \mid K \subseteq A \;\&\; K \text{ compact}\}$ holds, so that $\mu(A) = 0$ by (3.2).

COROLLARY. *The Haar measure* μ *in a locally compact, non-discrete group G satisfies* (M_3) *for any closed subset of G if and only if G is* σ-*compact*.

PROOF. "If". cf. the proof of (i) $\longrightarrow$ (iii) in the preceding Proposition. "Only if": follows immediately from (iii) $\Longrightarrow$ (i) of Proposition 3.2.

REMARK. This Corollary is exactly the Corollary in [1].

The Theorem in [1] states that in any regular space X such that there is a measure μ on the σ-ring of Borel sets satisfying (M1), (M2), and (M3) for every closed subset A of X, and whose support is all of X (i.e. for any open $U \subseteq X$: $U = \emptyset \Longleftrightarrow \mu(U) = 0$), the following property holds: any locally finite open covering of X has a countable subcovering.

The proof is essentially as follows (in our terminology): the proof of (ii) $\Longrightarrow$ (i) of the preceding Proposition yields that $\eta(X) \leq \aleph_0$; then the result follows from our Lemma 2.1 (The regularity of X seems to be superfluous, and we have shown, in fact, that any locally finite open covering of X is countable).

We conclude this Section with some remarks about locally compact abelian groups which have nothing to do with local finiteness. In the Appendix we shall prove that for any infinite locally compact space X the equality $\chi[C(X)] = L(X)$ is satisfied. Here $C(X)$ denotes the space of all bounded continuous real-valued functions on X endowed with the compact-open topology.

Now let G be non-discrete locally compact abelian group. By the duality theorem we may regard G as a subset of $C(G^\wedge)$, so that

$$\chi(G) \leq \chi[C(G^\wedge)] = L(G^\wedge)$$

(here we use only that G, hence $G^\wedge$, is not finite).

Conversely, the sets

$$\{x \mid x \in G^\wedge \ \& \ \forall t \in \overline{U} : \mid x(t) - 1 \mid < \tfrac{1}{2}\}$$

form a covering of $G^\wedge$ with compact sets if U runs through a neighbourhood base of the identity e of G, consisting of relatively compact neighbourhoods of e (cf. [5], 23.16, or any other proof that $G^\wedge$ is locally compact).

Since in any non-compact locally compact space X

$$L(X) = \min\{|\mathcal{K}| \mid \mathcal{K} \text{ is a covering of X by compact sets}\},$$

and $G^\wedge$ is non-compact because G is not discrete, it follows that $L(G^\wedge) \leq \chi(G)$.

Consequently, in any non-discrete, locally compact abelian group one has

(3.3) $\chi(G) = L(G^\wedge)$;

dualy one has, in any non-compact locally compact abelian group the equality

(3.4) $L(G) = \chi(G^\wedge)$.

Originally (3.3) is due to Hewitt and Stromberg, who proved that, in any locally compact abelian group G, $\chi(G)$ equals the minimal number of compact sets with which $G^\wedge$ may be covered (cf. [5], 24.48).

In Section 1 we made a classification of the class of locally compact groups on the hand of the inequalities between the several cardinal functions, in particular χ and L.
Using (3.3) and (3.4), the following statements are clear:

PROPOSITION 3.3 *Let G be a non-discrete, non-compact locally compact abelian group.*

(i) G *is of type* (A) *if and only* $G^\wedge$ *is of type* (C).

In this case, $d(G^\wedge) = d(G)$ *or* $d(G^\wedge) < d(G)$, *according to the situation that* $G^\wedge$ *is of type* (C_1) *or of type* (C_2).

(ii) G *is of type* (C) *if and only if* $G^\wedge$ *is of type* (A).

In this case, $d(G) = d(G^\wedge)$ or $d(G) < d(G^\wedge)$, *according to the situation that* G *is of type* (C_1) *or of type* (C_2).

(iii) G *is of type* (B) *if and only if* $G^\wedge$ *is of type* (B).

In all cases we have $w(G) = w(G^\wedge)$.

PROOF. We indicate only the proof of (i).
G is of type (A) if and only if

$$\chi(G) < d(G) = L(G) = w(G).$$

By (3.3) and (3.4) this is equivalent to

$$L(G^\wedge) < d(G) = \chi(G^\wedge) = w(G).$$

Consequently, $G^\wedge$ is of type (C). Hence we have

$$L(G^\wedge) < d(G^\wedge) \leq \chi(G^\wedge) = w(G^\wedge).$$

If we bear in mind that $\chi(G^\wedge) = L(G) = w(G) = d(G)$, everything is now trivial.

REMARK. For groups which are discrete or compact we have the following (G always abelian):

(a). If G is finite and discrete, $G^\wedge$ is topologically isomorphic with G, and both G and $G^\wedge$ are of type (A))[1].

(b). If G is infinite and discrete, $G^\wedge$ is infinite and compact. Now G is of type (A) and $G^\wedge$ is of type (B) or of type (C), according to the case that $G^\wedge$ is metrizable or not.

(c). If G is infinite and compact, $G^\wedge$ is infinite and discrete. Now G is of type (B) or (C) and $G^\wedge$ is of type (A).

However, in these cases we also have the equality $w(G) = w(G^\wedge)$.
In (a) this is trivial.
In (b), notice that (3.4) applies, so that

$$w(G) = L(G) = \chi(G^\wedge) = w(G^\wedge).$$

Notice that $w(G) = |G|$, so that $w(G^\wedge) = |G|$.

)[1] We do not consider the case (B), which applies only to a one-point group.

Dually, in the situation of (c) we have by (3.3)

$$w(G) = \chi(G) = L(G^\wedge) = w(G^\wedge),$$

where $w(G^\wedge) = |G|$.

So for any locally compact abelian group G we have

$$w(G) = w(G^\wedge),$$

and the proof is based on (3.3) and (3.4) and some simple cardinal arithmatic (in the proof of (3.3) and (3.4) the duality theorem for locally compact abelian groups is used). Another proof is included in [5], 24.14.

4. Appendix

The results in this Appendix are probably well-known.
However, the author was able to find in the literature only some statements about these results in compact spaces.

Whenever X is a topological space, C(X) denotes the space of all bounded continuous, real-valued functions on X, endowed with the compact-open topology. A local base at $f \in C(X)$ is formed by all sets

$$U_f(C,\varepsilon) := \{g \mid g \in C(X) \;\&\; \forall x \in C: \mid g(x) - f(x) \mid < \varepsilon\},$$

with C a compact subset of X and $\varepsilon > 0$. Observe that C(X) may be considered as a topological group (addition as a group operation), so that (1.8) and (1.9) apply with G replaced by C(X).

PROPOSITION 4.1. *Let X be an infinite locally compact topological space. Then* $\chi[C(X)] = L(X)$.

PROOF. Let $\{U_i \mid i \in I\}$ be an open covering of X such that each $U_i \neq \emptyset$,

U_i relatively compact, while $|I| \leq L(X)$.
Let $\mathcal{B}$ denote the collection of all finite unions of members of $\{U_i \mid i \in I\}$; then $|\mathcal{B}| \leq \aleph_0 . L(X) = L(X)$, and every compact subset of X is contained in some member of $\mathcal{B}$. Now the family

$$\{U_0(\overline{U}, \tfrac{1}{n}) \mid U \in \mathcal{B} \ \& \ n \in \mathbb{N}\}$$

is a base of the neighbourhood system of 0 in $C(X)$, and the cardinality of this local base is $\leq L(X)$.
Consequently, $\chi[C(X)] \leq L(X)$.
Conversely, let $\mathcal{V}$ be a local base at 0 in $C(X)$, such that $|\mathcal{V}| = \chi[C(X)]$. For each $V \in \mathcal{V}$, choose a compact subset K_V of X and an integer n_V such that $U_0(K_V, n_V^{-1}) \subseteq V$, and let $\mathcal{V}_0 := \{U_0(K_V, n_V^{-1}) \mid V \in \mathcal{V}\}$.

Now $\mathcal{V}_0$ is a local base at 0, and it is easy to see that any compact set K in X must be contained in some $K_V (V \in \mathcal{V})$ (otherwise some $U_0(K,1)$ contains no $V \in \mathcal{V}$). Hence

$$L(X) \leq |\{K_V \mid V \in \mathcal{V}\}| \leq |\mathcal{V}| = \chi[C(X)] .$$

<u>COROLLARY</u> 4.1 <u>Let</u> X <u>be a locally compact topological space. The space</u> $C(X)$ <u>is metrizable if and only if</u> X <u>is</u> σ-<u>compact</u>.

<u>PROOF.</u> The assertion is trivial for finite spaces X, so we may assume that X is infinite. $C(X)$ is metrizable if and only if $\chi[C(X)] = \aleph_0$, which is the case if and only if $L(X) = \aleph_0$; the latter condition is equivalent with the σ-compactness of X.

<u>PROPOSITION</u> 4.2 <u>Let</u> X <u>be an infinite locally compact space. Then</u>

$$d[C(X)] \leq w(X).$$

PROOF. First we consider the case that X is compact (then the result is well-known, cf. [7], Proposition 7.6.5).
Because X can be embedded in $[0,1]^{w(X)}$ ([3], Theorem 1.2.8), there is a subset $\Phi \subseteq C(X)$ which separates the points of X with $|\Phi| = w(X)$.
It follows from the Stone-Weierstrass theorem that the algebra A, generated by Φ is dense in $C(X)$. Since $|A| = \aleph_0 . |\Phi| = w(X)$, this shows that $d[C(X)] \leq w(X)$.
Conversely, any dense subset of $C(X)$ separates the points of X. Hence such a subset of $C(X)$ defines a Hausdorff topology in X which is weaker than the original compact topology of X. Consequently, any dense subset Ψ of $C(X)$ defines the topology of X; if we take such a Ψ with $|\Psi| = d[C(X)]$, it follows immediately that

$$w(X) \leq \aleph_0 . |\Psi| = d[C(X)]$$

because $\{f^{-1}(U) \mid U \in \mathcal{B}_0 \ \& \ f \in \Psi\}$ is a subbase for the topology of X ($\mathcal{B}_0$ is a countable base for the topology of $\mathbb{R}$). This proves our Proposition for compact spaces X.

In the general case, let $\mathcal{U} = \{U_i \mid i \in I\}$ be a family of open, relatively compact, non-empty sets in X such that each compact set $K \subseteq Z$ is contained in some U_i, and such that $|I| = L(X)$ (cf. the proof of Proposition 4.1). For each $i \in I$, let F_i be a set of functions which is dense in $C(\bar{U}_i)$ and such that $|F_i| = d[C(\bar{U}_i)]$.
Now $\bar{U}_i$ is compact, hence any $f \in C(\bar{U}_i)$ can be extended to an element of $C(X)$. For every $f \in F_i$, choose such an extension $\bar{f}$ of f, and let $G_i := \{\bar{f} \mid f \in F_i\}$.
Then $|G_i| = |F_i| = d[C(\bar{U}_i)]$. It is clear that the set $\bigcup\{G_i \mid i \in I\}$ is dense in $C(X)$: for any $g \in C(X)$ and any basical neighbourhood $U_g(K,\varepsilon)$ of g ($\varepsilon>0$ and $K \subseteq X$ compact) there is an $i \in I$ such that $K \subseteq \bar{U}_i$; now for some $f \in F_i$ we have $\sup\{ \, |f(x) - g(x)| \mid x \in \bar{U}_i\} < \varepsilon$, so that $\bar{f} \in U_g(\bar{U}_i,\varepsilon) \subseteq U_g(K,\varepsilon)$. Thus,

$$d[C(X)] \leq \left| \bigcup_{i\in I} G_i \right| \leq |I|.\sup_{i\in I} |G_i| = L(X).\sup_{i\in I} d[C(\bar{U}_i)] .$$

However, the result for compact spaces implies that, for every $i \in I$,

$$d[C(\bar{U}_i)] = w(\bar{U}_i) \leq w(X),$$

so that

$$d[C(X)] \leq L(X).\ w(X) = w(X).$$

REMARK. The inequality in Proposition 4.2 may be strict:
Let κ be an infinite cardinal number such that $\log \kappa < \kappa$ (e.g. $\kappa = 2^{\aleph_0}$), and let X be the discrete space with $|X| = \kappa$.
Now it is easy to see that $d[C(X)] \leq d(\mathbb{R}^X)$. Indeed, if F is dense in $\mathbb{R}^X$ and we replace each $f \in F$ by the functions f_n $(n\in\mathbb{N})$, defined by

$$f_n(x) = \begin{cases} f(x) & \text{whenever } -n \leq f(x) \leq n, \\ -n & \text{whenever } \quad f(x) < -n, \\ n & \text{whenever } \quad f(x) > n, \end{cases}$$

then we get a dense subset in $C(X)$ of cardinality $\leq \aleph_0 .\ |F|$ (a similar procedure shows, that $C(X)$ has the same density as the space of all, (possibly unbounded), continuous functions on X into $\mathbb{R}$ for any locally compact space X, if both function spaces are endowed with the compact open topology).
It follows from [6], 4.5 that $d(\mathbb{R}^X) = \log \kappa$, so that $d[C(X)] \leq \log \kappa <$ $< \kappa = w(X)$.

PROPOSITION 4.3. *Let X be any infinite locally compact space. Then*

$$w(X) = L(X).\ d[C(X)].$$

In particular, if X is σ-compact, then we have

$$w(X) = d[C(X)].$$

PROOF. We use the same notation as in Proposition 4.2.
Because $\mathcal{U}$ is an open covering of X, we have

$$w(X) \leq \sum_{i \in I} w(U_i) \leq |I| . \sup_{i \in I} w(\bar{U}_i) = L(X) . \sup_{i \in I} w(\bar{U}_i).$$

However, for each $i \in I$, we have $w(\bar{U}_i) = d[C(\bar{U}_i)]$ because $\bar{U}_i$ is compact (cf. the first part of the proof of Proposition 4.2), and $d[C(\bar{U}_i)] \leq d[C(X)]$, because $C(\bar{U}_i)$ is the continuous image of $C(X)$ under the restriction mapping $f \longmapsto f \mid \bar{U}_i$. Consequently,

$$w(X) \leq L(X) . d[C(X)] \leq L(X) . w(X) = w(X),$$

which proves our result.

COROLLARY. For any locally compact space X such that $L(X) < w(X)$, we have $d[C(X)] = w(X)$.

PROOF. If $L(X) < w(X)$, X must be infinite, hence $d[C(X)] \leq w(X)$. Suppose $d[C(X)] < w(X)$. Then it follows from Proposition 4.3 that $w(X) = L(X)$, a contradiction with $w(X) > L(X)$.

PROPOSITION 4.4. Let X be an infinite locally compact space. Then $w(X) = w[C(X)]$.

PROOF. We use formula (1.8) with $C(X)$ instead of G:

$$w[C(X)] = d[C(X)] . \chi[C(X)].$$

Here $\chi[C(X)] = L(X)$, by Proposition 4.1. Now we distinguish two possibilities:

(a) $d[C(X)] = w(X)$. Now $w[C(X)] = w(X) . L(X) = w(X)$.

(b) $d[C(X)] < w(X)$. In this case the preceding Corollary implies that

$$w(X) = L(X) = \chi[C(X)] \leq w[C(X)].$$

On the other hand,

$$w[C(X)] = d[C(X)].\ \chi[C(X)] \leq w(X).\ L(X) = w(X),$$

so that in this case we also have $w[C(X)] = w(X)$.

COROLLARY. *Let X be a σ-compact locally compact space. Then X has a countable base if and only if C(X) is a Lindelöf space.*

PROOF. We apply formula (1.9) with $C(X)$ instead of G; making use of the preceding results, we get

$$w(X) = L(X).\ L[C(X)],$$

whenever X is infinite. Since $L(X) \leq \aleph_0$, it follows that $w(X) = L[C(X)]$, so that, indeed, $w(X) \leq \aleph_0$ if and only if $L[C(X)] \leq \aleph_0$. For finite spaces X, $C(X)$ is homeomorphic with the σ-compact space $\mathbb{R}^n$ (with $n = |X|$), so that $C(X)$ is in this case a Lindelöf space as well.

REMARK. For finite, discrete spaces X we have

$$d[C(X)] = \chi[C(X)] = w[C(X)] = \aleph_0$$

because $C(X)$ is homeomorphic with $\mathbb{R}^n$ (with $n = |X|$).
So for these spaces the Proposition 4.1 through 4.4 are false.

The final remark we wish to make is, that we have not used the fact that the elements of $C(X)$ are bounded functions. Thus, if we had defined $C(X)$ as the space of all real-valued continuous functions on X, endowed with the compact-open topology, the same results would have been obtained.

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