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MATHEMATICS

ON THE PROPAGATION OF A DISCONTINUOUS ELECTROMAGNETIC WAVE

BY

BALTH. VAN DER POL † AND A. H. M. LEVELT

(Communicated by Prof. J. F. KOKSMA at the meeting of January 30, 1960) [1]

1. *Introduction*

In 1909 SOMMERFELD [1] found mathematical formulae describing the electromagnetic field of an infinitesimal electric dipole oscillating harmonically in the plane separating two homogeneous half spaces of different refractive indices. The axis of the dipole is perpendicular to the plane. Sommerfeld's method of solution can be found in several textbooks on electromagnetic theory and mathematical physics ([3], [4]).

Some years ago a variant of Sommerfeld's problem has been introduced ([2]): the oscillating dipole is replaced by a dipole, having moment 0 for $t<0$, and moment 1 for $t>0$. In [2] the problem was translated into the operational form, and several special cases were explicitly solved there. The present paper deals with the general case. A solution will be given in terms of complete elliptic integrals (section 4).

In section 2 a part of the results of [2] are given, being the starting point for the further analysis. Section 3, containing the solution of the mathematical part of the problem, is almost identical with the preliminary report [6]. Another solution can be found in an earlier version of that report ([5]).

2. *Sommerfeld's solution and the mathematical formulation of the discontinuous problem*

Let the dipole be placed at the origin of a system of Cartesian coordinates x, y, z, the axis of the dipole coinciding with the z-axis. We put $\varrho=\sqrt{x^2+y^2}$, $R=\sqrt{x^2+y^2+z^2}$. The subscript 1 is attached to quantities of the first half space $z>0$ (air), the subscript 2 to those of the second, denser, half space $z<0$ (earth). So for the dielectric constants we have $\varepsilon_2>\varepsilon_1$. We assume that the conductivities σ_1 and σ_2 vanish, and that the magnetic permeabilities μ_1 and μ_2 are both equal to 1. The electromagnetic field is uniquely determined by the so called Hertzian

[1]) The deplored death of Professor VAN DER POL on October 6, 1959, put an abrupt end to our fruitful co-operation. As a result this paper contains only a part of what might have been achieved if he had lived. To have worked with this great scientist is a privilege for which I am most grateful. A. H. M. L.

vector $\prod$. In Sommerfeld's solution $\prod$ is always parallel to the z-axis, and the z-component is denoted by $\prod_1$ and $\prod_2$ in the first and second half space respectively. The solution is

$$(2.1)\qquad \prod_1 = e^{-i\omega t}\cdot 2k_2^2 \int_0^\infty \frac{J_0(\lambda\varrho)\, e^{-z\sqrt{\lambda^2-k_1^2}}\,\lambda\, d\lambda}{k_2^2\sqrt{\lambda^2-k_1^2}+k_1^2\sqrt{\lambda^2-k_2^2}} \quad \text{for} \quad z>0,$$

$$(2.2)\qquad \prod_2 = e^{-i\omega t}\cdot 2k_1^2 \int_0^\infty \frac{J_0(\lambda\varrho)\, e^{z\sqrt{\lambda^2-k_2^2}}\,\lambda\, d\lambda}{k_2^2\sqrt{\lambda^2-k_1^2}+k_1^2\sqrt{\lambda^2-k_2^2}} \quad \text{for} \quad z<0,$$

where

$$\operatorname{Re}\sqrt{\lambda^2-k_1^2} \geqslant 0\ ,\ \operatorname{Re}\sqrt{\lambda^2-k_2^2} \geqslant 0$$

and

$$k_1 = \frac{\omega}{c}\sqrt{\varepsilon_1}\quad ,\quad k_2 = \frac{\omega}{c}\sqrt{\varepsilon_2}.$$

As usual ω is the angular frequency, and c is the propagation velocity in free space.

With Sommerfeld's solution as starting-point it was shown in [2] that the Hertzian vector $\prod^*$ of the discontinuous problem satisfies

$$(2.3)\qquad p\int_0^\infty e^{-pt}\prod_1^*(t)\,dt = 2\varepsilon_2 \int_0^\infty \frac{J_0(\lambda\varrho)\, e^{-z\sqrt{\lambda^2+\varepsilon_1 c^{-2}p^2}}\,\lambda\, d\lambda}{\varepsilon_2\sqrt{\lambda^2+\varepsilon_1 c^{-2}p^2}+\varepsilon_1\sqrt{\lambda^2+\varepsilon_2 c^{-2}p^2}}$$

for $z>0$ and $p>0$,

$$(2.4)\qquad p\int_0^\infty e^{-pt}\prod_2^*(t)\,dt = 2\varepsilon_1 \int_0^\infty \frac{J_0(\lambda\varrho)\, e^{z\sqrt{\lambda^2+\varepsilon_2 c^{-2}p^2}}\,\lambda\, d\lambda}{\varepsilon_2\sqrt{\lambda^2+\varepsilon_1 c^{-2}p^2}+\varepsilon_1\sqrt{\lambda^2+\varepsilon_2 c^{-2}p^2}}$$

for $z<0$ and $p>0$. These formulae can also be derived from (2.1) and (2.2) in the following heuristic way. The Hertzian vector for the oscillating dipole has a singularity of the type

$$(2.5)\qquad \prod_j = \frac{C_j}{R}\, e^{-i\omega t} \qquad (j=1,2)$$

in the origin. The dipole jumping from moment 0 to moment 1 at $t=0$ has a singularity in the origin of the type

$$(2.6)\qquad \prod_j^* = \begin{cases} 0 \ \text{for}\ t<0. \\ \dfrac{C_j}{R}\ \text{for}\ t>0, \end{cases}$$

If we try to get $\prod_j^*$ by superposing solutions $\prod_j$ with different ω's, such that the singularity in the origin is given by (2.6), our choice will be

$$(2.7)\qquad \prod_j^* = -\frac{1}{2\pi i}\int_{-\infty+i\delta}^{\infty+i\delta} \frac{\prod_j}{\omega}\, d\omega,$$

where $\delta>0$. Putting $\omega = iw$, we see that (2.7) is the complex inversion of the Laplace integrals (2.3), (2.4) respectively.

3. *Solution of the mathematical problem*

A slightly generalized version of the mathematical problem of section 2 is how to find a function $h(t)$, such that the given function

$$(3.1)\qquad f(p) = \int_0^\infty \frac{e^{-z\sqrt{x^2+a^2p^2}}J_0(\varrho x)x\,dx}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} \qquad (p>0).$$

is the Laplace transform

$$(3.2)\qquad f(p) = p\int_0^\infty e^{-pt}\,h(t)\,dt$$

of $h(t)$. Here ϱ, z, a, b, c, d are positive constants and $a \neq b$. Substituting $y = p^{-1}\sqrt{x^2+a^2p^2}$, we deduce from (3.1)

$$(3.3)\qquad f(p) = p\int_a^\infty \frac{e^{-zyp}J_0(p\varrho\sqrt{y^2-a^2})y\,dy}{cy+d\sqrt{y^2+b^2-a^2}}.$$

By the well-known formula

$$(3.4)\qquad J_0(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{e^{ixs}}{\sqrt{1-s^2}}\,ds,$$

we then have

$$(3.5)\qquad f(p) = p\int_a^\infty \frac{e^{-zyp}\,y\,dy}{cy+d\sqrt{y^2+b^2-a^2}}\,\frac{1}{\pi}\int_{-1}^{1}\frac{e^{isp\varrho\sqrt{y^2-a^2}}}{\sqrt{1-s^2}}\,ds.$$

Replacing s by the new variable t

$$(3.6)\qquad t = zy - i\varrho s\sqrt{y^2-a^2},$$

we obtain

$$(3.7)\qquad f(p) = \int_a^\infty dy\int_L \varphi(y,t)\,dt,$$

where $\varphi(y,t)$ is defined by

$$(3.8)\qquad \varphi(y,t) = \frac{p}{\pi i}\,\frac{y\,e^{-pt}}{\sqrt{\varrho^2(y^2-a^2)+(zy-t)^2}\,(cy+d\sqrt{y^2+b^2-a^2})}.$$

The t-integration contour in (3.7) is a line segment L connecting $zy-i\varrho\sqrt{y^2-a^2}$ and $zy+i\varrho\sqrt{y^2-a^2}$ (fig. 1). If y varies from a to ∞, the points $zy-i\varrho\sqrt{y^2-a^2}$ and $zy+i\varrho\sqrt{y^2-a^2}$ describe a branch H of a hyperbola in the complex t-plane. If y is fixed, the function

$$\frac{e^{-pt}}{\sqrt{\varrho^2(y^2-a^2)+(zy-t)^2}}$$

of t has no singularities in the region G to the right of H, and is $O(e^{-pt})$ if $t\to\infty$. We therefore have

$$(3.9)\qquad \int_L \varphi(y,t)\,dt = \int_{\mathrm{I}}\varphi(y,t)\,dt + \int_{\mathrm{II}}\varphi(y,t)\,dt,$$

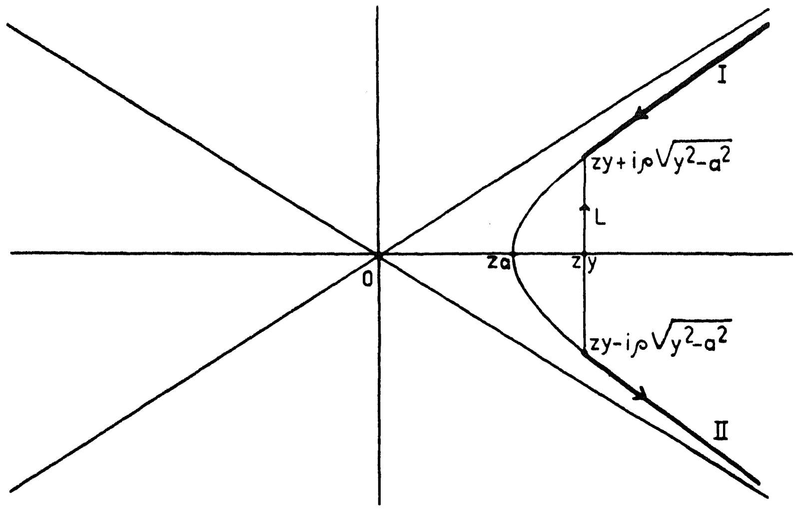

Fig. 1. The t-plane.

where the sign of $\sqrt{\varrho^2(y^2-a^2)+(zy-t)^2}$ has to be chosen in such a way that the square root is asymptotically equal to t if $t \to \infty$, $t \in G$. The countours I and II are parts of H as is shown in fig. 1, and have parametric representations

$$(3.10)\qquad \begin{cases} \mathrm{I}: t = t_1(u) = zu+i\varrho\sqrt{u^2-a^2},\ u \geqslant a; \\ \mathrm{II}: t = t_2(v) = zv-i\varrho\sqrt{v^2-a^2},\ v \geqslant a. \end{cases}$$

From (3.7), (3.9) and (3.10) we deduce

$$(3.11)\qquad \begin{cases} f(p) = \int\limits_a^\infty dy \int\limits_{\mathrm{I}} \varphi(y,t)\,dt + \int\limits_a^\infty dy \int\limits_{\mathrm{II}} \varphi(y,t)\,dt = \\ \quad = \int\limits_a^\infty dy \int\limits_\infty^y \varphi(y, t_1(u))\,t_1'(u)\,du + \int\limits_a^\infty dy \int\limits_y^\infty \varphi(u, t_2(v))\,t_2'(v)\,dv = \\ \quad = -\int\limits_a^\infty t_1'(u)\,du \int\limits_a^u \varphi(y, t_1(u))\,dy + \int\limits_a^\infty t_2'(v)\,dv \int\limits_a^v \varphi(y, t_2(v))\,dy. \end{cases}$$

The integrations may be interchanged, as is justified in the following way. If $t_1 \in \mathrm{I}$ we have

$$(3.12)\qquad \begin{cases} |\varrho^2(y^2-a^2)+(zy-t_1)^2| = \\ = |t_1-(zy+i\varrho\sqrt{y^2-a^2})|\,|t-(zy-i\varrho\sqrt{y^2-a^2})| \geqslant |zu-zy|\,|2\varrho\sqrt{y^2-a^2}|. \end{cases}$$

We also have

$$(3.13)\qquad |t_1'(u)| = \left|z+\frac{i\varrho u}{\sqrt{u^2-a^2}}\right| \leqslant z+\frac{\varrho y}{\sqrt{y^2-a^2}}.$$

From (3.12) and (3.13) it follows that

$$\psi(y) = \int_{y}^{\infty} |\varphi(y, t_1(u)) t_1'(u)|\, du \leqslant \left| \frac{p}{\pi} \frac{y}{cy+d\sqrt{y^2+b^2-a^2}} \right| \frac{1}{\sqrt{2\varrho z \sqrt{y^2-a^2}}} \cdot$$

$$\cdot \left(z + \frac{\varrho y}{\sqrt{y^2-a^2}} \right) \int_{y}^{\infty} \frac{e^{-pzu}}{\sqrt{u-y}}\, du.$$

So there is a constant C (independent of y) with

$$\psi(y) \leqslant C \frac{y^2 e^{-pzy}}{(cy+d\sqrt{y^2+b^2-a^2})\,(y^2-a^2)^{3/4}}.$$

Since $cy+d\sqrt{y^2+b^2-a^2} \geqslant db > 0$ if $y \geqslant a$, and since $a \neq 0, p > 0, z > 0$, we have

$$\int_{a}^{\infty} \psi(y)dy < \infty.$$

The integral over II can be dealt with similarly. If u ranges from a to ∞, $t_1(u)$ describes a contour W_1, which is the part of H above the real

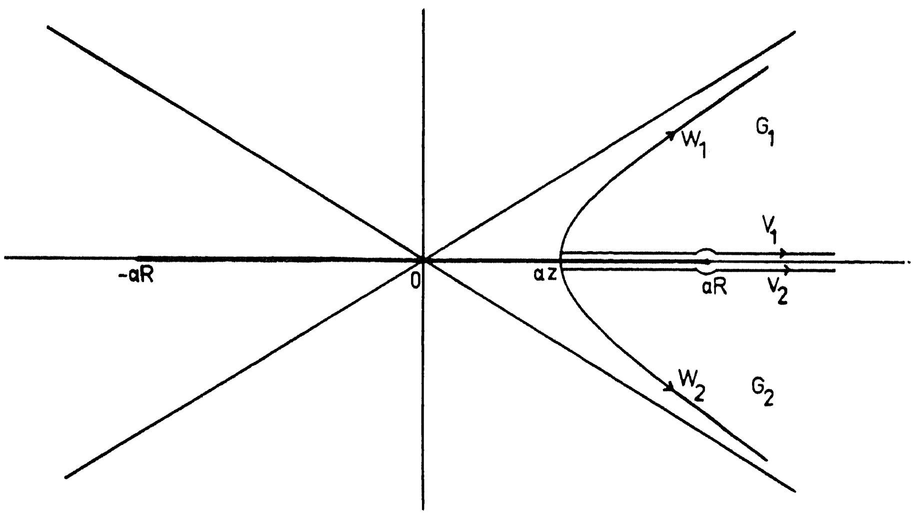

Fig. 2. The t-plane.

axis (fig. 2). If $t \in W_1$, the corresponding value of u will be given by

$$u(t) = \frac{tz - i\varrho\sqrt{t^2-a^2R^2}}{R^2}. \tag{3.14}$$

From now on we cut the t-plane along the real axis from $-aR$ to aR, taking $\sqrt{t^2-a^2R^2}$ positive if $t > aR$. Similarly, if v ranges from a to ∞, $t_2(v)$ describes a contour W_2, the part of H under the real axis, and now

$$v(t) = \frac{tz + i\varrho\sqrt{t^2-a^2R^2}}{R^2} \qquad (t \in W_2). \tag{3.15}$$

Hence (11) can be written as

$$f(p) = -\int_{W_1} dt \int_a^{u(t)} \varphi(y,t)\,dy + \int_{W_2} dt \int_a^{v(t)} \varphi(y,t)\,dy. \tag{3.16}$$

From now on y will also assume complex values. Let G_1 be the region bounded by W_1 and the part of the positive real axis from az to ∞. First we define

$$g(t) = \int_{W(t)} \varphi(y,t)\,dy, \tag{3.17}$$

for $t \in G_1$ in the following way. G_1 is conformally mapped onto a region G_1' of the y-plane by $y=u(t)$ ((3.14)). G_1' is also bounded by the positive real axis, and a hyperbolic arc, which is the image of the part of the real axis $t>aR$ (fig. 3).

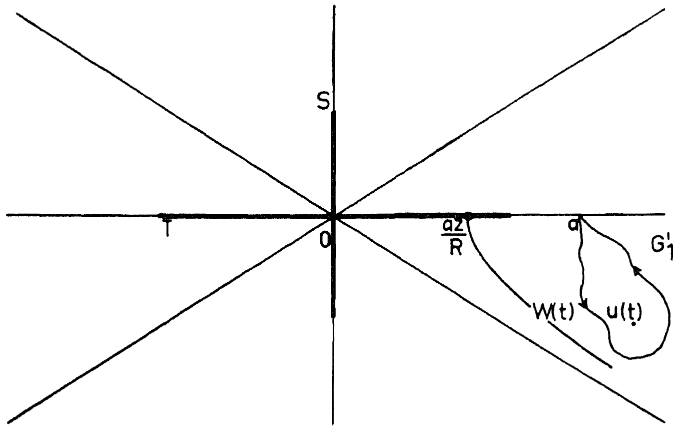

Fig. 3. The y-plane.

$\sqrt{y^2+b^2-a^2}$ is defined as follows.

I. If $a<b$, we cut the y-plane along the interval

$$S : [-i\sqrt{b^2-a^2},\, i\sqrt{b^2-a^2}]$$

on the imaginary axis.

II. If $a>$b, the real axis is cut along the interval

$$T : [-\sqrt{a^2-b^2},\, \sqrt{a^2-b^2}].$$

In both cases the square root is positive for large positive values of y. $W(t)$ is a simple curve in the y-plane. Starting in a, $W(t)$ encircles $u(t)$ in positive direction, ending in a again without leaving G_1'. Evidently, if t is fixed in G_1, only the root $u(t)$ of $\varrho^2(y^2-a^2)+(zy-t)^2=0$ is in G_1'.

On $W(t)$ we define the function $\sqrt{\varrho^2(y^2-a^2)+(zy-t)^2}$ by analytic continuation, taking the value $t-za$ at the initial point $y=a$ of $W(t)$. If $W(t)$ satisfies the above conditions, the integral on the right of (3.17) is independent of $W(t)$, and $g(t)$ is uniquely defined on G_1. One can easily prove that $g(t)$ is analytic on G_1. In fact, $g(t)$ can be analytically continued to the boundary of G_1, the point $t = Ra$ being excluded. If t is fixed and $t \neq Ra$, the conformal mapping $y = u(t)$ can be extended across the cut $(-aR, aR)$, and the roots of $\varrho^2(y^2-a^2)+(zy-t)^2 = 0$ are separated. If $u(t)$ is on the boundary of G_1', we can take a contour $W(t)$, which leaves G_1' only in a small neighbourhood of $u(t)$, but which apart from that satisfies the above conditions. In case II it may occur that $u(t) \in T$; then $\sqrt{y^2+b^2-a^2}$ has to be continued analytically along $W(t)$ across the cut T.

Finally we need an estimate of $|g(t)|$ if $t \in G_1$ and $t \to \infty$. It is not difficult to see that there exists a constant $k > 0$ so that

$$\left| \frac{y}{cy + d\sqrt{y^2+b^2-a^2}} \right| \leqslant k \qquad (y \in G_1'). \tag{3.18}$$

We can deform $W(t)$ into the line-segment

$$y = a + (u(t)-a)\, s \qquad (0 \leqslant s \leqslant 1). \tag{3.19}$$

Then, (3.17), (3.18) and (3.19) imply

$$\left\{ \begin{aligned} |g(t)| &\leqslant \frac{2pk}{\pi} e^{-p\,\mathrm{Re}\, t} \int_0^1 \frac{|u(t)-a|\, ds}{\sqrt{|a-u(t)|\,|1-s|\,|a(1-s)+u(t)\,(1+s)-(2tz/R^2)|}} \leqslant \\ &\leqslant \frac{2pl}{\pi} e^{-p\,\mathrm{Re}\, t} \sqrt{\frac{2|t|}{R}+a}, \end{aligned} \right. \tag{3.20}$$

if $|t|$ is sufficiently large (l is independent of t).

If $u(t)$ is on the real axis and $> a$, then

$$g(t) = 2 \int_a^{u(t)} \varphi(y, t)\, dy,$$

which can be proved by deforming $W(t)$ into the interval $[a, u(t)]$. Hence we get for the first term on the right-hand side of (3.16)

$$-\int_{W_1} dt \int_a^{u(t)} \varphi(y, t)\, dy = -\tfrac{1}{2} \int_{W_1} g(t)\, dt. \tag{3.21}$$

Now, by (3.20) and since $g(t)$ is analytic in G_1 we can replace W_1 on the right of (3.21) by the contour V_1 of fig. 2.
Hence

$$-\int_{W_1} dt \int_a^{u(t)} \varphi(y, t)\, dy = -\int_{V_1} dt \int_{W(t)} \varphi(y, t)\, dy. \tag{3.22}$$

The second term on the right of (3.16) can be transformed in exactly the same way. We introduce here a contour $W'(t)$ surrounding the point $v(t)$. $W'(t)$ is the image of $W(t)$ in the real axis. Hence

$$f(p) = -\tfrac{1}{2} \int_{V_1} dt \int_{W(t)} \varphi(y, t)\, dy + \tfrac{1}{2} \int_{V_2} dt \int_{W'(t)} \varphi(y, t)\, dy. \tag{3.23}$$

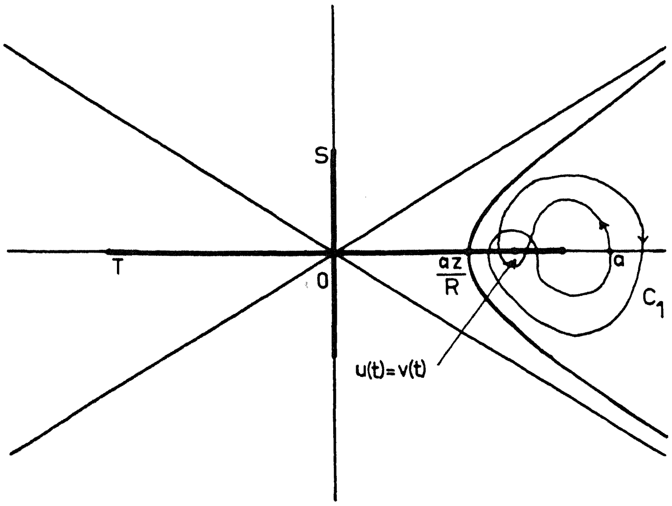

Fig. 4. The y-plane.

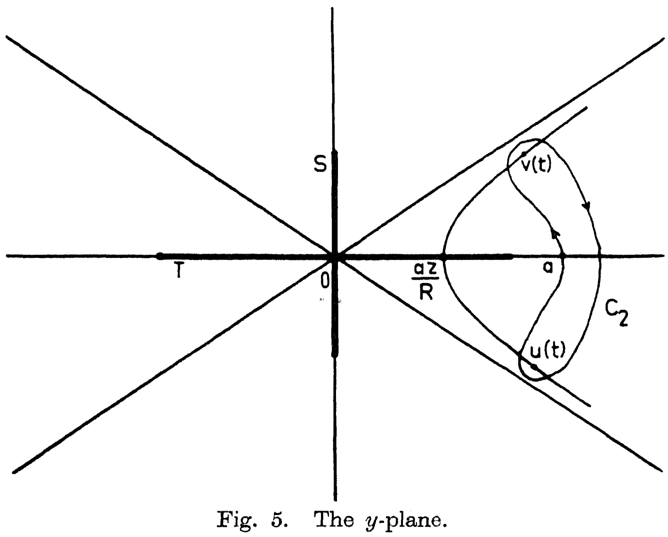

Fig. 5. The y-plane.

Adding the contours $W(t)$ and $W'(t)$ in the y-plane, we either obtain a contour C_1 (fig. 4) (if $az<t<aR$), or a contour C_2 (fig. 5) (if $t>aR$). From (3.23) it is clear, that the function

$$h(t) = \frac{1}{2\pi i} \int_{C_i} \frac{y\,dy}{(cy+d\sqrt{y^2+b^2-a^2})\sqrt{\varrho^2(y^2-a^2)+(zy-t)^2}}, \tag{3.24}$$

where $i=1$ if $az<t<aR$, $i=2$ if $t>aR$ and $h(t)=0$ if $t<az$, satisfies (3.2). (Both roots in the denominator of the integral are >0 if $y=a$).

It is possible to put the solution $h(t)$ of our problem in the form of complete elliptic integrals over intervals of the real axis. This can easily be done if we start from the formulae deduced above. We again distinguish the two cases I and II.

I. If $a<b$ and $t>Ra$ we deform the integration contour C_2 into the contour shown in fig. 6, taking into account the residue at $y=\infty$. After some calculations we find

$$(3.25)\quad h(t)=+\frac{1}{(c+d)R}-\frac{1}{\pi i}\int_{-\sqrt{b^2-a^2}}^{\sqrt{b^2-a^2}}\frac{xd\sqrt{b^2-a^2-x^2}\,dx}{\{(c^2-d^2)x^2+d^2(b^2-a^2)\}\sqrt{-R^2x^2+2iztx+t^2-\varrho^2a^2}},$$

where the roots are positive if $x=0$. It is not difficult to see that $h(t)$ assumes only real values.

If $t<Ra$ the contour C_2 can be shrunk to $u(t)=v(t)$, and so $h(t)=0$ in this case.

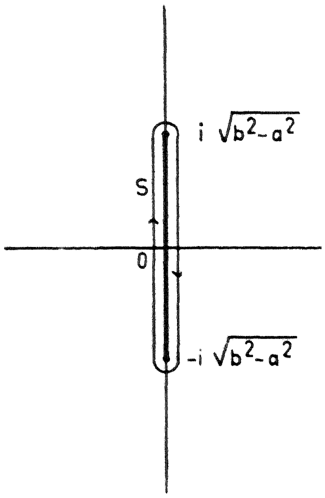

Fig. 6

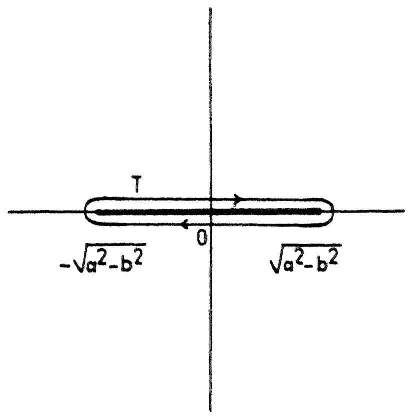

Fig. 7

II. If $a>b$, we consider first the case $t>Ra$. We deform the integration contour C_2 into the contour shown in fig. 7. Proceeding as in the case $a<b$ we find

$$(3.26)\quad h(t)=+\frac{1}{(c+d)R}+\frac{1}{\pi}\int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}}\frac{xd\sqrt{a^2-b^2-x^2}\,dx}{\{(c^2-d^2)x^2+d^2(a^2-b^2)\}\sqrt{R^2x^2-2ztx+t^2-\varrho^2a^2}},$$

where the roots are non-negative.

Next we look at the case $t<Ra$. Then we have to take C_1 in (3.24). It may happen that $u(t)=v(t)\notin T$. This will be the case, if the greater root $y=u_1$ of $\varrho^2(y^2-a^2)+(zy-t)^2=0$ is greater than $\sqrt{a^2-b^2}$. A necessary and sufficient condition is: either $Rb>\varrho a$ or $Rb<\varrho a$ and $t<z\sqrt{a^2-b^2}+\varrho b$. C_1 can be shrunk to $u(t)=v(t)$, hence $h(t)=0$. However, if $Rb<\varrho a$ and $z\sqrt{a^2-b^2}+\varrho b<t<Ra$, then C_1 can be deformed into the contour shown

in fig. 8. A simple calculation yields

$$(3.27)\qquad h(t) = +\frac{2}{\pi}\int_{u}^{\sqrt{a^2-b^2}} \frac{x d\sqrt{a^2-b^2-x^2}\,dx}{\{(c^2-d^2)x^2+d^2(a^2-b^2)\}\sqrt{R^2x^2-2ztx+t^2-\varrho^2a^2}},$$

where $u = \dfrac{zt+\varrho\sqrt{R^2a^2-t^2}}{R^2}$ and the roots are non-negative.

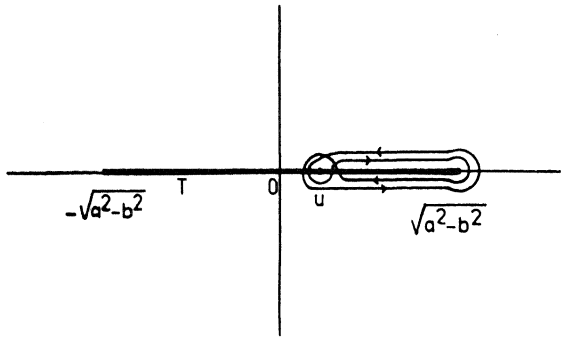

Fig. 8

4. *Results and discussion*

Applying the formulae of section 3 to (2.3) we find at once

$$(4.1)\qquad \Pi_1^* = 0 \text{ if } t < \frac{R}{c}\sqrt{\varepsilon_1},$$

$$(4.2)\qquad \Pi_1^* = \frac{2\varepsilon_2}{(\varepsilon_1+\varepsilon_2)R} - \frac{2\varepsilon_1\varepsilon_2}{\pi i(\varepsilon_2-\varepsilon_1)}\int_{-\sqrt{\varepsilon_2-\varepsilon_1}}^{\sqrt{\varepsilon_2-\varepsilon_1}} \frac{s\sqrt{(\varepsilon_2-\varepsilon_1)-s^2}\,ds}{\{(\varepsilon_1+\varepsilon_2)s^2+\varepsilon_1^2\}\sqrt{-R^2s^2+2izcts+c^2t^2-\varepsilon_1\varrho^2}}$$

if $t > \dfrac{R}{c}\sqrt{\varepsilon_1}$. The square roots are >0 if $s=0$.

The situation in the second half space is more complicated. Application of the formulae (3.26) and (3.27) yields

$$(4.3)\qquad \begin{cases} \Pi_2^* = 0 \text{ if } ct < R\sqrt{\varepsilon_2} \text{ and either } \dfrac{R}{\varrho} > \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}} \\ \qquad\qquad \text{or } \dfrac{R}{\varrho} < \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}} \text{ and } ct < z\sqrt{\varepsilon_2-\varepsilon_1}+\varrho\sqrt{\varepsilon_1}; \end{cases}$$

$$(4.4)\qquad \Pi_2^* = -\frac{4\varepsilon_1\varepsilon_2}{\pi(\varepsilon_2-e_1)}\int_{s(t)}^{\sqrt{\varepsilon_2-\varepsilon_1}} \frac{s\sqrt{(\varepsilon_2-\varepsilon_1)-s^2}\,ds}{\{(\varepsilon_1+\varepsilon_2)s^2-\varepsilon_2^2\}\sqrt{R^2s^2+2zcts+c^2t^2-\varrho^2\varepsilon_2}},$$

where

$$s(t) = R^{-2}\{-zct+\varrho\sqrt{R^2\varepsilon_2-t^2c^2}\},$$

if $R/\varrho < \sqrt{\varepsilon_2/\varepsilon_1}$ and $z\sqrt{\varepsilon_2-\varepsilon_1}+\varrho\sqrt{\varepsilon_1} < ct < R\sqrt{\varepsilon_2}$;

$$(4.5)\quad \Pi_2^* = \frac{2\varepsilon_1}{(\varepsilon_1+\varepsilon_2)R} - \frac{2\varepsilon_1\varepsilon_2}{\pi(\varepsilon_2-\varepsilon_1)} \int_{-\sqrt{\varepsilon_2-\varepsilon_1}}^{\sqrt{\varepsilon_2-\varepsilon_1}} \frac{s\sqrt{(\varepsilon_2-\varepsilon_1)-s^2}\,ds}{\{(\varepsilon_1+\varepsilon_2)s^2-\varepsilon_2^2\}\sqrt{R^2s^2+2zcts+c^2t^2-\varrho^2\varepsilon_2}},$$

if $ct > R\sqrt{\varepsilon_2}$.

In fig. 9 a plane through the z-axis is drawn, and the wave fronts I, II, and III are indicated. B is the Brewster cone, defined by $\sin\theta = \sqrt{\varepsilon_1/\varepsilon_2}$.

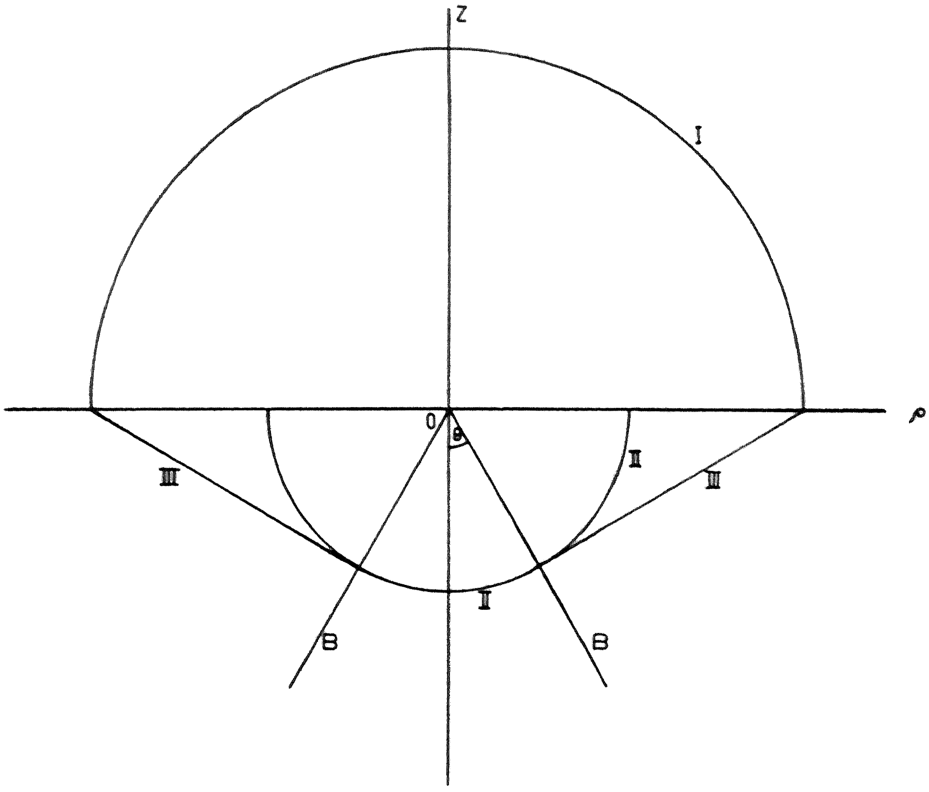

Fig. 9

In the air there is only one spherical wave front I, which follows from (4.1) and (4.2). In the ground there are two wave fronts: a spherical wave front II, which can be deduced from (4.3) and (4.5), and a conical wave front III, tangent to II along the Brewster cone. I and III intersect the plane $z=0$ along the same circle. All these facts follow from the formulae (4.3) and (4.4). The shape of III can be explained by the assumption that part of the electromagnetic disturbance travels through the plane $z=0$, and enters the second medium in a direction parallel to the Brewster cone. This is the quickest way for a disturbance to reach a fixed point in the second medium outside the Brewster cone.

Finally, the problem being solved, a shorter way to our result is opened. It can be verified, that

$$u = (R^2 s^2 + 2zcts + c^2 t^2 - \varrho^2 \varepsilon_j)^{-\frac{1}{2}}$$

satisfies the wave equation

$$\Delta u = \frac{\varepsilon_j}{c^2} \frac{\partial^2 u}{\partial t^2}.$$

$\prod_j^*$ is a solution of this equation satisfying certain boundary conditions, which will not be specified here. One can try to get solutions of the form

$$\prod_j^* = \int_{W_j} \frac{\varphi_j(s)\, ds}{\sqrt{R^2 s^2 + 2zcts + c^2 t^2 - \varrho^2 \varepsilon_j}},$$

where the function $\varphi_j(s)$ and the complex contour W_j have to be chosen in such a way, that $\prod_j^*$ satisfies the boundary conditions.

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