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On generalized sums of powers of complex numbers.

by

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1. Introduction.

In his book "Eine neue Methode in der Analysis und dezen Anwendungen" [1] P. Turán proved the following Theorems:

I. (Satz VII)

Let $b_1, \dots, b_n; z_1, \dots, z_n$ be complex numbers and $m \geq -1$ an integer. Then there exists an integer ν with

$$m + 1 \leq \nu \leq m + n,$$

such that

$$(1.1). \quad |b_1 z_1^\nu + \dots + b_n z_n^\nu| \geq \left(\frac{n}{2e(m+n)} \right)^n \cdot |b_1 + \dots + b_n| \cdot \min_{j=1,\dots,n} |z_j|^\nu .$$

II. (Satz IX)

Let $z_1, \dots, z_n$ be complex numbers with $|z_1| \geq |z_2| \geq \dots \geq |z_n|$, and $b_1, \dots, b_n$ arbitrary complex numbers; $m \in \mathbb{Z}$, $m \geq -1$.

Then there exists an integer ν with

$$m + 1 \leq \nu \leq m + n,$$

such that

$$(1.2). \quad |b_1 z_1^\nu + \dots + b_n z_n^\nu| \geq \left(\frac{n}{24e^2(m+2n)} \right)^n \cdot \min_{j=1,\dots,n} |b_1 + \dots + b_j| \cdot \max_{j=1,\dots,n} |z_j|^\nu .$$

In 1958 I. Dancs [2] proved that the factor $\left(\frac{n}{2e(m+n)} \right)^n$ in (1.1) may be replaced by $\frac{1}{2e} \left(\frac{n}{2e(m+n)} \right)^{n-1}$. In 1959 E. Makai [3] and in 1960 N.G. de Bruijn [4] found independently the best possible value for this constant, viz.

$$\left\{ \sum_{j=0}^{n-1} 2^j \binom{m+j}{j} \right\}^{-1} .$$

For applications, this best possible value is not very handy and in practice, Turán's or Dancs' result will do very well.

After a first improvement of the factor $\left(\frac{n}{24e^2(m+2n)}\right)^n$ in (1.2) by V.T. Sós and P. Turán [5], in 1964 I. Dancs [6] proved the following generalization of theorem II:

(IIa) Let $z_1, \ldots, z_n$ be complex numbers with $|z_i| \leq 1$ $(i=1,2, \ldots, n)$ and $0=|1-z_1| \leq |1-z_2| \leq \ldots \leq |1-z_n|$. Let m be a non-negative integer; let B_j be polynomials with complex coefficients and of degree k_j with $k_j \leq m+2$ $(j=1, \ldots, n)$ and let $k=k_1+ \ldots + k_n+n$. Then there exists an integer ν with

$$m + 1 \leq \nu \leq m + k,$$

such that

$$|B_1(\nu)z_1^\nu + \ldots + B_n(\nu)z_n^\nu| \geq \frac{1}{2k} \left(\frac{k-1}{8e(m+k)}\right)^{k-1} \min_{j=1,\ldots,n} |B_1(0)+ \ldots +B_j(0)|.$$

In this report the following generalization of I and refinement of (IIa) will be proved:

<u>Theorem 1</u> Let $z_1, \ldots, z_n$ be complex numbers $\neq 0$ and m an integer, $m \geq -1$. Let B_j be polynomials with complex coefficients and of degree $k_j (j=1, \ldots,n)$. Let $k=k_1+ \ldots +k_n+n$. Then there exists an integer ν with

$$m + 1 \leq \nu \leq m + k,$$

such that

$$|B_1(\nu)z_1^\nu + \ldots + B_n(\nu)z_n^\nu| \geq \left(\frac{k-1}{2e(m+k)}\right)^{k-1} \cdot |B_1(0) + \ldots + B_n(0)| \cdot \min_{j=1,\ldots,n} |z_j|^\nu.$$

<u>Theorem 2</u> Let $z_1, \ldots, z_n$ be complex numbers $\neq 0$ with $|z_i| \leq 1$ $(i=1,\ldots,n)$ and $0=|1-z_1| \leq |1-z_2| \leq \ldots \leq |1-z_n|$. Let m, B_j and k be as in theorem 1. Then there exists an integer ν with

$$m + 1 \leq \nu \leq m + k,$$

such that

$$|B_1(\nu)z_1^\nu + \ldots +B_n(\nu)z_n^\nu| \geq \frac{1}{4} \left(\frac{k-1}{8e(m+k)}\right)^{k-1} \min_{j=1,\ldots,n} |B_1(0)+ \ldots + B_j(0)|.$$

The proof of these theorems is based on a combination of ideas of I. Dancs and N.G. de Bruijn, see [6] resp. [4].

2. Proof of theorem 1.

The following well-known lemma will be essential in the proof of both theorems 1 and 2.

Lemma 2.1 (Schwarz-Stieltjes).

Let $g \in R[t]$ and $h \neq 0$.

Define $\Delta g(t) = g(t+h) - g(t)$ and $\Delta^{\mu} g(t) = \Delta(\Delta^{\mu-1} g(t))$; $\mu = 2,3, \ldots$.

Then if $h > 0$ there exists θ: $a < \theta < a+h\mu$ or if $h < 0$ there exists θ: $a+\mu h < \theta < a$

such that

$$(\Delta^{\mu} g(t))_{t=a} = h^{\mu} \cdot g^{(\mu)}(\theta) \;, \quad \mu=1,2,3, \ldots$$

Proof of theorem 1.

We may assume without loss of generality that $\min\limits_{j=1,\ldots,n} |z_j| = 1$.

Consider the points $z_1, z_1(1+\varepsilon), \ldots, z_1(1+k_1\varepsilon), z_2, \ldots, z_n, z_n(1+\varepsilon), \ldots, z_n(1+k_n\varepsilon)$, where $\varepsilon > 0$ and $z_1, \ldots, z_n$ are ordered in such a way that $1 = |z_1| \leq |z_2| \leq \ldots \leq |z_n|$. They form a set of points $\{\xi_j\}_{j=1}^{k}$.

Order this set such that $1 = |\xi_1| \leq |\xi_2| \leq \ldots \leq |\xi_k|$.

Define $\quad u_j = -\frac{1}{\xi_j} \quad , \quad j=1, \ldots, k,$

and

$$(2.1) \qquad f(z) = \prod_{j=1}^{k} (1+zu_j) = \prod_{j=1}^{k} \left(1-\frac{z}{\xi_j}\right).$$

The function f has no zeros in the domain $|z| < 1$, so we can write

$$\frac{1}{f(z)} = \sum_{i=0}^{\infty} a_i^{(1)} z^i \quad , \; |z| < 1.$$

We put

$$(2.2) \qquad h_m(z) = (-1)^{m-1} \left\{1-f(z) \sum_{i=0}^{m} a_i^{(1)} z^i\right\}.$$

It follows from (2.1) and from the definition of the coefficients $a_\mu^{(1)}$ that we can write (2.2) as

$$(2.3)\qquad h_m(z) = \sum_{\nu=m+1}^{m+k} a_\nu^{(2)} z^\nu.$$

Further we have $h_m(\xi_j) = (-1)^{m-1}$ for $j=1, \ldots, k$ and hence

$$(2.4)\qquad h_m(z_j(1+\mu_j\varepsilon)) = (-1)^{m-1}, \qquad j=1, \ldots, n; \mu_j=0,1, \ldots, k_j.$$

If we apply lemma 2.1 to $h_m(z_j t)$ with $h=\varepsilon, a=1$, then we obtain: for all j,μ_j $(\mu_j \neq 0)$ there exists $\theta_{j\mu_j}$ with $1 < \theta_{j\mu_j} < 1+\mu_j\varepsilon$ such that

$$z_j^{\mu_j}\varepsilon^{\mu_j} h_m^{(\mu_j)}(\theta_{j\mu_j} z_j) = (\Delta^{\mu_j} h_m(tz_j))_{t=1} =$$

$$= h_m(z_j(1+\mu_j\varepsilon)) - \binom{\mu_j}{1} h_m(z_j(1+(\mu_j-1)\varepsilon)) + \ldots + (-1)^{\mu_j} h_m(z_j),$$

$$\text{for } j=1, \ldots, n; \mu_j=1, \ldots, k_j.$$

Substitution of (2.4) gives:

$$(2.5)\qquad h_m^{(\mu_j)}(\theta_{j\mu_j} z_j) = 0, \qquad j=1, \ldots, n; \mu_j=1, \ldots, k_j.$$

In the sequel we follow the usual convention that

$$\binom{\nu}{\mu_j} = 0 \quad \text{if } \nu < \mu_j.$$

From (2.3) and (2.5) it follows that

$$\sum_{\nu=m+1}^{m+k} a_\nu^{(2)} \binom{\nu}{\mu_j} z_j^\nu \theta_{j\mu_j}^\nu = 0, \qquad j=1, \ldots, n; \mu_j=1, \ldots, k_j.$$

If we define

$$\delta_{j\mu_j}(\varepsilon) = \sum_{\nu=m+1}^{m+k} a_\nu^{(2)} \binom{\nu}{\mu_j} z_j^\nu \{\theta_{j\mu_j}^\nu - 1\},$$

then

$$(2.6) \qquad \sum_{\nu=m+1}^{m+k} a_\nu^{(2)} \binom{\nu}{\mu_j} z_j^\nu + \delta_{j\mu_j}(\varepsilon) = 0, \qquad j=1, \ldots, n; \mu_j=1, \ldots, k_j.$$

For $\delta_{j\mu_j}(\varepsilon)$ we have the estimation

$$|\delta_{j\mu_j}(\varepsilon)| \leq 2^{m+k} |z_n|^{m+k} \{(1+k\varepsilon)^{m+k}-1\} \sum_{\nu=m+1}^{m+k} |a_\nu^{(2)}|.$$

We use the abbreviation

$$\delta(\varepsilon) = 2^{m+k} |z_n|^{m+k} \{(1+k\varepsilon)^{m+k}-1\}.$$

Hence

$$\delta(\varepsilon) \longrightarrow 0 \text{ if } \varepsilon \longrightarrow 0.$$

Write B_j in the form

$$(2.7) \qquad B_j(t) = \sum_{\mu_j=0}^{k_j} b_{j\mu_j} \binom{t}{\mu_j}, \qquad j=1, \ldots, n.$$

Multiplying (2.6) by $b_{j\mu_j}$ and summing over $\mu_j=1, \ldots, k_j$ we obtain

$$\sum_{\mu_j=1}^{k_j} \sum_{\nu=m+1}^{m+k} a_\nu^{(2)} \binom{\nu}{\mu_j} z_j^\nu b_{j\mu_j} + \sum_{\mu_j=1}^{k_j} \delta_{j\mu_j}(\varepsilon) b_{j\mu_j} = 0.$$

From this and (2.7) it follows that

$$\sum_{\nu=m+1}^{m+k} \{B_j(\nu)-B_j(0)\} a_\nu^{(2)} z_j^\nu + \sum_{\mu_j=1}^{k_j} \delta_{j\mu_j}(\varepsilon) b_{j\mu_j} = 0, \quad j=1, \ldots, n.$$

Summing over $j=1, \ldots, n$ and substituting

$S_\nu = \sum_{j=1}^{n} B_j(\nu) z_j^\nu$ gives

$$\sum_{\nu=m+1}^{m+k} S_\nu a_\nu^{(2)} - \sum_{j=1}^{n} \sum_{\nu=m+1}^{m+k} B_j(0) a_\nu^{(2)} z_j^\nu + \sum_{j=1}^{n} \sum_{\mu_j=1}^{k_j} \delta_{j\mu_j}(\varepsilon) b_{j\mu_j} = 0.$$

Substituting (2.3) and (2.4) we obtain

$$\sum_{j=1}^{n} B_j(0).(-1)^{m-1} = \sum_{\nu=m+1}^{m+k} S_\nu\, a_\nu^{(2)} + \sum_{j=1}^{n} \sum_{\mu_j=1}^{k_j} \delta_{j\mu_j}(\varepsilon) b_{j\mu_j}.$$

Therefore we have

$$\left|\sum_{j=1}^{n} B_j(0)\right| \leq \sum_{\nu=m+1}^{m+k} |S_\nu|\,|a_\nu^{(2)}| + \sum_{j=1}^{n} \sum_{\mu_j=1}^{k_j} \delta(\varepsilon)|b_{j\mu_j}| \sum_{\nu=m+1}^{m+k} |a_\nu^{(2)}|$$

$$\leq \{\max_{m+1\leq\nu\leq m+k} |S_\nu| + \delta(\varepsilon) \sum_{j=1}^{n} \sum_{\mu_j=1}^{k_j} |b_{j\mu_j}|\} \sum_{\nu=m+1}^{m+k} |a_\nu^{(2)}|;$$

i.e.,

$$(2.8) \qquad \max_{m+1\leq\nu\leq m+k} |S_\nu| \geq \frac{1}{\sum_{\nu=m+1}^{m+k} |a_\nu^{(2)}|} \cdot \left|\sum_{j=1}^{n} B_j(0)\right| + \sigma(\varepsilon).$$

Now recall that we had defined $h_m(z) = \sum_{\nu=m+1}^{m+k} a_\nu^{(2)} z^\nu$ from $f(z) = \prod_{j=1}^{k} (1+zu_j)$. Remark that $a_\nu^{(2)} = a_\nu^{(2)}(\xi_1, \ldots, \xi_k)$.

From this definition it follows that the determination of an upper bound for $\sum_{\nu=m+1}^{m+k} |a_\nu^{(2)}|$ can be done in the same way as in [4]. (Note that N.G. de Bruijn only uses the given number of points and that these points have absolute value ≥ 1).

Hence we obtain

$$(2.9) \qquad \sum_{\nu=m+1}^{m+k} |a_\nu^{(2)}| \leq \sum_{j=0}^{k-1} 2^j \binom{m+j}{j} \leq \left(\frac{2e(m+k)}{k-1}\right)^{k-1}.$$

If we substitute (2.9) in (2.8) and let $\varepsilon \to 0$, then we obtain the statement of the theorem.

3. Preliminary lemmas.

Before giving the proog of theorem 2 we need several lemmas.

Lemma 3.1 For $f \in \mathbb{R}[x]$, f monic and of degree n, the following statement is true:

$$\max_{a \leq x \leq b} |f(x)| \geq 2\left(\frac{b-a}{4}\right)^n .$$

Proof: The statement follows immediately from [8], VI §7, Aufgabe 62.

Lemma 3.2 Let δ be a positive real number.

Let $f(z) = \prod_{j=1}^{n} (z-z_j)$ and $\phi(x) = \prod_{j=1}^{n} (x-|1-z_j|)$ with $z_1, z_2, \ldots, z_n$

given complex numbers.

Let r_0 be defined by

$$\max_{0 \leq x \leq \delta} |\phi(x)| = |\phi(r_0)| \text{ , } r_0 \text{ minimal, } r_0 \geq 0.$$

Then we have for z on the circle $|1-z| = r_0$ that

$$|f(z)| \geq 2\left(\frac{\delta}{4}\right)^n .$$

Proof: $|f(z)| \geq |\phi(|1-z|)|$. Apply lemma 3.1 to $\phi(x)$ with a=0 and b=δ.

Corollary 3.2 $r_0 \neq |1-z_j|$, j=1, ..., n.

In the following we suppose that the numbers $z_1, z_2, \ldots, z_n$ satisfy $|z_i| \leq 1$ (i=1, ...,n) and that $0 = |1-z_1| \leq |1-z_2| \leq \ldots \leq |1-z_n|$.

Furthermore, every empty product that may occur in the sequel has to be read as 1.

Lemma 3.3 Let δ be a real number, $0 < \delta < 1$. Let r_0 be defined by lemma 3.2 and let the natural number l be chosen such that

$$0=|1-z_1| \leq |1-z_2| \leq \ldots \leq |1-z_l| < r_0 < |1-z_{l+1}| \leq \ldots \leq |1-z_n|.$$

Let $0 \leq \mu \leq l$ and let $j_1, j_2, \ldots, j_\mu$ be μ distinct numbers chosen from $\{1,2, \ldots, l\}$. Then we have for $|1-z|=r_0$

$$\left| \sum_{j=l+1}^{n} (z-z_j) \prod_{i=1}^{\mu} (z-z_{j_i}) \right| \geq 2 \left(\frac{\delta}{4}\right)^n .$$

Proof: Define $P(x) = \dfrac{\phi(x)}{\prod_{j_k} (x-|1-z_{j_k}|)}$, where j_k runs through the sequence $\{1,2, \ldots, l\} \setminus \{j_i\}_{i=1}^{\mu}$. Hence $|1-z_{j_k}| < r_0 < 1$, for all j_k from this sequence.

Using lemma 3.2 we have for $|1-z| = r_0$

$$\left| \sum_{j=l+1}^{n} (z-z_j) \prod_{i=1}^{\mu} (z-z_{j_i}) \right| \geq |P(r_0)| > |\phi(r_0)| \geq 2\left(\frac{\delta}{4}\right)^n .$$

Lemma 3.4 Let the function g be holómorphie inside and on the closed Jordancurve L and let $z_1, z_2, \ldots, z_n$ be given points inside L. Then

$$g(z) = e_0 + e_1(z-z_1) + \ldots + e_{l-1}(z-z_1) \ldots (z-z_{l-1}) +$$

$$+ (z-z_1) \ldots (z-z_l) g_{l-1}(z)$$

$$\text{for } l=1,2, \ldots, n,$$

with

$$e_\mu = \frac{1}{2\pi i} \oint_L \frac{g(z)}{(z-z_1)\ldots(z-z_{\mu+1})} dz , \quad \mu = 0,1, \ldots, l-1,$$

and

$$g_{l-1}(z) = \frac{1}{2\pi i} \oint_L \frac{g(t)}{(t-z)(t-z_1)\ldots(t-z_l)} dt.$$

Let P be a polynomial of degree not exceeding $l-1$, with $P(z_i)=g(z_i)$, $i=1,2, \ldots, l$. Then

$$P(z) = e_0 + e_1(z-z_1) + \ldots + e_{l-1}(z-z_1) \ldots (z-z_{l-1}).$$

Proof: The proof is straightforward; see [9], Ch I §7.

Lemma 3.5 (A. Markov).
Let $Q \in \mathbb{R}[x]$ and of degree k. Let further

$$|Q(x)| \leq 1 \quad \text{for } 0 \leq x \leq \delta \text{ , where } \delta \text{ is a positive number.}$$

Then

$$|Q'(x)| \leq \frac{2k^2}{\delta}, \text{ for } 0 \leq x \leq \delta.$$

Proof. The statement follows immediately by applying |8|, VI, Aufgabe 83 to the polynomial P(y), defined by

$$P(y) = Q(\frac{\delta}{2}(y+1)).$$

Lemma 3.6 Let $z_1, z_2, \ldots, z_n \in \mathbb{C}$ and let δ be a real number with $0 < \delta < 1$.

Define γ by

$$\gamma(x) = \prod_{j=1}^{n} (x - |1-z_j|)^{1+k_j}$$, where the k_j's are non-negative integers.

Further, let r_0 be defined by

$$|\gamma(r_0)| = \max_{0 \leq x \leq \delta} |\gamma(x)|, \ r_0 \text{ minimal}, \ r_0 \geq 0.$$

Then the following statement is true.
If $|z_i - z_j| < \frac{\delta}{2k^2}$, then

$$|1-z_i| < r_0 \Longleftrightarrow |1-z_j| < r_0.$$

Proof. Let $k = n + k_1 + \ldots + k_n$ be the degree of γ.
Applying lemma 3.5 with $Q = \frac{\gamma}{M}$, where $M = |\gamma(r_0)|$, gives

$$(3.1) \qquad |\gamma'(x)| \leq \frac{2Mk^2}{\delta} \qquad \text{for } 0 \leq x \leq \delta.$$

From $\gamma(t) = \gamma(r_0) + \int_{r_0}^{t} \gamma'(u)du$

we obtain that if x_0 is a zero of γ with $0 \leq x_0 \leq \delta$, then

$$\gamma(r_0) = -\int_{r_0}^{x} \gamma'(u)du.$$

Hence $M = |\gamma(r_0)| \leq |x_0-r_0| \max_{0 \leq x \leq \delta} |\gamma'(u)| \leq \frac{2Mk^2}{\delta} |x_0-r_0|$.

From this it follows that $|x_0-r_0| \geq \frac{\delta}{2k^2}$ and we may conclude that γ cannot have a zero in the interval

$$(r_0-\frac{\delta}{2k^2}, r_0+\frac{\delta}{2k^2}) \cap [0,\delta].$$

Let z_i and z_j be such that

$$(3.2) \qquad |z_i-z_j| < \frac{\delta}{2k^2},$$

and suppose that $x_i = |1-z_i| < r_0$ and that $x_j = |1-z_j| \geq r_0$.

First suppose that $r_0 \leq {}^{\delta}/2$.
From corollary 2.3 it follows that $x_j \neq r_0$.

Since x_j is a zero of γ it cannot lie in $(r_0, r_0 + \frac{\delta}{2k^2})$,
hence $x_j \geq r_0 + \frac{\delta}{2k^2}$. But then we would have

$$|z_i-z_j| \geq |x_i-x_j| \geq \frac{\delta}{2k^2}, \text{ in contradiction with (3.2).}$$

Similarly we are led to a contradiction in the case that $r_0 \geq \frac{\delta}{2}$.

This proves lemma 3.6.

<u>Lemma 3.7</u> Let $\delta, z_1, \ldots, z_n, k, \gamma, r_0, M$ be defined as in lemma 3.6 and let ε be a positive real number.

Let

$$\psi(x) = \prod_{j=1}^{n} \prod_{\mu_j=0}^{k_j} (x - |1-(1-\mu_j\varepsilon)\, z_j|),$$

and let

$$\xi_{j\mu_j} = (1-\mu_j\varepsilon) z_j, \qquad j=1,2,\ldots,n;\mu_j=0,1,\ldots,k_j.$$

If $|\xi_{j\mu_j} - \xi_{j\nu_j}| < \frac{\delta}{4k^2}$ and if ε is sufficiently small then we have

$$|1 - \xi_{j\mu_j}| < r_0 \Longleftrightarrow |1-\xi_{j\nu_j}| < r_0.$$

Proof. The functions ψ and ψ' are continuous functions of ε with

$$\lim_{\varepsilon \to 0} \psi(x) = \gamma(x), \text{ uniformly in } x \text{ on } [0,\delta]$$

and

$$\lim_{\varepsilon \to 0} \psi'(x) = \gamma'(x), \text{ uniformly in } x \text{ on } [0,\delta].$$

Thus for all $\eta > 0$ there exists $\varepsilon_1 = \varepsilon_1(\delta,\eta)$ such that

$$\begin{cases} |\psi(x)-\gamma(x)| < M\eta & \text{for } \varepsilon < \varepsilon_1,\ x \in [0,\delta] \\ |\psi'(x)-\gamma'(x)| < M\eta & \text{for } \varepsilon < \varepsilon_1,\ x \in [0,\delta]. \end{cases}$$

Let x_0 be a zero of ψ with $0 \leq x_0 \leq \delta$.
On the one hand we have

(3.3) $$|\psi(r_0)| \geq M(1-\eta),$$

on the other hand

$$\psi(t) = \psi(r_0) + \int_{r_0}^{t} \psi'(u)du$$

and hence

(3.4) $$|\psi(r_0)| \leq |x_0-r_0| \cdot \max_{0 \leq u \leq \delta} |\psi'(u)| \leq$$

$$\leq \max_{0 \leq u \leq \delta} \{|\gamma'(u)|+M\eta\} \leq |x_0-r_0| \; \{\frac{2Mk^2}{\delta} + M\eta\}.$$

Choose η sufficiently small; then from (3.3) and (3.4) it follows that

$$|x_0-r_0| \geq \frac{\delta}{2k^2}(1-2\eta) \geq \frac{\delta}{4k^2}.$$

Hence for $\varepsilon < \varepsilon_1$ the function ψ has no zeros in the interval

$$(r_0-\frac{\delta}{4k^2}, r_0+\frac{\delta}{4k^2}) \cap [0,\delta].$$

Let $\varepsilon < \varepsilon_1$; suppose that $|\xi_{j\mu_j} - \xi_{j\nu_j}| < \frac{\delta}{4k^2}$, but that

$$x_{j\mu_j} = |1-\xi_{j\mu_j}| < r_0 \text{ and } \quad x_{j\nu_j} = |1-\xi_{j\nu_j}| \geq r_0.$$

Now in the same way as in the previous lemma by this supposition we are led to a contradiction.

Lemma 3.8 Let $\delta, z_1, \ldots, z_n, k, \gamma, r_0, M, \psi$ be defined as in the previous lemma. Let r be defined by

$$|\psi(r)| = \max_{0 \leq x \leq \delta} |\psi(x)|, \ r \text{ minimal}, \ r \geq 0.$$

Then there exists a real number $\varepsilon_2 > 0$ such that for all $\varepsilon < \varepsilon_2$ the function γ has no zeros between r and r_0.

Proof.

Let us recall the definitions of γ and ψ:

$$\gamma(x) = \prod_{j=1}^{n} (x-|1-z_j|)^{1+k_j}$$

$$\psi(x) = \prod_{j=1}^{n} \prod_{\mu_j=0}^{k_j} (x-|1-(1-\mu_j\varepsilon)z_j|).$$

Since the function ψ also depends on ε, we write it as $\psi_\varepsilon(x)$.
Let $M = |\gamma(r_0)|$ and $M^*(\varepsilon) = |\psi_\varepsilon(r)|$.
Since $\psi_\varepsilon(x) \longrightarrow \gamma(x)$ for $\varepsilon \longrightarrow 0$ uniformly in x for $x \in [0,\delta]$, it follows that $M^*(\varepsilon) \longrightarrow M$ for $\varepsilon \longrightarrow 0$.
From the definition of r we see that $r=r(\varepsilon)$; further it follows from the definitions of γ and ψ that

$$\psi_0(x) = \gamma(x).$$

Hence $\psi_0(r(0)) = \gamma(r(0))$.

On the other hand from $M^{*}(\varepsilon) \longrightarrow M$ for $\varepsilon \longrightarrow 0$ we see

$$|\psi_0(r(0))| = M = |\gamma(r_0)|.$$

Thus since r and r_0 both are taken minimal and positive $r(0)=r_0$.

In other words $\lim_{\varepsilon \to 0} r(\varepsilon)=r_0$.

This yields: there exists a real positive number ε_2 such that

$$x_l < r(\varepsilon) < x_{l+1} \quad \text{for } \varepsilon < \varepsilon_2,$$

from which the statement follows.

4. Proof of theorem 2.

Define $\xi_{j\mu_j} = (1-\mu_j\varepsilon)z_j, \quad j=1, \ldots, n; \mu_j=0,1, \ldots, k_j,$

$$f(z) = \prod_{j=1}^{n} \prod_{\mu_j=0}^{k_j} (z-(1-\mu_j\varepsilon)z_j),$$

$$\psi(x) = \prod_{j=1}^{n} \prod_{\mu_j=0}^{k_j} (x-|1-\xi_{j\mu_j}|),$$

$$g(z) = \prod_{j=1}^{n} (z-z_j)^{1+k_j},$$

$$\gamma(x) = \prod_{j=1}^{n} (x-|1-z_j|)^{1+k_j}.$$

Let δ be any real number, with $0 < \delta < 1$.

Define r_0 by

$$M = |\gamma(r_0)| = \max_{0 \leq x \leq \delta} |\gamma(x)|, \; r_0 \text{ minimal}, \; r_0 \geq 0.$$

Let l be the natural number defined by

$$0=|1-z_1| \leq |1-z_2| \leq \ldots \leq |1-z_l| < r_0 < |1-z_{l+1}| \leq \ldots \leq |1-z_n|.$$

Let $\varepsilon_1, \varepsilon_2$ be defined by the lemmas 3.7 and 3.8.

Now choose $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \frac{\delta}{4k^3}\}$.

Hence we can apply lemma 3.8 and we obtain

$$0=|1-\xi_{10}| \leq |1-\xi_{20}| \leq \dots \leq |1-\xi_{l0}| < \frac{r}{r_0} < |1-\xi_{l+1,0}| \leq \dots \leq |1-\xi_{n_0}|.$$

Since $\varepsilon < \frac{\delta}{4k^3}$ we have

$$|\xi_{j\mu_j} - \xi_{j\nu_j}| < \frac{\delta}{4k^2}, \quad j=1, \dots, n; \mu_j, \nu_j = 0,1, \dots, k_j,$$

and therefore by applying lemma 3.7 we obtain

$$(4.1)\begin{cases} |1-\xi_{j\mu_j}| < r_0, & j=1, \dots, l; \mu_j=0,1, \dots, k_j \\ |1-\xi_{j\mu_j}| > r_0, & j=l+1, \dots, n; \mu_j=0,1, \dots, k_j. \end{cases}$$

Order the set $\{\xi_{j\mu_j}\}$ $j=1, \dots, n; \mu_j=0,1, \dots, k_j$ in such a way that

$$0=|1-\xi_1| \leq \dots \leq |1-\xi_s| < \frac{r}{r_0} < |1-\xi_{s+1}| \leq \dots \leq |1-\xi_k|.$$

We see from (4.1) that $s=l+k_1+ \dots +k_l$.

Define

$$(4.2) \quad F_1(z) = \prod_{i=s+1}^{k} (z-\xi_i) = \sum_{i=0}^{k-s} c_i^{(1)} z^{k-s-i}.$$

It is easily seen that

$$(4.3) \quad |c_i^{(1)}| < \binom{k-s}{i}, \quad i=0,1, \dots, k-s.$$

Let $F_2(z)$ be a polynomial of degree $\leq s-1$ with

$$(4.4) \quad F_2(\xi_i) = \frac{1}{\xi_i^{m+1} F_1(\xi_i)}, \quad i=1, \dots, s.$$

From lemma 3.4 it follows that

$$(4.5) \quad F_2(z)=e_0+e_1(z-\xi_1)+ \dots +e_{s-1}(z-\xi_1) \dots (z-\xi_{s-1}) \text{ with}$$

$$e_\nu = \frac{1}{2\pi i}\oint_{|1-z|=r_0} \frac{dz}{z^{m+1}\ F_1(z)(z-\xi_1)\dots(z-\xi_{\nu+1})}, \quad \nu=0,1,\ \dots,\ s-1.$$

Lemma 3.2 yields

$$|\gamma(|1-z|)| \geq 2(\tfrac{\delta}{4})^k, \quad |1-z|=r_0.$$

Further, since $\varepsilon < \varepsilon_1$ we have the following inequalities

$$|f(z)| \geq |\psi(|1-z|)| \geq |\gamma(|1-z|)| - M\eta \geq 2(\tfrac{\delta}{4})^k - M\eta, \text{ for } |1-z|=r_0.$$

According to lemma 3.3 the same holds true for the function

$$F_1(z)(z-\xi_1)\dots(z-\xi_{\nu+1}), \qquad \nu=0,1,\ \dots,\ s-1.$$

i.e.

$$|F_1(z)(z-\xi_1)\dots(z-\xi_{\nu+1})| \geq 2(\tfrac{\delta}{4})^k - M\eta, \text{ for } |1-z|=r_0;\nu=0,1,\ \dots,\ s-1.$$

From the representation of the coefficients e_ν as a contour integral we derive

$$(4.6) \quad |e_\nu| \leq \frac{1}{2(\frac{\delta}{4})^k - M\eta} \cdot \frac{1}{(1-r_0)^{m+1}} \cdot r_0 \leq 2(\tfrac{4}{\delta})^{k-1} \frac{1}{(1-\delta)^{m+1}} \{1+M\eta(\tfrac{\delta}{4})^k\},$$

$\nu=0,1,\ \dots,\ s-1$; η sufficiently small.

Write the function $F_2(z)$ as

$$(4.7) \quad F_2(z)=\sum_{\nu=0}^{s-1} c_\nu^{(2)} z^\nu.$$

Using (4.5) and (4.6) we obtain

$$(4.8) \quad |c_\nu^{(2)}| \leq |e_\nu| + |e_{\nu+1}|\binom{\nu+1}{1} + \dots + |e_{s-1}|\binom{s-1}{s-\nu-1} \leq$$

$$\leq \binom{s}{\nu+1} \frac{1}{(1-\delta)^{m+1}}\ 2(\tfrac{4}{\delta})^{k-1}\ \{1 + M\eta(\tfrac{4}{\delta})^k\}, \qquad \nu=0,1,\ \dots,\ s-1.$$

Define

$$(4.9)\quad F_3(z)=z^{m+1}F_1(z)F_2(z) = \sum_{\nu=m+1}^{m+k} c_\nu^{(3)} z^\nu.$$

Note that $c_\nu^{(3)}=c_\nu^{(3)}(\varepsilon)$.

From (4.9), (4,2) and (4,7), it follows that

$$\sum_{\nu=m+1}^{m+k} |c_\nu^{(3)}| \le \sum_{i=0}^{k-s} |c_i^{(1)}| . \sum_{\nu=0}^{s-1} |c_\nu^{(2)}| .$$

Using (4.3 and (4.8) we obtain

$$(4.10)\quad \sum_{\nu=m+1}^{m+k} |c_\nu^{(3)}| \le 4\left(\frac{8}{\delta}\right)^{k-1} \frac{1}{(1-\delta)^{m+1}} \left\{1+Mn\left(\frac{4}{\delta}\right)^k\right\}.$$

It follows from (4.2),(4.4) and (4.9) that

$$F_3(\xi_i) = \begin{cases} 1 & \text{for } i=1, \ldots, s \\ 0 & \text{for } i=s+1, \ldots, k. \end{cases}$$

i.e.

$$(4.11)\quad F_3((1-\mu_j\varepsilon)z_j) = \begin{cases} 1 & \text{for } j=1, \ldots, l;\mu_j=0,1, \ldots, k_j \\ 0 & \text{for } j=l+1, \ldots, n;\mu_j=0,1, \ldots, k_j. \end{cases}$$

Now apply lemma 2.1 to $F_3(z_jt)$ with $a=1$ and with $h=-\varepsilon$; $\mu=\mu_j, j=1, \ldots,n;\mu_j=1, \ldots,k_j$. Then it follows that there exists $\theta_{j\mu_j}$ with $1-\mu_j\varepsilon < \theta_{j\mu_j} < 1$ such that

$$(-\varepsilon)^{\mu_j}\left[\frac{\partial^{\mu_j}}{\partial t^{\mu_j}} F_3(z_jt)\right]_{t=\theta_{j\mu_j}} =$$

$$= F_3((1-\mu_j\varepsilon)z_j) - \binom{\mu_j}{1}F_3((1-(\mu_j-1)\varepsilon)z_j)+ \ldots +(-1)^{\mu_j}F_3(z_j),$$

$$\text{for } j=1, \ldots,n;\mu_j=1, \ldots, k_j.$$

By (4.11) the latter expression is 0. Thus we find that

$$\sum_{\nu=m+1}^{m+k} c_\nu^{(3)} \binom{\nu}{\mu_j} z_j^{\nu} \theta_{j\mu_j}^{\nu} = 0, \qquad j=1, \ldots, n; \mu_j=1, \ldots, k_j.$$

As in the proof of theorem 1, we define the following function

$$(4.12)\quad \delta_{j\mu_j}(\varepsilon) = \sum_{\nu=m+1}^{m+k} \binom{\nu}{\mu_j} c_\nu^{(3)} z_n^{\nu} \{\theta_{j\mu_j}^{\nu} - 1\}.$$

Then we have

$$(4.13)\quad \sum_{\nu=m+1}^{m+k} \binom{\nu}{\mu_j} c_\nu^{(3)} z_j^{\nu} + \delta_{j\mu_j}(\varepsilon) = 0, \qquad j=1, \ldots, n; \mu_j=1, \ldots, k_j.$$

From (4.12) it follows that

$$(4.14)\quad |\delta_{j\mu_j}(\varepsilon)| \leq 2^{m+k}(1-(1-k\varepsilon)^{m+k}) \sum_{\nu=m+1}^{m+k} |c_\nu^{(3)}|,$$

$$j=1, \ldots, n; \mu_j=1, \ldots, k_j; \varepsilon \text{ sufficiently small.}$$

Write the polynomials B_j in the form

$$(4.15)\quad B_j(t) = \sum_{\mu_j=0}^{k_j} b_{j\mu_j} \binom{t}{\mu_j}, \qquad j=1, \ldots, n.$$

Multiplying (4.13) by $b_{j\mu_j}$ and summing over $\mu_j=1, \ldots, k_j$ yields

$$(4.16)\quad \sum_{\mu_j=1}^{k_j} b_{j\mu_j} \sum_{\nu=m+1}^{m+k} \binom{\nu}{\mu_j} c_\nu^{(3)} z_j^{\nu} + \sum_{\mu_j=1}^{k_j} b_{j\mu_j} \delta_{j\mu_j}(\varepsilon) = 0, \quad j=1, \ldots, n.$$

Substituting (4.15) in (4.16) and summing over j=1, ..., n yields

$$(4.17)\quad \sum_{\nu=m+1}^{m+k} \sum_{j=1}^{n} B_j(\nu) c_\nu^{(3)} z_j^{\nu} + \sum_{j=1}^{n} \sum_{\mu_j=1}^{k_j} b_{j\mu_j} \delta_{j\mu_j}(\varepsilon) + $$

$$- \sum_{j=1}^{n} \sum_{\nu=m+1}^{m+k} c_\nu^{(3)} b_{j0} z_j^{\nu} = 0.$$

Using (4.11) we see that

$$\sum_{j=1}^{n} \sum_{\nu=m+1}^{m+k} c_\nu^{(3)} b_{j0} z_j^\nu = \sum_{j=1}^{n} b_{j0} F_3(z_j) = \sum_{j=1}^{l} b_{jo} = \sum_{j=1}^{l} B_j(0).$$

Therefore

$$(4.18) \quad \sum_{\nu=m+1}^{m+k} \left(\sum_{j=1}^{n} B_j(\nu) z_j^\nu \right) c_\nu^{(3)} + \sum_{j=1}^{n} \sum_{\mu_j=1}^{k_j} b_{j\mu_j} \delta_{j\mu_j}(\varepsilon) = \sum_{j=1}^{l} B_j(0).$$

Now we use the estimations (4.10) and (4.14).
Hence we obtain

$$\min_{j=1,\ldots,n} |B_1(0) + \ldots + B_j(0)| \leq \sum_{j=1}^{l} |B_j(0)| \leq$$

$$\leq \sum_{\nu=m+1}^{m+k} \left| \sum_{j=1}^{n} B_j(\nu) z_j^\nu \right| |c_\nu^{(3)}| + \sum_{j=1}^{n} \sum_{\mu_j=1}^{k_j} |b_{j\mu_j}| |\delta_{j\mu_j}(\varepsilon)| \leq$$

$$\leq \left\{ \max_{m+1 \leq \nu \leq m+k} \left| \sum_{j=1}^{n} B_j(\nu) z_j^\nu \right| + 2^{m+k}(1-(1-k\varepsilon)^{m+k}) \sum_{j,\mu_j} |b_{j\mu_j}| \right\} \sum_{\nu=m+1}^{m+k} |c_\nu^{(3)}| \leq$$

$$\left\{ \max_{m+1 \leq \nu \leq m+k} \left| \sum_{j=1}^{n} B_j(\nu) z_j^\nu \right| + \alpha(\varepsilon) \right\} 4.\left(\frac{8}{\delta}\right)^{k-1} \frac{1}{(1-\delta)^{m+1}} \left\{ 1+M\eta\left(\frac{4}{\delta}\right)^k \right\},$$

where $\alpha(\varepsilon)=2^{m+k}(1-(1-k\varepsilon)^{m+k}) \sum_{j,\mu_j} | b_{j,\mu_j} |$.

From the definition of $\alpha(\varepsilon)$ we see

$\alpha(\varepsilon)=o(\varepsilon)$ for $\varepsilon \downarrow 0$.

We recall that δ was an arbitrarily chosen real number, $0 < \delta < 1$, that η had to be chosen sufficiently small, i.e. $\eta < \eta_0(\delta)$ and that ε had to be taken small accordingly, $\varepsilon < \varepsilon_0(\eta,\delta)$.

First let $\varepsilon \downarrow 0$. This yields

$$\min_{j=1,\ldots,n} |B_1(0) + \ldots + B_j(0)| \leq$$

$$\leq \{\max_{m+1\leq\nu\leq m+k} | \sum_{j=1}^{n} B_j(\nu)z_j^{\nu}|\}.4.(\frac{8}{\delta})^{k-1} \frac{1}{(1-\delta)^{m+1}} \{1+Mn(\frac{4}{\delta})^{k}\}.$$

Then let $\eta \longrightarrow 0$. This yields

(4.19)

$$\min_{j=1,\ldots,n} |B_1(0) + \ldots + B_j(0)| \leq 4(\frac{8}{\delta})^{k-1} \frac{1}{(1-\delta)^{m+1}} \max_{m+1\leq\nu\leq m+k} | \sum_{j=1}^{n} B_j(\nu)z_j^{\nu}|.$$

Finally, we take δ in the most appreciate way. If we determine the maximum of the function $x^{k-1}(1-x)^{m+1}$ we find that for $m \geq 0$, $k \geq 2$ the best choice for δ is

$$\delta = \frac{k-1}{m+k} .$$

Substituting this in (4.19) we obtain the statement of the theorem. For m=-1 or k=1, theorem 2 is trivial.
This completes the proof.

Litterature

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