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Lattice points in unbounded point sets

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Lattice points in unbounded point sets [1)]

by

C.G. Lekkerkerker

1. Introduction. In the last two volumes of the "American Mathematical Monthly", in the department of "Advanced Problems and Solutions", the following two problems concerning divergent integrals and series were proposed by K.L. Chung and G.R. MacLane respectively [2)].

Problem A. If $f(x)$ is continuous and non-negative in $[0,\infty)$ and $\int_0^\infty f(x)dx=\infty$, then there exists an $x>0$ such that $\sum_{n=1}^{\infty} f(nx)=\infty$.

Problem B. Find a function $f(x)$, upper semi-continuous [3)] and non-negative on $[0,\infty)$, bounded on each finite interval $(0,T)$, such that $\int_0^\infty f(x)dx=\infty$ and $\sum_{n=1}^{\infty} f(nx)<\infty$ for every $x>0$.

Problem A is itself a generalization of the following problem, proposed in the same periodical by D.J. Newman and W.E. Weissblum [4)]:

Problem C. Given an unbounded set of positive reals. Prove that there exists a real number such that infinitely many integral multiples of it lie in the set.

Solutions to Problems C and A have also been published [5)]. In this report we shall deduce slightly more general properties and also derive some results of a related type. Next, we shall give appropriate extensions for n-dimensional integrals. Here the concept

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1) This report is the fruit of discussions with Prof.de Bruijn, Mr. Kesten and Prof. Koksma, who contributed much to the results exposed (see also footnote 6)).

2) See problem 4670, Am.Math.Monthly, 63, 47 and 190 (1956); problem 4727, ibidem, 64, 117 (1957).

3) i.e. $\limsup_{x\to a} f(x) \leq f(a)$ for every a.

4) See problem 4605, Am. Math. Monthly, 61, 572 (1954).

5) Same journal, 62, 738 (1955); 64, 119-120 (1957).

of set of the multiples of a positive number will be replaced by the more general concept of a lattice in n-dimensional space and use will be made of some results in the field of the geometry of numbers. In particular, we shall apply Siegel's refinement of the Minkowski-Hlawka theorem.

2. Results for onedimensional sets and integrals. In the following V will always be a set of non-negative reals. It is always assumed that V is Lebesgue measurable; sometimes we shall require that V be even Jordan measurable. We shall denote by μV the (Lebesgue, c.q. Jordan) measure of V. Now, for given V, we are interested in the sets W_1, W_2, W_3 of the numbers $x > 0$ satisfying resp. the requirement that

(1) $kx \in V$ for infinitely many positive integers k

(2) $kx \in V$ only for finitely many positive integers k

(3) $kx \in V$ for no positive integer k.

There are four theorems, which run as follows:

Theorem 1. If $\mu V < \infty$, then (2) holds for almost all x.

Theorem 2. If $\mu V = \infty$ and V is Jordan measurable, then there is at least one number $x > 0$ for which (1) holds.

Theorem 3. There exists a Jordan measurable set V (consisting of an infinite sequence of disjoint intervals) with $\mu V = \infty$, such that (2) holds for almost all x and, moreover, (3) holds on a set of infinite measure.

Theorem 4. If $0 < \rho < 1$ and if V has density $\geq \rho$ on each of an infinite sequence of intervals of length 1, then (1) holds for almost all x in (0,1).

The first two of these theorems can be generalized to statements involving an arbitrary function. Let f(x) be any non-negative, measurable function on the interval $[0, \infty)$ and put

$$F(x) = \sum_{n=1}^{\infty} f(nx).$$

Then the following two theorems hold:

Theorem 1'. If $\int_0^{\infty} f(x)dx < \infty$, then $F(x) < \infty$ for almost all x.

Theorem 2'. If f(x) is Riemann integrable and $\int_0^{\infty} f(x)dx = \infty$, then there exists an $x > 0$ with $F(x) = \infty$.

If in these theorems one takes for $f(x)$ the characteristic function of a set V (measurable in the sense of Lebesgue, c.q. Jordan), then one gets back the theorems 1 and 2. So we need only to prove the theorems 1',2',3 and 4. [6)]

Before giving the proofs of these theorems we draw some conclusions and make some additional remarks. Theorems 2 and 4 deal with the set W_1 of numbers $x > 0$ possessing property (1) and assert that, under certain conditions, this set is nonempty, c.q. covers the whole interval $[0,\infty)$ apart from a set of measure zero. Theorem 3 says that, under the conditions of theorem 2, the set W_1 may well be a set of measure zero. Further, theorem 3 learns that in theorem 2 one cannot omit the condition that V be Jordan measurable. For, deleting from a set V satisfying the conditions of theorem 3 a suitably chosen set of measure zero, one retains a Lebesgue measurable set V_1 of infinite measure, such that no real number $x > 0$ possesses property (1).

A set V which certainly satisfies the condition imposed on V in theorem 4 is e.g. obtained in the following way. Let V_1 be an arbitrary subset of $[0,\infty)$ of positive measure μV_1 and let $V_m = m\, V_1$ $(m=1,2,\dots)$. Then, as is easily seen, $V = \bigcup_{m=1}^{\infty} V_k$ satisfies the conditions of theorem 4. One can even prove that the complement of V, say W, satisfies the relation

$$\lim_{t\to\infty} \mu\,(W \cap (t,t+p)) = 0 \qquad \text{for each } p > 0.$$

We further remark that our theorems give the solutions of problems A and B mentioned in the introduction. Actually, theorem 2' solves Problem A, even for Riemann integrable, non-negative functions. Next, a solution to Problem B is obtained from theorem 3 in the following way. Let V be the union of infinitely many disjoint intervals, such that $\mu V = \infty$ and (2) is valid for almost all x. Without restriction we may suppose that these intervals are all closed. Further let W_1 be the set of numbers x for which (1) holds. Then $\mu W_1 = 0$ and so there exists an open set W_1^* of finite measure which contains $\bigcup_{k=1}^{\infty} kW_1$. Then $V^* = V/W_1^*$ is a closed set of infinite measure,and

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6) Theorems 1 and 4 were obtained by Prof.J.F. Koksma, theorem 2' is due to Prof.N.G. de Bruijn, whereas the example leading to theorem 3 was given by Mr H. Kesten.

no number $x>0$ possesses property (1). Now the characteristic function of a closed set clearly is upper semi-continuous. It follows that the characteristic function of V^* satisfies the assertions of problem B.

Next we shall indicate a more special class of sets V, for which sharper conclusions hold. To this end we refer to a theorem in the field of diophantine approximation, proved by Koksma [1] and Cassels [2] [7]. This theorem can be formulated as follows:

Theorem 5. Let $\varphi(q)$ be a monotonely decreasing function of the integer variable $q>0$, which tends to zero for $q\to\infty$. Let $L=\sum_{q=1}^{\infty}\varphi(q)$. Then one has

a) If $L=\infty$, then for almost all $\alpha>0$ there are infinitely many pairs of positive integers p,q with $|q\alpha-p|<\varphi(q)$

b) If $L<\infty$, then for almost all $\alpha>0$ there are only finitely many pairs (p,q) with $|q\alpha-p|<\varphi(q)$.

From this result one can deduce the following

Theorem 6. Let $\varphi(q)$ be a monotonely decreasing function of the integer variable $q>0$, which tends to zero for $q\to\infty$. Let V be the union of the intervals $(q-\varphi(q),\ q+\varphi(q))$, where q runs through the positive integers. Then one has

α) If $\mu V=\infty$, then almost all numbers $x>0$ possess the property (1)

β) If $\mu V<\infty$, then for almost all numbers $x>0$ (2) holds.

In fact, let $\varphi(q)$ be a function of the type considered and let L and V be defined as above. Let a be an arbitrary positive number and suppose that $\mu V=\infty$. Then also $L=\infty$. Now apply a) to the function $a\varphi(q)$ instead of $\varphi(q)$. This learns that for almost all $\alpha>0$ one has

(5) $|q\alpha-p|<a\varphi(q)$ for infinitely many pairs (p,q).

A fortiori, (5) holds for almost all $\alpha>a$. Hence, dividing through by α and putting $\frac{1}{\alpha}=x$, one sees that for almost x with $0<x<\frac{1}{a}$ it is true that

(6) $|q-px|<\varphi(q)$ for infinitely many pairs (p,q).

This means that for almost all x with $0<x<\frac{1}{a}$ the set V defined in the theorem contains infinitely many multiples of x. Since a is arbitrary, this proves α).

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7) See also Cassels [3] , Ch.VII, Theorem I.

Next suppose that $\mu V < \infty$. Then also $L < \infty$. Then, by b), if $a > 0$ is arbitrary, one has for almost all $\alpha > 0$

(7) $|q\alpha - p| < a\,\varphi(q)$ only for finitely many pairs (p,q).

It follows that for almost all $x > \frac{1}{a}$ it is true that

(8) $|q-px| < \varphi(q)$ only for finitely many pairs (p,q).

From this, since a is arbitrary, β) follows.

Koksma [1] and Cassels [2] also generalized theorem 5 in the sense that they admitted q to take only certain sets of positive integral values. Thus they arrived to a statement of the following form [8]:

For a wide class of sequences of distinct positive integers $\{\lambda_q\}$ it is true that the inequality

$$|\lambda_q \alpha - p| < \varphi(q),$$

where $\varphi(q)$ is any monotonely decreasing function, has an infinity of integer solutions $p,q > 0$ for almost all or almost no $\alpha > 0$ according as $\sum \varphi(q)$ diverges or converges.

This statement leads to an analogous generalization of theorem 6. We do not carry this out. We rather draw attention to a peculiarity established by Cassels [2]. Cassels showed namely that there are sequences $\{\lambda_q\}$ of distinct positive integers (e.g. increasing sequences) with the following property [9]:

There is a monotonic function $\varphi(q)$ decreasing to zero such that $\sum_{q=1}^{\infty} \varphi(q) = \infty$ but that for each $a > 0$ the inequality

(9) $$0 \leq \lambda_q \alpha - p < a\,\varphi(q)$$

has an infinity of solutions for almost no α.

Now for any such sequence $\{\lambda_q\}$ and a corresponding function $\varphi(q)$ take V to be the union of the intervals $(\lambda_q, \lambda_q + \varphi(q))$. Further let W be the set of numbers $\alpha > 0$, such that, for all $a > 0$, the inequality (9) has an infinity of solutions. Then it follows from the above property that $\mu W = 0$. Next, $\mu V = \infty$. Finally, by a reasoning as in the above deduction of theorem 6, one sees that W is precisely the set of numbers $x > 0$ possessing property (1).

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8) See Koksma [1], Théorème 4; Cassels [2], Theorems III and IV.

9) See Cassels [2], Theorem VII. The sequences $\{\lambda_q\}$ having the property discussed can be specified explicitly.

The above set V satisfies the assertions stated in theorem 3 except possibly for the last one. But the proof given in the next section will be of a simpler nature. On the other hand, for suitable $\{\lambda_q\}$ and $\varphi(q)$, the set V constructed here is the union of a sequence of (disjoint) intervals which in their natural arrangement are of steadily decreasing length.

3. Proofs of theorems 1',2',3,4.

Proof of theorem 1'. Let $0 < a < b$ and let p be a positive integer with $pa > b$. Then, for any positive integer n, we have

$$\int_a^b \sum_{k=1}^{n} f(kx)dx = \sum_{k=1}^{n} \frac{1}{k}\int_{ka}^{kb} f(x)dx$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k}\int_{ka}^{pka} f(x)dx = \sum_{m=1}^{\infty} \sum_{k\leq m,\, pk>m} \frac{1}{k}\int_{ma}^{(m+1)a} f(x)dx$$

$$< 2\log p \sum_{m=1}^{\infty} \int_{ma}^{(m+1)a} f(x)dx = 2\log p \int_a^{\infty} f(x)dx.$$

Hence, if we put $\int_0^{\infty} f(x)dx = \gamma$ (so that $\gamma < \infty$), we have

$$\int_a^b \sum_{k=1}^{n} f(kx)dx \leq 2\gamma \log p \text{ for each positive integer } n.$$

Then, by a property of Lebesgue integrals [10], since $f(x) \geq 0$,

$$\lim_{n\to\infty}\int_a^b \sum_{k=1}^{n} f(kx)dx = \int_a^b \lim_{n\to\infty} \sum_{k=1}^{n} f(kx)dx = \int_a^b \sum_{k=1}^{\infty} f(kx)dx,$$

and moreover, $\sum_{k=1}^{\infty} f(kx)$ is finite for almost all x in the interval (a,b). Hence, since a and b are arbitrary, the last sum is finite for almost all $x > 0$. This proves the theorem.

Proof of theorem 2'. Let $0 < a < b$ and let N be a large positive integer. We have

$$\int_a^b \sum_{k=1}^{N} f(kx)dx = \sum_{k=1}^{N} \frac{1}{k}\int_{ka}^{kb} f(t)dt$$

$$= \int_a^{Nb} f(t)\left\{\sum_{1\leq k\leq N,\, ka\leq t,\, kb\geq t} \frac{1}{k}\right\}dt$$

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10) See E.C. Titchmarsh, Theory of functions, Oxford 1939, assertion (i) of theorem 10.82. Compare also footnote 13).

$$\geqq \int_a^{Na} f(t) \sum_{t/b \leqq k \leqq t/a} \frac{1}{k} \, dt .$$

Now, for fixed a and b, there is a positive number t_o such that

$$\sum_{t/b \leqq k \leqq t/a} \frac{1}{k} > \frac{1}{2} \log \frac{b}{a} \quad \text{if } t \geqq t_o .$$

Then, for $N \geqq t_o/a$,

$$\int_a^b \sum_{k=1}^{N} f(kx)dx > \frac{1}{2} \log \frac{b}{a} \int_{t_o}^{Na} f(t)dt.$$

The last integral tends to ∞ for $N \to \infty$, by the hypotheses of the theorem. So we may conclude that for each pair of positive numbers a and b with $b > a$ we have

(10) $$\lim_{N \to \infty} \int_a^b \sum_{k=1}^{N} f(kx)dx = \infty .$$

We now consider the sets V_m defined by

(11) $$V_m = \left\{ x \mid \sum_{k=1}^{\infty} f(kx) < m \right\} \qquad (m=1,2,\ldots).$$

We shall show that each set V_m is nowhere dense, i.e. that, for arbitrary m, each interval on the positive real axis contains a subinterval which is disjoint with V_m. Suppose the contrary is true. Then, for some m,a,b, the set V_m is everywhere dense in the interval (a,b). Put $W=V_m \cap (a,b)$. Then W is a subset of (a,b), which is everywhere dense in (a,b), whereas

$$\sum_{k=1}^{\infty} f(kx) < m \quad \text{for } x \in W.$$

A fortiori, if N is any positive integer,

$$\sum_{k=1}^{N} f(kx) < m \text{ for } x \in W.$$

Hence, since f(x) is Riemann integrable,

$$\int_a^b \sum_{k=1}^{N} f(kx)dx \leqq m\ (b-a)$$

for each positive integer N. This contradicts (10) and so proves that each set V_m is nowhere dense.

Consequently, by a well-known property of nowhere dense point sets, the union of the sets V_m does not cover the whole interval $[0,\infty)$ [11]. In other words, there is a number $x > 0$ not belonging to any set V_m. So, for this number x, $\sum_{k=1}^{\infty} f(kx)=\infty$. This proves the theorem.

Proof of theorem 3. We consider an arbitrary sequence of positive numbers ε_k $(k=1,2,\ldots)$ satisfying

(12) $$0 < \varepsilon_k < 1 , \quad \sum_{k=1}^{\infty} \varepsilon_k < \infty .$$

For any such sequence $\{\varepsilon_k\}$ we can find a sequence of positive integers n_k $(k=1,2,\ldots)$, such that

(13) $$n_k > n_{k-1} + 1 , \quad \sum_{k=2}^{\infty} \varepsilon_k \log \frac{n_k}{n_{k-1}+1} = \infty .$$

Now let V be the union of the intervals V_{km} given by

(14) $$V_{km} = \left(\frac{n_k}{m} , \frac{n_k+\varepsilon_k}{m} \right) ,$$

where $m=1,2,\ldots,\left[\frac{n_k}{n_{k-1}+1} \right]$; $k=1,2,\ldots (n_0=0)$.

For fixed k the intervals V_{km}, with $m \leq n_k$, are disjoint, because $0 < \varepsilon_k < 1$. Further, if k,m and m' are positive integers with

$$m \leq n_k/(n_{k-1}+1) , \quad m' \leq n_{k+1}/(n_k+1) ,$$

then $$\frac{n_k+\varepsilon_k}{m} < n_k+1 = \frac{n_{k+1}}{n_{k+1}/(n_k+1)} \leq \frac{n_{k+1}}{m'}$$

and so, for fixed k, those intervals (14) for which m has one of the assigned values are lying to the left of those with index k+1 instead of k and one of the corresponding values of the index m. Hence the intervals V_{km}, which together constitute V, are mutually disjoint. Consequently,

$$\mu V = \sum_{k=1}^{\infty} \varepsilon_k \sum_{m=1}^{[n_k/(n_{k-1}+1)]} \frac{1}{m} > \sum_{k=2}^{\infty} \varepsilon_k \log \frac{n_k}{n_{k-1}+1} ,$$

and so, by (13), $\mu V = \infty$.

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11) Actually, the complement of this union is everywhere dense.

We next prove that V has the remaining properties desired. Let us denote by V^* the union of the intervals $(n_k, n_k+\varepsilon_k)$ $(k=1,2,\ldots)$ and by W_1, W_2, W_3 the sets of numbers $x>0$ possessing the properties (1), (2), (3) respectively. It is clear, from the construction of V, that $x \notin W_2$, i.e. $x \in W_1$, if and only if infinitely many multiples of x belong to V^*. Next, by (2), $\mu V^* = \sum_{k=1}^{\infty} \varepsilon_k < \infty$. Then it follows from theorem 1 that W_1 has measure zero. This proves that almost all $x>0$ possess property (2).

Further, $x \notin W_3$, if and only if some multiple of x belongs to V^*. Now consider an arbitrary interval (a,b), where $b>a>0$. The set of numbers $x>0$, which belong to (a,b) and have a multiple in an assigned interval $(n_k, n_k+\varepsilon_k)$, has measure $\leq$

$$\varepsilon_k \sum_{mb \geq n_k,\ ma \leq n_k+\varepsilon_k} \frac{1}{m} < \left(2 + \log \frac{b}{a}\right) \varepsilon_k .$$

Hence the set of numbers $x>0$, which belong to (a,b) and have a multiple in V^*, has measure $< (2+\log \frac{b}{a}) \sum_{k=1}^{\infty} \varepsilon_k$. So we have

$$\mu(W_3 \cap (a,b)) > b-a-\left(2+\log \frac{b}{a}\right) \sum_{k=1}^{\infty} \varepsilon_k .$$

Here the right-hand member can be made arbitrarily large by a suitable choice of a and b. Hence $\mu W_3 = \infty$. This completes the proof of theorem 3.

<u>Proof of theorem 4</u>. For $p=1,2,\ldots$ let U_p be defined by

$$U_p = \left\{ x \mid 0 < x < 1,\ kx \notin V \text{ for } k > p \right\}.$$

Suppose that $\mu\left(\bigcup_{p=1}^{\infty} U_p\right) > 0$. Then there is a positive integer p with $\mu U_p > 0$. Then there are a number c with $0<c<1$ and a number $\delta > 0$, such that the density of U_p on each interval which contains the point c and is itself contained in the interval $(c-\delta, c+\delta)$, is greater than $1-\frac{1}{2}\rho$ [12].

Now choose a positive integer k with

$$k\delta > 1 , \quad k > p$$

and take a positive number $A > k$ such that V has density $\geq \rho$ on the

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12) See e.g. J.F. Koksma, Diophantische Approximationen, Berlin Springer 1936, Satz 31, p.44 and Bemerkung IV, p.45.

interval $(A,A+1)$. There exists a positive integer l with $A < lc < A+1$, since $0 < c < 1$. Then $l > k$, because $l > lc > A > k$. Hence $l\delta > 1$, hence

$$l(c-\delta) < A < A+1 < l(c+\delta).$$

Thus we find that c belongs to the interval $(A/l, (A+1)/l)$ and that the last interval is contained in $(c-\delta, c+\delta)$.

In virtue of the last fact, and by the choice of c and δ, the density of U_p on the interval $(A/l, (A+1)/l)$, hence also the density of $l\,U_p$ on $(A,A+1)$, is at least $1-\frac{1}{2}\rho$. Further, by the choice of A, the density of V on this interval is at least ρ. Since $l\,U_p$ and V are disjoint, if $l > p$, this is a contradiction. Hence we must have $\mu(\bigcup_{p=1}^{\infty} U_p)=0$. This means that almost no x in $(0,1)$ has only finitely many multiples in V, and so proves the theorem.

4. Generalizations for n-dimensional point sets. In the following n is a fixed integer ≥ 2 and V is an arbitrary Lebesgue measurable set in the n-dimensional Euclidean space R_n of points $x=(x_1,x_2,\dots,x_n)$. We shall use vector notation. Further, n points $x^{(1)},x^{(2)},\dots,x^{(n)}$ in R_n will be called independent if they do not lie in an $(n-1)$-dimensional hyperplane through the origin O.

Our aim is to generalize the theorems 1-4. In doing this, instead of the set of multiples of a number $x > 0$, we shall have to do with the set of all points x of the form

$$x = u_1x^{(1)} + u_2x^{(2)} + \dots + u_n x^{(n)} ,$$

where $\{x^{(1)},x^{(2)},\dots,x^{(n)}\}$ is any system of n independent points and $u_1,u_2,\dots,u_n$ take all integral values. In other words, we shall have to do with lattices Λ, with an arbitrary basis $\{x^{(1)},x^{(2)},\dots,x^{(n)}\}$. We adopt the following notations:

$\mathcal{U}$ = lattice of the points with integral coordinates
X = matrix with columns $x^{(1)},x^{(2)},\dots,x^{(n)}$, or also the affine transformation of R_n determined by this matrix.

Then the lattice Λ with basis $\{x^{(1)},x^{(2)},\dots,x^{(n)}\}$ can be written in the form $\Lambda = X\mathcal{U}$.

For what follows we have to define measure and integration in the set of lattices. This was done by Siegel [4] in the following way. Let Ω be the set of all nonsingular $n \times n$ matrices and let Γ be the set of all integral unimodular matrices in Ω. Two lattices

$\Lambda_1 = X\mathcal{U}$ and $\Lambda_2 = Y\mathcal{U}$, where $X, Y \in \Omega$, are identical, if and only if there exists a matrix $C \in \Gamma$ such that $X = YC$. Hence the set of all lattices can be identified with the factor group Ω/Γ. As a fundamental region of Ω with respect to Γ one can take the set of matrices $X \in \Omega$, for which sp $X \geq 0$ and the positive quadratic form with coefficient matrix $S = X'X$ is reduced in the sense of Minkowski. Let us denote this region by G, and by G^* the set of matrices $X \in \Omega$ with $X \in G$, $|\det X| = 1$. The Euclidean metric in the n^2-dimensional space of all $n \times n$ matrices X induces a measure in G. Further, if $\overline{G}$ is the set of matrices $X \in \Omega$ with $X \in G$, $|\det X| \leq 1$, then the (n^2-1)-uple integral of some function F(X) over G^* is given by

$$(15) \qquad \int_{G^*} F(X)\, dX = n^2 \int_{\overline{G}} F(|\det X|^{-1/n} X)\, dX.$$

The integrals in (15) can be interpreted as integrals over certain sets of lattices. It now has a sense to make "almost all" or "almost no" statements about lattices. Siegel proves that the (n^2-1)-dimensional volume of G^*, given by the integral (15), if one takes $F(X) \equiv 1$, has a finite value, v say (the quantity v can be computed explicitly). Next, he proves

<u>Theorem 7.</u> Let f(x) be a bounded, Riemann integrable function on R_n vanishing outside some finite region. For arbitrary $X \in \Omega$, let $F(X) = \sum_{u \in \mathcal{U}} f(Xu)$. Then one has

$$(16) \qquad \int_{G^*} F(X)\, dX = v \int_{R_n} f(x)\, dx.$$

From (16) one can immediately derive a similar relation for the integral of F(X) over $\overline{G}$ or $t\overline{G}$, where $t > 0$. If, for $t > 0$, we put

$$f(ty) = g(y), \quad \sum_{u \in \mathcal{U}} g(Xu) = G(X),$$

then we have

$$\int_{R_n} f(x)dx = t^n \int_{R_n} g(y)dy, \quad G(X) = F(tX),$$

$$\int_{tG^*} F(X)dX = t^{n^2-1} \int_{G^*} G(Y)dY.$$

Hence, applying (16) to g(y), we get

$$\int_{tG^*} F(X)dX = t^{n^2-1} v \int_{R_n} g(y)dy = t^{n^2-n-1} v \int_{R_n} f(x)dx.$$

Hence,

$$(16') \qquad \int_{t\overline{G}} F(X)dX = \int_0^t s^{n^2-n-1} ds \,.\, v \int_{R_n} f(x)dx = \frac{v}{n^2-n} t^{n^2-n} \int_{R_n} f(x)dx.$$

We now can state and prove the following generalization of theorem 1':

<u>Theorem 8</u>. Let $f(x)$ be a non-negative, Riemann integrable function, defined on R_n. Suppose that $\int_{R_n} f(x)dx < \infty$. Then for almost all lattices Λ one has $\sum_{x \in \Lambda} f(x) < \infty$.

<u>Proof.</u> Let N be a large positive number and let t be a positive number. Let $|x| = \max |x_i|$, if $x=(x_1,x_2,\dots,x_n)$, and put

$$\int_{R_n} f(x)dx = \gamma \quad (\text{so that } \gamma < \infty),$$

$$f_N(x) = \begin{cases} f(x) & \text{if } |x| \leq N \\ 0 & \text{if } |x| > N, \end{cases}$$

$$F_N(X) = \sum_{u \in \mathcal{U}} f_N(Xu) \qquad (X \in \Omega).$$

In virtue of the relation (16')

$$\int_{t\overline{G}} F_N(X)dX = \frac{v}{n^2-n} t^{n^2-n} \int_{R_n} f_N(x)dx$$

$$\leq \frac{v}{n^2-n} t^{n^2-n} \int_{R_n} f(x)dx = \gamma \frac{v}{n^2-n} t^{n^2-n}.$$

Now $F_N(X)$ is a steadily increasing function of N. Hence we have

$$\lim_{N \to \infty} \int_{t\overline{G}} F_N(X)dX \leq \gamma \frac{v}{n^2-n} t^{n^2-n},$$

where the limit on the left exists. Next, in virtue of the monotony of the sequence $\{F_N(X)\}$ and a well-known property of Lebesgue integrals [13],

$$\int_{t\overline{G}} \lim_{N \to \infty} F_N(X)dX = \lim_{N \to \infty} \int_{t\overline{G}} F_N(X)dX.$$

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13) Cf.e.g. A.C. Zaanen, Linear Analysis, Amsterdam Groningen 1953, Ch.3, § 3, Theorem 2 (p.43).

It follows that $\lim_{N\to\infty} F_N(X) = F(X)$ is finite almost everywhere in $t\bar{G}$. Since $\sum_{x\in\Lambda} f(x) = \sum_{u\in\mathcal{U}} f(Xu) = F(X)$, if $\Lambda = X\mathcal{U}$, and $t > 0$ is arbitrary, this proves the theorem.

Next, we generalize theorem 2'. Here our considerations will involve a reasoning similar to a reasoning of Davenport and Rogers [5] in their work on the theorem of Minkowski-Hlawka [14]. We have the following

<u>Theorem 9</u>. Let $f(x)$ be a non-negative, Riemann integrable function, defined on R_n. Suppose that $\int_{R_n} f(x)dx = \infty$. Then there is a lattice Λ with determinant 1 such that

$$\sum_{x\in\Lambda} f(x) = \infty .$$

<u>Proof</u>. We can cover R_n by finitely many cones with vertex at 0 and given semi-angle $< \pi/2$. Further, the conditions and the assertion of the theorem are not affected if one applies a linear transformation to x. Then it is no loss of generality to suppose that $f(x)$ vanishes outside a cone

$$\mathcal{C} : x_1^2 + x_2^2 + \dots + x_{n-1}^2 \leqq \beta^2 x_n^2,$$

where β^2 is some positive number. This cone intersects each hyperplane $x_n = a$ $(a > 0)$ in a bounded set, viz. a circle with radius βa. Now consider the integral

$$(17) \qquad V(a) = \int\!\!\int\!\!\dots\!\int_{x_1^2+x_2^2+\dots+x_{n-1}^2 \leqq \beta^2 a^2} f(x_1,x_2,\dots,x_{n-1},a)dx_1dx_2\dots dx_{n-1}.$$

By the theory of Riemann integrals, if A is any positive number, this integral exists for all a with $0 < a < A$, except for a set of Jordan measure zero.[15] Thus, by (17), the quantity $V(a)$ is defined for all $a > 0$, apart from a certain exceptional set S, which on each finite interval has Jordan measure zero. It is convenient to put

$$(17') \qquad V(a) = 0 \quad \text{if } a \in S.$$

The function $V(a)$, thus defined, is $\geqq 0$ for all $a > 0$. Also, by the theory of Riemann integrals, this function is Riemann integrable and

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14) The subsequent formula (19) is analogous to lemma 1 in the paper cited.

15) See E.W. Hobson, The theory of functions of a real variable, vol.I 3d ed., Cambridge 1927, pp.509-516.

we have [15)]

$$\iint\cdots\int_{x_n \leqq a} f(x_1,x_2,\ldots,x_{n-1},x_n)\, dx_1 dx_2\ldots dx_{n-1}dx_n$$

$$= \iint\cdots\int_{\substack{x_1^2+x_2^2+\ldots+x_{n-1}^2 \leqq \beta^2 x_n^2 \\ x_n \leqq a}} f(x_1,x_2,\ldots,x_{n-1},x_n)dx_1dx_2\ldots dx_{n-1}dx_n$$

$$= \int_0^a V(y)dy \qquad (a > 0).$$

Then, in virtue of our hypotheses, the last integral tends to infinity for $a \longrightarrow \infty$. In other words,

$$(18) \qquad \int_0^{\infty} V(y)dy = \infty .$$

We now consider an arbitrary (n-1)-dimensional lattice $\mathcal{L}$ in the hyperplane $x_n=0$, of determinant $d(\mathcal{L})$. We put $\alpha=(d(\mathcal{L}))^{-1}$ and denote by g any point of the form $g=(g_1,g_2,\ldots,g_{n-1}, \alpha)$. We shall show that we can choose g in such a way that, if Λ is the lattice generated by $\mathcal{L}$ and the point g,

$$(19) \qquad \sum_{x\in\Lambda\ ,x_n\neq 0} f(x) \geqq \alpha \sum_{m=1} V(m\alpha),$$

where V(a) is defined by (17) and (17') and where it is asserted that the sum on the left is infinite if the sum on the right is infinite.

If we apply a linear transformation to the variables $x_1,x_2,\ldots,x_{n-1}$, then the inequality (19) goes over in an inequality of the same form. Hence it is sufficient to prove (19) in the case that $\mathcal{L}$ is the lattice of points in $x_n=0$ with integral coordinates. Then $\alpha=1$ and the general point of Λ is $(u_1+mg_1,u_2+mg_2,\ldots,u_{n-1}+mg_{n-1},m)$, where $u_1,u_2,\ldots,u_{n-1}$, m are integers. Hence, since $f(x)=0$ for $x_n < 0$, the sum on the left of (19) is

$$S(g_1,g_2,\ldots,g_{n-1}) = \sum_{m=1}^{\infty} \sum_{u_1,u_2,\ldots,u_{n-1}} f(u_1+mg_1,u_2+mg_2,\ldots,u_{n-1}+mg_{n-1},m)$$

Next, $V(m)=0$ or

$$V(m) = \iint\limits_{x_1^2+x_2^2+\ldots+x_{n-1}^2 \leqq (m/3)^2} \cdots \int f(x_1,x_2,\ldots,x_{n-1},m)\,dx_1dx_2\ldots dx_{n-1}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}$$

$$= m^{-n+1}\int_0^m \int_0^m \cdots \int_0^m \sum_{u_1,u_2,\ldots,u_{n-1}} f(u_1+g_1,u_2+g_2,\ldots u_n+g_{n-1},m)\,dg_1dg_2\ldots dg_{n-1}$$

$$= \int_0^1 \int_0^1 \cdots \int_0^1 \sum_{u_1,u_2,\ldots,u_{n-1}} f(u_1+mg_1,u_2+mg_2,\ldots,u_{n-1}+mg_{n-1},m)\,dg_1dg_2\ldots dg_{n-1} .$$

Denoting the sum in the last member by $S_m(g_1,g_2,\ldots,g_{n-1})$, we have $S(g_1,g_2,\ldots,g_{n-1}) = \sum_{m=1}^{\infty} S_m(g_1,g_2,\ldots,g_{n-1})$. Hence, taking the lower integral of $S(g_1,g_2,\ldots,g_{n-1})$ over the unit cube, we have [16)]

$$\underline{\int_0^1\int_0^1 \cdots \int_0^1} S(g_1,g_2,\ldots,g_{n-1})\,dg_1dg_2\ldots dg_{n-1} \geqq \sum_{m=1}^{\infty} V(m),$$

no matter whether the expressions on the left and the right are finite or infinite. This proves (19).

The proof of the theorem is now easily completed. The function $V(a)$ was shown to be non-negative and Riemann integrable and to satisfy (18). Hence, in virtue of the corresponding theorem for the one-dimensional case, viz. theorem 2, there is an $\alpha > 0$ such that $\sum_{m=1}^{\infty} V(m\alpha)=\infty$. Hence, if $\mathcal{L}$ is any lattice in the hyperplane $x_n=0$, of determinant $d(\mathcal{L})=1/\alpha$, then, for a suitable point $g=(g_1,g_2,\ldots,g_{n-1},\alpha)$, we have

$$\sum_{x\in\Lambda} f(x) \geqq \sum_{x\in\Lambda,\,x_n\neq 0} f(x) = \infty .$$

This proves the theorem.

We now come to the analogues of theorems 3 and 4. It turns out that Cassels' considerations mentioned at the end of section 2 are well

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16) Though $S(g_1,g_2,\ldots,g_{n-1})$ may be infinite for one or more sets $(g_1,g_2,\ldots,g_{n-1})$, this lower integral is defined without ambiguity, since we have to do with non-negative functions only.

suited in order to generalize theorem 3. We shall prove

Theorem 10. There exists a Jordan measurable set V in R_n, of infinite measure, which has the property that almost all lattices have only finitely many points in V.

Proof. Let $\{\lambda_m\}$ be any ascending sequence of positive integers, and let μ_m denote the number of fractions j/λ_m $(0 < j < \lambda_m)$ which are not of the form k/λ_q, $q < m$. Now, as Cassels proved, we can choose the sequence $\{\lambda_m\}$ in such a way that [18]

(20) $\quad \sum_{m \leqq M} \mu_m/\lambda_m = o(M)$ as $M \longrightarrow \infty$.

Further, for any such sequence $\{\lambda_m\}$, there exists a sequence of positive numbers $\psi(m)$ decreasing to zero such that $\sum \psi(m)$ is divergent, but

$$\sum \frac{\mu_m}{\lambda_m} \psi(m)$$

is convergent. We now consider the set V in R_n which is the union of the parallelepipeds

$$V_m = \lambda_m e^{(n)} + P_m \qquad (m=1,2,\ldots),$$

where $e^{(n)}$ is the nth unit vector $(0,0,\ldots,1)$ and P_m is the set of points $y=(y_1,y_2,\ldots,y_n)$ with

$$|y_i| \leqq 1 \quad (i=1,2,\ldots,n-1) \ , \quad |y_n| \leqq \psi(m).$$

Then V_m has content $2^n\psi(m)$, and so V is of infinite content. We shall prove that almost all lattices have only finitely many points in V.

Let $\bar{\Lambda}$ be an arbitrary lattice, with basis $\{\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(n)}\}$. By permuting, if necessary, the points $\bar{x}^{(i)}$ we may obtain that

$$\det(\bar{x}^{(1)}, \bar{x}^{(2)},\ldots,\bar{x}^{(n-1)}, e^{(n)}) \neq 0.$$

Then there are a positive number r and n-1 neighbourhoods $N_1,N_2,\ldots,N_{n-1}$ of $\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(n-1)}$ respectively, such that

$$\det(x^{(1)},x^{(2)},\ldots,x^{(n-1)},e^{(n)}) \neq 0 \text{ if } x^{(i)} \in N_i \ (i=1,2,\ldots,n-1)$$

and even for all systems $\{x^{(1)},x^{(2)},\ldots,x^{(n-1)}\}$ with $x^{(i)} \in N_i$, the parallelotope consisting of the points

$$x=\eta_1 x^{(1)}+\eta_2 x^{(2)}+\ldots+\eta_{n-1} x^{(n-1)}+\eta_n e^{(n)} \quad (|\eta_i| \leqq 1 \text{ for } i=1,2,\ldots,n)$$

contains the sphere

$$x_1^2+x_2^2+\ldots+x_n^2 \leqq r^2.$$

18) See Cassels [2], theorem VI and lemma 7.

Further let a be an arbitrary positive number and let H denote the set of lattices $\Lambda = X\mathcal{U}$, with basis $\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$, such that

$$x^{(i)} \in N_i \quad (i=1,2,\ldots,n-1),$$

$$\frac{\det (x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}, x^{(n)})}{\det (x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}, e^{(n)})} = \rho \quad \text{with } a \leqq \rho \leqq 2a.$$

Here the point $x^{(n)}$ may be chosen, in a unique way, such that we have a relation of the following form:

$$(21) \qquad \rho e^{(n)} = \theta_1 x^{(1)} + \theta_2 x^{(2)} + \ldots + \theta_{n-1} x^{(n-1)} + x^{(n)},$$

with $0 \leqq \theta_i < 1$ $(i=1,2,\ldots,n-1)$. For fixed $x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}$, we shall denote by $Q=Q(x^{(1)}, x^{(2)}, \ldots, x^{(n-1)})$ the parallelotope of points $x^{(n)}$, such that (21) holds and Λ belongs to H. Further, we shall write $\frac{1}{\rho} = \beta$, so that $\frac{1}{2a} \leqq \beta \leqq \frac{1}{a}$.

Next, if $\Lambda = X\mathcal{U}$ belongs to H, we denote by $\gamma_m(X)$ the number of points $u=(u_1, u_2, \ldots, u_n) \in \mathcal{U}$ for which

$$Xu = u_1 x^{(1)} + u_2 x^{(2)} + \ldots + u_n x^{(n)} \in V_m$$

and the fraction u_n/λ_m is not of the form v/λ_k, $k < m$. Finally, we put

$$\Gamma_M(X) = \sum_{m \leqq M} \gamma_m(X) \qquad (M=1,2,\ldots).$$

We wish to derive an estimate for $\int_H \Gamma_M(X)dX$. First, we deduce, for arbitrary $\Lambda \in H$, an estimate for the components of any integral vector u with $Xu \in V_m$. We have

$$u_1 x^{(1)} + u_2 x^{(2)} + \ldots + u_{n-1} x^{(n-1)} + u_n x^{(n)} = \lambda_m e^{(n)} + y,$$

where $y \in P_m$. Further, P_m is contained in the sphere $x_1^2 + x_2^2 + \ldots + x_n^2 \leqq n$, and so $y \in P_m$ can be written as

$$y = \frac{\sqrt{n}}{r} (\eta_1 x^{(1)} + \eta_2 x^{(2)} + \ldots + \eta_{n-1} x^{(n-1)} + \eta_n e^{(n)}),$$

where $|\eta_i| \leqq 1$ for $i=1,2,\ldots,n$. Hence, by (21),

$$u_1 x^{(1)} + u_2 x^{(2)} + \ldots + u_{n-1} x^{(n-1)} + u_n x^{(n)}$$

$$= (\lambda_m \beta \theta_1 + \frac{\sqrt{n}}{r} \eta_n \beta \theta_1 + \frac{\sqrt{n}}{r} \eta_1) x^{(1)} + \ldots .$$

$$+ (\lambda_m \beta \theta_{n-1} + \frac{\sqrt{n}}{r} \eta_n \beta \theta_{n-1} + \frac{\sqrt{n}}{r} \eta_{n-1}) x^{(n-1)} + (\lambda_m + \frac{\sqrt{n}}{r} \eta_n) \beta x^{(n)}.$$

Hence

$$u_i = \lambda_m/\beta\,\theta_i + \frac{\sqrt{n}}{r}\,\eta_n/\beta\,\theta_i + \frac{\sqrt{n}}{r}\,\eta_i \quad (i=1,2,\ldots,n-1),$$

$$u_n = (\lambda_m + \frac{\sqrt{n}}{r}\,\eta_n)/\beta,$$

and so

$$-\frac{(a+1)\sqrt{n}}{ra} < u_i < \frac{1}{a}\lambda_m + \frac{(a+1)\sqrt{n}}{ra} \quad (i=1,2,\ldots,n-1),$$

$$\frac{1}{2a}\left(\lambda_m - \frac{\sqrt{n}}{r}\right) \leqq u_n \leqq \frac{1}{a}\left(\lambda_m + \frac{\sqrt{n}}{r}\right).$$

Next, let $\chi_m(x)$ be the characteristic function of V_m. Then,

$$\gamma_m(X) = \sum_{u_1,u_2,\ldots,u_n}{}^{*} \chi_m(Xu),$$

where the asterisk indicates that the summation is extended only over the integral vectors $u=(u_1,u_2,\ldots,u_n)$ for which the fraction u_n/λ_m is not of the form v/λ_k, $k<m$ (so that $u_n \neq 0$). For fixed u_i with $u_n \neq 0$, we have

$$\int_Q \chi_m(Xu)dx^{(n)} = \int_Q \chi_m(u_1x^{(1)}+u_2x^{(2)}+\ldots+u_nx^{(n)})dx^{(n)}$$

$$\leqq \int_{R_n} \chi_m(u_1x^{(1)}+u_2x^{(2)}+\ldots+u_nx^{(n)})dx^{(n)}$$

$$= \left|\frac{1}{u_n{}^n}\right| \cdot 2^n\psi(m).$$

Hence

$$\int_Q \gamma_m(X)dx^{(n)} = \sum_{u_n}{}^{*} \sum_{u_1,u_2,\ldots,u_{n-1}} \int_Q \chi_m(Xu)dx^{(n)}$$

$$\leqq \sum_{u_n}{}^{*} \left|\frac{1}{u_n}\right|^n \left(\frac{1}{a}\lambda_m + \frac{2(a+1)\sqrt{n}}{ra} + 1\right)^{n-1} 2^n\psi(m).$$

Now, in the last sum, the number of terms is $< \frac{2\sqrt{n}}{ar} + \frac{a+1}{a}\mu_m$, and further $\left|\frac{1}{u_n}\right| \leqq \frac{2a}{\lambda_m} + O(\frac{1}{\lambda_m{}^2})$, where the constant in the O-term does not depend on m. So we find

$$\int_Q \gamma_m(X)dx^{(n)} \leqq \left\{4^n(a+1)(\mu_m + O(1))\left(\frac{1}{\lambda_m} + O(\frac{1}{\lambda_m{}^2})\right)\right\}\psi(m).$$

This gives

(22) $$\int_H \Gamma_M(X)dX = O\Big(\sum_{m \leqq M} \frac{\mu_m}{\lambda_m}\,\psi(m)\Big),$$

where the constants in the O-symbol only depends on the numbers a and r and the contents of $N_1, N_2, \ldots, N_{n-1}$, but not on M.

Consequently, by the choice of the $\psi(m)$, $\int_H \Gamma_M(X)dX$ is bounded by a constant not depending on M. Let M tend to infinity. Then it follows that almost all lattices $\Lambda = X\mathcal{U}$ in H contain only finitely many points Xu such that, for some m, $Xu \in V_m$ and u_n/λ_m is not of the form v/λ_k, $k < m$. Here, if for a given lattice $\Lambda = X\mathcal{U}$, the number β is defined by (21), then u_n satisfies $|\beta\lambda_m - u_n| \leq \frac{\sqrt{n}}{r}\beta$. If β is irrational, then a given value of u_n/λ_m corresponds to at most a finite number of solutions of this inequality. Further, for fixed m and u_n, there are only finitely many integral vectors $u=(u_1,u_2,\ldots,u_n)$ with $Xu \in V_m$. We may conclude that almost all lattices Λ in H have only finitely many points in V.

By the arbitrariness of the points $\overline{x}^{(i)}$ and the number a, the assertion of the theorem now follows.

For the generalization of theorem 4 we need the following

Lemma. Let C be the cube $|x_i| \leq 1$ $(i=1,2,\ldots,n)$ and let Λ be a lattice which has a basis contained in C. Then each cube of the form

$$a_i \leq x_i \leq a_i + n \quad (i=1,2,\ldots,n),$$

where a_i are arbitrary real numbers, contains a point of Λ.

Proof. We use some well-known concepts from the geometry of numbers. In particular, we consider the n successive minima of the cube C with respect to the given lattice Λ, say $\lambda_1, \lambda_2, \ldots, \lambda_n$. Thus λ_i is the smallest positive number such that $\lambda_i C$ contains at least i linearly independent points of Λ $(i=1,2,\ldots,n)$. Next, we consider the inhomogeneous minimum of C with respect to Λ, say σ; it is the smallest positive number such that for each $g \in R_n$ the cube $\sigma C + g$ contains a point of Λ. We can easily obtain an estimate for σ.

First, since C contains a basis, and so a set of n linearly independent points of Λ, the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all ≤ 1. Secondly, by a well-known result [19],

$$\sigma \leq \tfrac{1}{2}(\lambda_1 + \lambda_2 + \ldots + \lambda_n).$$

Hence, $\sigma \leq n/2$. This implies the truth of the lemma.

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19) See H. Minkowski, Geometrie der Zahlen, Leipzig Berlin 1910, Kap.V, or Jarník [6].

We now prove

Theorem 11. Let V be a Lebesgue measurable set in R_n. Let E be the cube $0 \leq x_i \leq 1$ $(i=1,2,\ldots,n)$. Let ρ be a positive number < 1 and suppose that there are infinitely many disjoint cubes $g^{(k)}+E$ $(g^{(k)} \in R_n)$ on which V has density $\geq \rho$. Then almost all lattices have infinitely many points in V.

Proof. Let C be the cube $|x_i| \leq 1$ $(i=1,2,\ldots,n)$. We consider the set H of all systems of n points $\{x^{(1)},x^{(2)},\ldots,x^{(n)}\}$ (or matrices with columns $x^{(1)},x^{(2)},\ldots,x^{(n)}$) for which

$$\det (x^{(1)},x^{(2)},\ldots,x^{(n)}) \neq 0, \; x^{(i)} \in C \; (i=1,2,\ldots,n).$$

We use the natural measure in H considered as a set in n^2-dimensional space. Further, we arrange in some order the points $u=(u_1,u_2,\ldots,u_n) \in \mathcal{U}$ denoting them in this order by $u^{(1)},u^{(2)},\ldots$. Next, for $p=1,2,\ldots$, we denote by W_p the set of all $X \in H$ for which

$$Xu^{(q)} \notin V \quad \text{for } q > p.$$

We first prove that the sets W_p are measurable. For fixed u, let Z(u) be the set of all $X \in H$ with $Xu=u_1x^{(1)}+u_2x^{(2)}+\ldots+u_nx^{(n)} \in V$. Here V is a measurable set in R_n and the set of the X for which Xu is a fixed point in V is the intersection of H and a certain (n^2-n)-dimensional plane of given direction. Hence Z(u) is measurable. Hence $W_p = \bigcap_{q>p} (H/Z(u^{(q)}))$ is measurable.

Now suppose that $\mu W_p > 0$, for some p. We shall show that this leads to a contradiction. For fixed $x^{(1)},x^{(2)},\ldots,x^{(n-1)}$, let $S=S(x^{(1)},x^{(2)},\ldots,x^{(n-1)})$ denote the set of points $x^{(n)} \in W$ with $\{x^{(1)},x^{(2)},\ldots,x^{(n)}\} \in W_p$. Since W_p is measurable, this set is measurable for almost all $x^{(1)},x^{(2)},\ldots,x^{(n-1)}$. Further, since $\mu W_p > 0$, we have $\mu S > 0$ on a set of systems $\{x^{(1)},x^{(2)},\ldots,x^{(n-1)}\}$ of positive measure. Hence we can choose n-1 linearly independent points $\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(n-1)} \in W$ in such a way that

1) the subset $S(\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(n-1)})$ of W has a positive Lebesgue measure

2) the distances of the given points $g^{(k)}$ from the (n-1)-dimensional hyperplane through 0, $\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(n-1)}$ are not bounded.

Next, as in the proof of theorem 4, in virtue of 1), there exist a point $\bar{x}^{(n)}=(a_1,a_2,\ldots,a_n) \in W$ and a postive number δ with the following properties:

a) the cube $|x_i - a_i| \leq \delta$ $(i=1,2,\ldots,n)$ is contained in W and for each point x in this cube $\det(\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(n-1)},x)\neq 0$

b) on each cube $b_i \leq x_i \leq c_i$ $(i=1,2,\ldots,n)$ with $a_i-\delta \leq b_i \leq a_i \leq c_i \leq a_i+\delta$ $(i=1,2,\ldots,n)$ the set $S(\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(n-1)})$ has density $\geq 1-\frac{1}{2}n^{-n}\rho$.

Let $\bar{\Lambda}$ be the lattice with basis $\{\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(n)}\}$. Consider the cubes $g^{(k)}+nE$. On each of these cubes, in virtue of our hypotheses, the set V has density $\geq n^{-n}\rho$. Further, by the lemma, each of these cubes contains a point of $\bar{\Lambda}$. Thus, for k=1,2,..., let $g^{(k)}+nE$ contain the point $\bar{X}u^{(l)}=u_{1l}\bar{x}^{(1)}+u_{2l}\bar{x}^{(2)}+\ldots+u_{nl}\bar{x}^{(n)}$, where l depends on k, say $l=\varphi(k)$. In virtue of 2), the sequence of numbers u_{nl} $(l=\varphi(k);\ k=1,2,\ldots)$ is not bounded. Hence the values $l=\varphi/k)$ also are not bounded.

Now choose k such that $l>p$ and $u_{nl}\delta > n$. If $x^{(n)}$ describes the cube $|x_i-a_i|\leq \delta(i=1,2,\ldots,n)$, then the point $x=u_{1l}\bar{x}^{(1)}+u_{2l}\bar{x}^{(2)}+\ldots+u_{n-1,l}\bar{x}^{(n-1)}+u_{nl}x^{(n)}$ describes a cube, whose center lies in $g^{(k)}+nE$ and whose edges have length $u_{nl}.2\delta > 2n$, and so covers $g^{(k)}+nE$. Consequently, x describes the cube $g^{(k)}+nE$, if $x^{(n)}$ varies in some cube $P: b_i \leq x_i \leq c_i$ $(i=1,2,\ldots,n)$ contained in $|x_i-a_i| \leq \delta(i=1,2,\ldots,n)$. In virtue of b), the set $S(\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(n-1)})$ has density $\geq 1-\frac{1}{2}n^{-n}\rho$ on this cube. Also, the set of points

$$x = u_{1l}\bar{x}^{(1)}+u_{2l}\bar{x}^{(2)}+\ldots+u_{n-1,l}\bar{x}^{(n-1)}+u_{nl}x^{(n)},$$

with $x^{(n)}\in P\cap S(\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(n-1)})$, which is contained in $g^{(k)}+nE$, has density $\geq 1-\frac{1}{2}n^{-n}\rho$ on $g^{(k)}+nE$. Further, by the definition of W_p and $S(\bar{x}^{(1)},\bar{x}^{(2)},\ldots,\bar{x}^{(n-1)})$ and the relation $l>p$, this set is disjoint with V. But V has density $\geq n^{-n}\rho$ on $g^{(k)}+nE$. This is a contradiction.

The above contradiction shows that $\mu W_p=0$ for all p. Hence $\mu(\bigcup_{p=1}^{\infty} W_p)=0$. This means that for almost all $X\in H$ there is an infinity of points $u\in\mathcal{U}$ with $Xu\in V$. The same result can be proved, if instead of C we work with αC, where α is a positive number. Hence, for almost all systems X of n linearly independent points there is an infinity of points Xu in V. Since the set of all lattices can be identified with a certain subset of the set of these systems X including the metric used, this proves the theorem.

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