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On the measure of the vectorial sum of two-dimensional point sets

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by

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Let P,Q be two bodies in R_n and let P+Q be the vectorial sum. Prof. van der Corput raised the question, whether one can derive an upper bound for the volume of P+Q in terms of quantities, each of which only depends on one of the sets P,Q. Actually he thought of the quantities $V_{i_1,i_2,\dots,i_k}(P)$, etc., defined below. This led to the conjecture that such an upper bound is given by the relation (2) (see below). In this report we give the proof of this formula in the case n=2 and also make some general remarks.

We consider point sets in the plane. We use a fixed Cartesian coordinate system. Points will be denoted by $x=(x_1,x_2)$, $y=(y_1,y_2)$, etc. We write x+y for the vectorial sum of x and y and denote by $|x|$ the distance from x to the origin.

The notion of component of a point set is important for our purpose. Here it may be defined as follows. Two points x,y of a point set P are called P-connected, if for each $\varepsilon > 0$ we can find a finite chain of points $x^{(0)}=x, x^{(1)}, \dots, x^{(n)}=y$, such that $|x^{(k+1)}-x^{(k)}| < \varepsilon$ and $x^{(k)} \in P$ for $k=0,1,\dots,n-1$. This relation between two points of P determines uniquely a subdivision of P into subsets, such that two points of P are connected if and only if these points belong to the same subset; these subsets are called the components of P. The number of components of a point set P may be denoted by $V_{12}(P)$; it may be finite or infinite.

Next we define for each point set P two quantities which as to their nature stand between the measure of P and the above defined quantity $V_{12}(P)$. Let the coordinates of a point x be denoted by x_1,x_2. For each real c let $\Pi_1(P;c)$ be the intersection of P and the vertical line $x_1=c$ and put

$$m_1(P;c) = V_{12}(\Pi_1(P;c)).$$

The least upper bound of the integral $\int_{-\infty}^{\infty} f(t)\,dt$ for those non-negative measurable functions f(t), which vanish outside a finite interval and which satisfy the relation

$$0 \leq f(t) \leq m_1(P;t) \qquad \text{for all } t,$$

will be denoted by $V_1(P)$. Thus, if $m_1(P;t)$ is a measurable function of t, then $V_1(P)$ simply is given by

$$V_1(P) = \int_{-\infty}^{\infty} m_1(P;t)\,dt\ .$$

Similarly we define $\Pi_2(P;c)$ as the intersection of P and the horizontal line $x_2=c$ and denote by $V_2(P)$ the least upper bound of

$$\int_{-\infty}^{\infty} f(t)\,dt$$

for the non-negative, measurable functions f(t), which vanish outside a finite interval and for which

$$0 \leqslant f(t) \leqslant m_2(P;t) = V_{12}(\Pi_2(P;t)).$$

We remark that the quantities $V_1(P)$, $V_2(P)$, $V_{12}(P)$ may be infinite even if P is bounded.

Finally we use the <u>outer measure</u> of P. In the case that P is bounded this outer measure is defined as the lower bound of the area of the point sets P^*, which contain the point set P and which consist of a finite number of rectangles with sides parallel to the coordinate axes.

We now can state the theorem a proof of which is the main object of this note.

<u>Theorem</u>. Let P, Q be two bounded, closed point sets in the plane. Suppose that P and Q have a finite or enumerable system of components. Let the quantities $V_1(P)$, $V_2(P)$, $V_{12}(P)$ be defined as above, and similarly the quantities $V_1(Q)$, $V_2(Q)$, $V_{12}(Q)$. Let V(P), V(Q), V(P+Q) be the outer measures of P, Q, P+Q respectively.

Then we have

(1) $V(P+Q) \leqslant V(P)V_{12}(Q)+V_1(P)V_2(Q)+V_2(P)V_1(Q)+V_{12}(P)V(Q)$,

if in the right hand member we use the convention $0.\infty=0$, $a.\infty=\infty$ if $a > 0$.

<u>Remark 1</u>. With the above convention the theorem is no longer true if we omit the condition that the number of components of P and Q is at most enumerable. For let P be the set of points $x=(x_1,x_2)$ with $0 \leqslant x_1 \leqslant 1$, $0 \leqslant x_2 \leqslant 1$, such that both x_1 and x_2 can be written as an infinite decimal in the scale of 3, where the digits are all 0 or 2 (so that the projections of P on the x_1-axis and the x_2-axis form the so-called discontinuum of Cantor). And let Q be the set of points $y=(y_1,y_2)$ with $0 \leqslant y_1 \leqslant 1$, $0 \leqslant y_2 \leqslant 1$, such that both y_1 and y_2 can be written as an infinite decimal in the scale of 3, where the digits are all 0 or 1. The number of components of P and of Q is not enumerable. It is evident that each point $z=(z_1,z_2)$ of the square $0 \leqslant z_1 \leqslant 1$, $0 \leqslant x_2 \leqslant 1$ can be written as $z=(x_1,x_2)+(y_1,y_2)$, where x_1,x_2,y_1,y_2 have the form specified above. Hence P+Q overlaps this square, so that $V(P+Q) \geqslant 1$

Further the sets P, Q are bounded, closed and have Jordan measure 0. Likewise the projections of P and Q on the coordinate axes have, as one-dimensional sets, Jordan measure 0. Hence, according to our definitions, the numbers $V_1(P)$, $V_2(P)$, $V_1(Q)$, $V_2(Q)$ are all equal to zero. Hence, according to our convention, the right hand member of (1) must be interpreted to be equal to zero. Consequently (1) is not true for the pair P, Q.

<u>Remark 2</u>. In the inequality (1) the equality sign cannot be omitted For if P and Q are rectangles with sides parallel to the coordinate axes with sides a, b and c, d respectively, then, as is easily verified, both members of (1) are equal to $(a+c)(b+d)$. A less trivial example is obtained as follows.

Let $S(p,q)$ be the square $p \leq x_1 \leq p+1$, $q \leq x_2 \leq q+1$ and let k and l be positive integers. Then let Q be the square $S(0,0)$ and let P consist of the kl squares

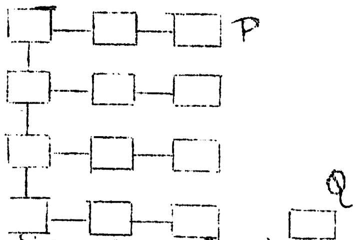

fig.1. (k = 3, l = 4)

$$S(0,0),\ S(2,0),\ S(4,0),\ \ldots,\ S(2k-2,0),$$
$$S(0,2),\ S(2,2),\ S(4,2),\ \ldots,\ S(2k-2,2),$$
$$- - - - - - - - - - - - - - - - - -$$
$$S(0,2l-2),\ S(2,2l-2),\ S(4,2l-2),\ \ldots,\ S(2k-2,2l-2),$$

which are connected by a number of horizontal or vertical line-sequents in such a way that this number is minimal and that P is connected (see fig. 1). Then $P+Q$ is a rectangle with sides $2k, 2l$, so that $V(P+Q) = 4kl$. We further find

$$V(Q) = V_1(Q) = V_2(Q) = V_{12}(Q) = 1,$$
$$V(P) = kl \quad , \quad V_{12}(P) = 1,$$
$$V_1(P) = (2k-1)l \, , \; V_2(P) = kl+l-1 \, .$$

Consequently for these sets P, Q the right hand member of (1) has the value

$$kl + (2k-1)l + kl + l-1 + 1 = 4kl.$$

Hence (1) holds with the equality sign.

We shall say a few words about the corresponding problem in $R_n (n \geq 3)$. Let P, Q be two bounded point sets in R_n. We use a fixed Cartesian coordinate system and denote the points of R_n by $x = (x_1, x_2, \ldots, x_n)$, etc. We define $P+Q$ in the same way as in the two-dimensional case.

Consider a set of positive integers $i_1, i_2, \ldots, i_k$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ $(k=1,2,\ldots,n-1)$. Then for any real numbers $c_1, c_2, \ldots, c_k$ we denote by $m_{i_1, i_2, \ldots, i_k}(P; c_1, c_2, \ldots, c_k)$ the number of components of the k-dimensional intersection of P and the subspace $x_{i_1} = c_1, x_{i_2} = c_2, \dots, x_{i_k} = c_k$. Assign in a suitable way a value to the integral

$$\int_{-\infty}^{\infty} \cdots \int m_{i_1, i_2, \dots, i_k}(P; t_1, t_2, \dots, t_k)\,dt_1 dt_2 \dots dt_k,$$

such that this integral reduces to the Lebesque (or Riemann) integral, if these integrals exist, and denote this value by $V_{i_1, i_2, \dots, i_k}(P)$.

In this terminology it is natural to denote by V(P) the number of components of P and by $V_{12\dots n}(P)$ the measure (contrary to our definitions in the two-dimensional case). In the same way

$$V(Q),\ V_{i_1, i_2, \dots, i_k}(Q),\ V_{12\dots n}(Q),\ V_{12\dots n}(P+Q)$$

can be defined. We now state the following

Conjecture. If P and Q are n-dimensional point sets, which satisfy certain not too restrictive conditions, then we have the formula

$$(2)\quad V_{12\dots n}(P+Q) \leqslant \sum V_{i_1, i_2, \dots, i_k}(P)\, V_{j_1, j_2, \dots, j_{n-k}}(Q),$$

where the sum is extended over all sets of positive integers $i_1, i_2, \dots, i_k;\ j_1, j_2, \dots, j_{n-k}$ $(k=0,1,2,\dots,n)$ with

$$1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n,\quad 1 \leqslant j_1 < j_2 < \dots < j_{n-k} \leqq n,$$
$$i_p \neq j_q \quad \text{for} \quad p=1,2,\dots,k;\ q=1,2,\dots,n-k.$$

We remark that the conjecture is right in the case of bounded convex bodies. In this case the intersection of P(Q) and a subspace consists of at most one component. Let P^* be the smallest parallelotope $a_i \leqslant x_i \leqslant b_i (i=1,2,\dots,n)$, which contains P. In the same way defir Q^*. Then we find

$$(3)\quad V_{i_1, i_2, \dots, i_k}(P) = V_{i_1, i_2, \dots, i_k}(P^*),$$

$$V_{i_1, i_2, \dots, i_k}(Q) = V_{i_1, i_2, \dots, i_k}(Q^*)$$

for each set of positive integers $i_1, i_2, \dots, i_k$ with

$$k \geqslant 1,\ 1 \leqslant i_1 < i_2 < \dots\ \ i_k \leqslant n.$$

Now we use a result of the theory of convex bodies[1] which runs as

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1) See T. Bonnesen-W. Fenchel, Theorie der konvexen Körper, Chelsea (1948), in particular p. 38, formula (1) and p. 41, property 5.

follows. Let λ, μ be positive numbers, let λP be the set of points λx with $x \in P$ and let μQ be the set of points μx with $x \in Q$. Then there exist non-negative numbers $A_0(P,Q), A_1(P,Q), \dots, A_n(P,Q)$, not depending on λ, μ, such that

(4) $$V_{12\dots n}(\lambda P + \mu Q) = \sum_{i=0}^{n} A_i(P,Q)\, \lambda^i \mu^{n-i}.$$

Furthermore, if P' and Q' are bounded, convex bodies with P P', Q Q', then we have

$$A_i(P,Q) \leqslant A_i(P',Q').$$

Taking $\lambda = \mu = 1$ we find

(5) $$V_{12\dots n}(P+Q) - V_{12\dots n}(P) - V_{12\dots n}(Q) \leqslant \sum_{i=1}^{n-1} A_i(P^*, Q^*).$$

For, if λ or μ tend to zero we get from (4)

$$V_{12\dots n}(P) = A_n(P,Q),\quad V_{12\dots n}(Q) = A_0(P,Q).$$

Similarly we have

(6) $$V_{12\dots n}(P^*) = A_n(P^*,Q^*),\quad V_{12\dots n}(Q^*) = A_0(P^*,Q^*).$$

Applying (4) with $\lambda = \mu = 1$ and with P,Q replaced by P^*, Q^* we further get

$$V_{12\dots n}(P^* + Q^*) = \sum_{i=0}^{n} A_i(P^*, Q^*).$$ Hence, in virtue of (5) and (6), we find

(7) $$V_{12\dots n}(P+Q) - V_{12\dots n}(P) - V_{12\dots n}(Q) \leqslant V_{12\dots n}(P^*+Q^*) - V_{12\dots n}(P^*) -$$
$$- V \dots$$

Denote by $d_1, d_2, \dots, d_n$ the lengths of the edges of P^* and by $e_1, e_2, \dots, e_n$ the lengths of the edges of Q^*. Then $P^* + Q^*$ is a rectangular parallelotope with edges $d_i + e_i$. Hence we get, using (7),

(8) $$V_{12\dots n}(P+Q) - V_{12\dots n}(P) - V_{12\dots n}(Q)$$

$$\leqslant \prod_{i=1}^{n} (d_i + e_i) - \prod_{i=1}^{n} d_i - \prod_{i=1}^{n} e_i.$$

We further find for $k=1,2,\dots,n-1$

(9) $$V_{i_1, i_2, \dots, i_k}(P^*) = d_{i_1} d_{i_2} \dots d_{i_k},$$

$$V_{j_1, j_2, \dots, j_{n-k}}(Q^*) = e_{j_1}, e_{j_2} \dots e_{j_{n-k}}.$$

Denoting by $\sum'$ a sum over all sets $i_1, i_2, \dots, i_k$; $j_1, j_2, \dots, j_{n-k}$ $(k=1,2,\dots,n-1)$ with

$$1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n,\quad 1 \leqslant j_1 < j_2 < \dots < j_{n-k} \leqslant n,$$

$i_p \neq j_q$ for $p=1,2,\ldots,k$ and $q=1,2,\ldots,n-k$, we find, using (3) and (9),

$$(10)\quad \sum{}' V_{i_1, i_2,\ldots,i_k}(P)\, V_{j_1, j_2,\ldots,j_{n-k}}(Q)$$

$$= \sum{}' V_{i_1, i_2,\ldots,i_k}(P^*)\, V_{j_1, j_2,\ldots,j_{n-k}}(Q^*)$$

$$= \sum{}' d_{i_1} d_{i_2} \ldots d_{i_k}\, e_{j_1} e_{j_2} \ldots e_{j_{n-k}}$$

$$= \prod_{i=1}^{n} (d_i + e_i) - \prod_{i=1}^{n} d_i - \prod_{i=1}^{n} e_i .$$

The inequality (2) follows at once from (10) and (8).
Finally we remark that it is easy to find more -dimensional point sets analoguous to the point sets treated in remark 2, for which (2) holds with the equality sign.

<u>Proof of the theorem</u>.

1. <u>Proof for sets P,Q of a special form</u>.

Let ρ be a positive number. In the following we call a ρ-<u>cell</u> each square

$$(11)\quad p\rho \leqslant x_1 \leqslant (p+1)\rho \quad , \quad q\rho \leqslant x_2 \leqslant (q+1)\rho \qquad (p,q \text{ integral}),$$

denoted by $S(p,q;\rho)$. We write $S(p,q;1) = S(p,q)$.

Suppose that, for some $\rho > 0$, P is the union of a finite number of ρ-cells. The intersection of P and the strip $q_0\rho \leqslant x_2 \leqslant (q_0+1)\rho$ (q_0 integral) is the union of the ρ-cells $S(p,q;\rho)$ of P with $q=q_0$. The components of this intersection will be called ρ-<u>beams</u> of P.

In this section we consider sets with the following properties.

1º the set is connected
2º for some $\rho > 0$ it is the union of a finite number of ρ-cells
3º the intersection of two different ρ-beams never consists of a single point
4º the boundary of the set is connected (or, stated otherwise, the set is simply connected).

And we shall prove the relation (1) in the case that P <u>possesses the properties</u> 1º - 4º <u>and</u> Q <u>possesses the properties</u> 1º - 2º.

Let r be a positive number and let $\overline{P},\overline{Q}$ be the set of points rx with $x \in P$ and the set of points rx with $x \in Q$ respectively. Both members of (1) are multiplied by r^2, if we replace P,Q by $\overline{P},\overline{Q}$. So, without loss of generality, we may suppose $\rho = 1$. Then both P and Q consist of 1-cells or of 1-beams (briefly called cells and beams).

Denote by $P \dotplus Q$ the union of the cells $S(p,q)$, for which there exist two cells $S(p',q')$, $S(p'',q'')$ with

$S(p',q') \subset P$, $S(p'',q'') \subset Q$, $(p',q') + (p'',q'') = (p,q)$.

Put

$$\tau_1 Q = P^{(1)} \dotplus Q, \text{ where } P^{(1)} = S(0,0) \cup S(1,0)$$
$$\tau_2 Q = P^{(2)} \dotplus Q, \text{ where } P^{(2)} = S(0,0) \cup S(0,1).$$

We arrange the beams of P into a sequence $B_1, B_2, \ldots, B_k$ in the following way. Choose B_1 arbitrarily and, if $B_1, B_2, \ldots, B_{i-1}$ are chosen, take as B_i one of the remaining beams, such that the intersection of B_i and $\bigcup_{j=1}^{i-1} B_j$ is not empty $(i=2,3,\ldots,k)$. On account of 3º this intersection is a line-segment, the length of which is a positive integer; it never happens that B_i is connected with two of the beams $B_1, B_2, \ldots, B_{i-1}$ for otherwise the boundary of $\bigcup_{j=1}^{i} B_j$, and in view of the maximality of the beams also the boundary of $\bigcup_{j=1}^{k} B_j$: should not be connected. Consequently each set $\bigcup_{j=1}^{i} B_j$ $(i=1,2,\ldots,k)$ possesses the properties 1º - 4º.

Let Q^* be an arbitrary set which possesses the properties 1º and 2º (with $\rho=1$). Denote the beams of Q^* by $C_1, C_2, \ldots, C_l$. Let L be the set of pairs $\{i,j\}$ of positive integers i,j with $1 \leqslant i < j \leqslant l$, such that C_i and C_j have a non-empty intersection. Let l' be the number of these pairs. According to the connectedness of Q^* we have the relation

(12) $$l' \geqslant l-1.$$

Let C_i consist of m_i cells $(i=1,2,\ldots,l)$ and for $\{i,j\} \in L$ let $n_{i,j}$ denote the length of the intersection of C_i and C_j (so that $n_{i,j}$ is a non-negative integer). Clearly

(13) $$V_1(Q^*) = \sum_{i=1}^{l} m_i - \sum_{\{i,j\} \in L} n_{i,j}, \quad V_2(Q^*) = l.$$

The set $\tau_1 Q^*$ is obtained from Q^* by adding to each of the l beams one new cell to the right of it. As a first consequence we conclude from this fact and the second relation (13) that

(14) $$V(\tau_1 Q^*) - V(Q^*) = V_2(Q^*), \quad V_2(\tau_1 Q^*) \leqslant V_2(Q^*).$$

Next we find

$$V_1(\tau_1 Q^*) = \sum_{i=1}^{l} (m_i+1) - \sum_{\{i,j\} \in L} (n_{i,j}+1).$$

Hence, on account of (13) and (12), we obtain,

(15) $$V_1(\tau_1 Q^*) - V_1(Q^*) = \sum_{i=1}^{l} 1 - \sum_{\{i,j\}\in L} 1 = l-l' \leq 1.$$

For reasons of symmetry the formulae, obtained from (14) and (15) by permuting the indices 1 and 2, are also true. We only need the relation

(14') $$V(\tau_2 Q^*) - V(Q^*) = V_1(Q^*).$$

Now by induction on the number of beams of P, we shall prove the following formula

(16) $$V(P \dotplus Q) \leq V(P) + V(Q) + V_1(P)V_2(Q) + V_2(P)V_1(Q).$$

First suppose k=1. Put $V_1(P) = a$. Then, apart from a translation, $P \dotplus Q$ is the set $\tau_1^{\,a-1}Q$ (if $\tau_1^{\,0}Q \equiv Q$, $\tau_1^{\,n}Q \equiv \tau_1(\tau_1^{\,n-1}Q)$ for n=1,2,...). Now for n=0,1,2,... the set $\tau_1^{\,n} Q$ possesses the properties 1º, 2º. Hence we find, by repeated application of both relations (14),

$$\begin{aligned} V(P \dotplus Q) &= V(\tau_1^{\,a-1}Q) = V(\tau_1^{\,a-2}Q) + V_2(\tau_1^{\,a-2}Q) \\ &\leq V(\tau_1^{\,a-2}Q) + V_2(Q) \leq \dots \\ &\leq V(Q) + (a-1)\, V_2(Q). \end{aligned}$$

On the other hand the right hand member of (16) becomes

$$a + V(Q) + aV_2(Q) + V_1(Q) > V(Q) + (a-1)\, V_2(Q).$$

This proves (16) in the case k=1.

Next suppose k > 1 and suppose that (16) holds if P is replaced by a set P^*, which has the properties 1º - 4º and consists of k-1 beams. Arrange the beams of P, such as is explained above and put $P^* = \bigcup_{j=1}^{k-1} B_j$. Then B_k has a non-empty intersection with exactly one of the beams of P^*; denote this beam by B. Without loss of generality we may suppose that we have the situation, given by figure 2. Denote by B_k' the union of the cells of B_k, which have a side in common with a cell of B, and denote by B' the union of the corresponding cells of B. Denote by B_k'' the union of the remaining cells of B_k. Further let a,b be the number of cells of B_k', B_k'' respectively; we have a > 0, b ≥ 0. Finally let S be the last cell of B_k' and put $P^{**} = P^* \cup B_k'$.

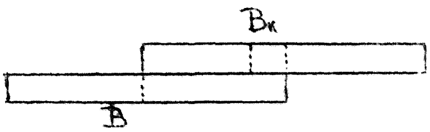

fig 2

A cell of $P^{**} \dotplus Q$, which does not belong to $P^* \dotplus Q$, must be a cell of $B_k' \dotplus Q$; furthermore this cell certainly does not belong to $B' \dotplus Q \subset P^* \dotplus Q$.

Hence we find

$$V(P^{**} \dotplus Q) - V(P^{*} \dotplus Q) \leq V((B' \cup B_k') \dotplus Q) - V(B' \dotplus Q).$$

Evidently

$$(B' \cup B_k') \dotplus Q = t_2(B' \dotplus Q)$$

and, apart from a translation,

$$B' \dotplus Q = t_1^{a-1}Q.$$

Hence, applying (14') and (15), we find

$$V((B' \cup B_k') \dotplus Q) - V(B' \dotplus Q) = V(t_2 t_1^{a-1}Q) - V(t_1^{a-1}Q) =$$

$$= V_1(t_1^{a-1}Q) \leq V_1(Q) + a-1,$$

hence

$$V(P^{**} \dotplus Q) - V(P^{*} \dotplus Q) \leq V_1(Q) + a-1.$$

Similarly a cell of $P \dotplus Q$, which does not belong to $P^{**} \dotplus Q$, is a cell of $B_k'' \dotplus Q$, which is not contained in $S \dotplus Q$.
So we find

$$V(P \dotplus Q) - V(P^{**} \dotplus Q) \leq V((S \cup B_k'') \dotplus Q) - V(S \dotplus Q)$$

$$= V(t_1^{b} Q) - V(Q),$$

hence, applying the relation (14)

$$V(P \dotplus Q) - V(P^{**} \dotplus Q) \leq bV_2(Q).$$

Summarizing we get

(17) $$V(P \dotplus Q) - V(P^{*} \dotplus Q) < a + V_1(Q) + b\, V_2(Q).$$

On the other hand P^{*} possesses properties 1º - 4º and consists of k-1 beams. So, by the induction hypothesis, we have

(18) $$V(P^{*} \dotplus Q) \leq V(P^{*}) + V(Q) + V_1(P^{*})V_2(Q) + V_2(P^{*})V_1(Q).$$

The right hand members of (16) and (18) differ by

$$\left\{V(P) - V(P^{*})\right\} + \left\{V_1(P) - V_1(P^{*})\right\} V_2(Q) + \left\{V_2(P) - V_2(P^{*})\right\} V_1(Q),$$

which is at least equal to

$$a + b + b\, V_2(Q) + V_1(Q) > a + V_1(Q) + b\, V_2(Q).$$

Using this fact and the relations (17) and (18) we obtain (16).

This completes the proof of (16).

Since P and Q are connected, the numbers $V_{12}(P)$, $V_{12}(Q)$ are equal t. Hence the right hand members of (16) and (1) are identical. It remains to prove, that in (16) we may replace the quantity $V(P \dotplus Q)$ by $V(P + Q)$.

Let N be a positive integer.
Let P_N be the set of points Nx with $x \in P$ and Q_N the set of points Nx with $x \in Q$. These sets are the union of a finite number of cells. Clearly P_N possesses the properties 1º - 4º and Q_N the properties 1º, 2º. Hence we get from (16)

$$(19) \quad \frac{1}{N^2} V(P_N \dotplus Q_N) \leqslant V(P) + V(Q) + V_1(P)\, V_2(Q) + V_2(P)\, V_1(Q).$$

Consider the sets P+Q, $P_N+Q_N = (P+Q)_N$. These sets consist of a finite number of cells. Denote by λ the length of the boundary of P+Q. Then P_N+Q_N has a boundary of length $N\lambda$. Each cell of $P_N \dotplus Q_N$ is a cell of P_N+Q_N and a cell of P_N+Q_N, which does not belong to $P_N \dotplus Q_N$, necessarily falls along the boundary of P_N+Q_N. Hence the number of these cells is at most equal to $N\lambda$. Henceforth

$$0 \leqslant V(P+Q) - \frac{1}{N^2} V(P_N \dotplus Q_N) \leqslant \frac{1}{N}\lambda,$$

which implies

$$(20) \qquad \lim_{N \to \infty} \frac{1}{N^2} V(P_N \dotplus Q_N) = V(P+Q).$$

The relation (1) is a consequence of (19) and (20).

2. Elimination of the condition 4º.

In this section we prove the relation (1) in the case that P and Q possess properties 1º - 3º (with $\rho = 1$).

The boundary of P consists of a finite number of line-segments. On account of 3º no three of these line-segments have a common end-point. So the boundary of P consists of a finite number (at least 1) of closed curves (broken lines) without double-points. Since P is connected, one of these closed curves, Γ_0 say, has the property, that the other ones, let us say $\Gamma_1, \Gamma_2, \dots, \Gamma_t$ $(t \geqslant 0)$, lie inside Γ_0. Consequently there exists a point set P_0, which possesses the properties 1º - 4º, such that P is obtained from P_0 by removing from its interior a finite number of open point sets $P_1, P_2, \dots, P_t$ $(t \geqslant 0)$, each of which has a closure which possesses the properties 1º - 4º.

In the case t=0 the assertion holds, in virtue of the result of section 1. So we may suppose $t \geqslant 1$. Put $\bigcup_{i=1}^{t} P_i = \Pi$, so that $P = P_0 / \Pi$.

Since Q is connected and closed, the projections of Q on the x_1-axis and the x_2-axis are closed intervals. From the definition of $V_1(Q)$, $V_2(Q)$ it follows that the lengths of these intervals are at most equal to $V_1(Q)$, $V_2(Q)$ respectively. Hence there exists a rectangle Q' with $Q \subset Q'$, such that the sides of Q' are parallel to the coordinate axes and have length $V_1(Q)$, $V_2(Q)$.

Clearly

(21) $V_1(Q') = V_1(Q)$, $V_2(Q') = V_2(Q)$.

Without loss of generality we may suppose that the origin is the left lower vertex of Q'.

Further let P' be a rectangle with $P \subset P'$ and with sides parallel to the coordinate axes (see fig. 3). Put

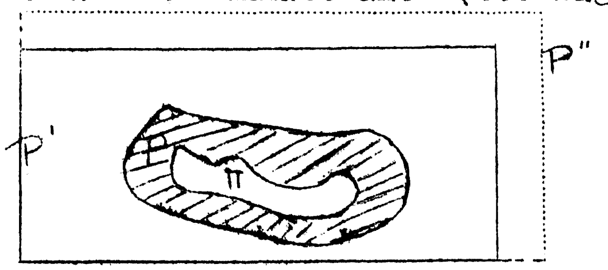

fig 3

$$P'' = P' + Q'$$
$$\Pi_1 = (P+Q) \cap \Pi$$
$$\Pi_2 = (P'/\Pi + Q') \cap \Pi.$$

On account of $P \subset P'/\Pi$, $Q \subset Q'$ we have

(22) $$\Pi_1 \subset \Pi_2.$$

Clearly $(P'/\Pi + Q')/\Pi = (P' + Q')/\Pi$,

hence

$$P'/\Pi + Q' = (P'/\Pi + Q')/\Pi \cup (P'/\Pi + Q') \cap \Pi$$
$$= P''/\Pi \cup \Pi_2,$$

hence

(23) $V(P'/\Pi + Q') = V(P'') - V(\Pi) + V(\Pi_2)$.

Since $(P+Q)/\Pi$ is contained in $(P_0 + Q)/\Pi$, we have

$$P+Q = (P+Q)/\Pi \cup ((P+Q) \cap \Pi) \subset (P_0+Q)/\Pi \cup \Pi_1,$$

hence

$$V(P+Q) \leqslant V(P_0+Q) - V(\Pi) + V(\Pi_1).$$

So on account of (22) we get

(24) $V(P+Q) - V(P_0+Q) \leqslant - V(\Pi) + V(\Pi_2)$.

For shortness write

$$V_h(P) = a_h,\ V_h(P') = a_h',\ V_h(\Pi) = \alpha_h,\ V_h(Q) = V_h(Q') = b_h \quad (h=1,2).$$

Then we clearly have

$$V(P') = a_1'a_2',\ V(Q') = b_1b_2,\ V(P'') = (a_1' + b_1)(a_2' + b_2),$$
$$V_h(P_0) = a_h - \alpha_h,\ V_h(P'/\Pi) = a_h' + \alpha_h \quad (h=1,2).$$

We further get

(25) $$\left\{V(P)+V(Q)+V_1(P)V_2(Q)+V_2(P)V_1(Q)\right\} - \left\{V(P_0)+V(Q)+V_1(P_0)V_2(Q) + V_2(P_0)V_1(Q)\right\}$$
$$= V(P)-V(P_0)+ \left\{V_1(P)-V_1(P_0)\right\} V_2(Q)+ \left\{V_2(P)-V_2(P_0)\right\} V_1(Q)$$
$$= -V(\Pi) + \alpha_1 b_2 + \alpha_2 b_1.$$

We now apply twice the result of section 1. First, since Q' possesses the properties 1º - 4º, we may deduce

$$V(P'/\Pi + Q') \leqslant V(P'/\Pi) + V(Q') + V_1(P'/\Pi)V_2(Q) + V_2(P'/\Pi)V_1(Q')$$

$$= -V(\Pi) + a_1'a_2' + b_1b_2 + (a_1' + \alpha_1)b_2 + (a_2' + \alpha_2)b_1.$$

Hence, on account of (23), we get

$$-V(\Pi) + V(\Pi_2) = V(P'/\Pi + Q') - V(P'')$$

$$\leqslant -(a_1' + b_1)(a_2' + b_2) - V(\Pi) + a_1'a_2' + b_1b_2 + (a_1' + \alpha_1)b_2 + (a_2' + \alpha_2)b_1$$

$$= -V(\Pi) + \alpha_1 b_2 + \alpha_2 b_1,$$ hence on account of (24)

(26) $V(P+Q) - V(P_0+Q) \leqslant - V(\Pi) + \alpha_1 b_2 + \alpha_2 b_1.$

Secondly, since P_0 possesses the properties 1º - 4º, we find

(27) $V(P_0+Q) \leqslant V(P_0) + V(Q) + V_1(P_0)V_2(Q) + V_2(P_0)V_1(Q).$

The relation (1) follows at once from (25), (26), (27).

3. Proof of the theorem in the case $V_{12}(P) = V_{12}(Q) = 1$.

Let ε be a positive number. There exists a set P^*, which is the union of a finite number of rectangles

R_i: $a_i \leqslant x_1 \leqslant b_i$, $c_i \leqslant x_2 \leqslant d_i$ $(i=1,2,\ldots,k)$,

such that two different rectangles R_{i_1}, R_{i_2} have no inner points in common and such that

$$P \subset P^* = \bigcup_{i=1}^{k} R_i \ , \ V(P^*) = \sum_{i=1}^{k} V(R_i) < V(P) + \varepsilon .$$

We may suppose that none of the intersections $P \cap R_i$ is empty and that k is at least 2. For if k=1 for each choice of ε, then P reduces to a single point, in which case the theorem is trivially true.

Consider a particular rectangle R_i and put $P_i = P \cap R_i$. Each point of P_i is P-connected with a point of the boundary of R_i, since $k \geqslant 2$ and P is closed and connected. Let $L^{(1)}$, $L^{(2)}$, $L^{(3)}$, $L^{(4)}$ be the four sides of R_i, let $S_i^{(t)}$ be the set of points x which belong to P_i and are P-connected with a point of $L^{(t)}$ $(t=1,2,3,4)$, and let $T_i^{(t)}$ be the projection of $S_i^{(t)}$ on $L^{(t)}$ $(t=1,2,3,4)$.

Let t have a fixed value (1,2,3 or 4). The set $S_i^{(t)}$ is closed, as well as $T_i^{(t)}$. Now $T_i^{(t)}$ is a bounded subset of some straight line H. Hence, on this line H, the set $T_i^{(t)}$ is Lebesque measurable, with measure $\mu(T_i^{(t)})$, say. The complementary set $H/T_i^{(t)}$ is an open subset of H and consists of a finite or enumerable system of mutually disjunct, open intervals. Consequently $T_i^{(t)}$ can be overlapped by a finite number of mutually disjunct, closed intervals $I_1^{(t)}, I_2^{(t)}, \ldots$ $\ldots, I_{l_t}^{(t)}$, which are all contained in the side $L^{(t)}$ and which have a total length

$$\sum_{j=1}^{l_t} \mu(I_j^{(t)}) < \mu(T_i^{(t)}) + \frac{1}{2k}\varepsilon .$$

Let $R_{i,1}^{(t)}, R_{i,2}^{(t)}, \ldots, R_{i,l_t}^{(t)}$ be the rectangles with minimal area, which are contained in R_i, such that $I_j^{(t)}$ is a side of $R_{i,j}^{(t)}$ $(j=1,2,\ldots,l_t)$ and such that $S_i^{(t)}$ is contained in the union of these rectangles. Then, if H is a horizontal line, $L^{(t)}$ is one of the horizontal sides of R_i and H intersects $R_{i,j}^{(t)}$, H also intersects $R_{i,j}^{(t)} \cap P_i$. If, on the other hand, $L^{(t)}$ is one of the vertical sides of R_i and H is a horizontal line which intersects $R_{i,j}^{(t)}$, then H also intersects $R_{i,j}^{(t)} \cap P_i$, except when H contains a point of the one-dimensional set

$$(\bigcup_{j=1}^{l_t} I_j^{(t)})/T_i^{(t)}$$

on the side $L^{(t)}$ with measure $< \frac{1}{2k}\varepsilon$.

Put

$$S_i^* = \bigcup_{t=1,2,3,4} \bigcup_{j=1}^{l_t} R_{i,j}^{(t)}.$$

Clearly P_i is contained in S_i^*. Let H be an arbitrary horizontal or vertical line and let $K_1, K_2, \ldots, K_s$ be the components of $H \cap S_i^*$. These components do overlap the components of $H \cap P_i$ and any component K_r has a non-empty intersection with P_i, except possibly when $L^{(t)}$ is one of the sides of R_i parallel to H, K_r is contained in one of the rectangles $R_{i,1}^{(t)}, R_{i,2}^{(t)}, \ldots, R_{i,l_t}^{(t)}$ and has a point in common with

$$(\bigcup_{j=1}^{l_t} I_j^{(t)})/T_i^{(t)}.$$ Hence we find

$$V_h(S_i^*) < V_h(P_i) + \frac{1}{k}\varepsilon \quad (h=1,2).$$

Finally put $S^* = \bigcup_{i=1}^{k} S_i^*$. Then we have

$$S_i^* = S^* \cap R_i \;,\; P_i = P \cap R_i \subset S_i^*.$$

Hence it follows from the definition of V_1, V_2 and from the fact that $P_i \cap P_j$ as a one-dimensional set is Lebesque measurable $(i \neq j)$ that

$$\sum_{i=1}^{k} V_h(S_i^*) - V_h(S^*) = \sum\sum_{1\leq i<j\leq k} V_h(S_i^* \frown S_j^*)$$

$$\geqslant \sum\sum_{1\leq i<j\leq k} V_h(P_i \frown P_j) = \sum_{i=1}^{k} V_h(P_i) - V_h(P),$$

hence

$$V_h(S^*) \leqslant V_h(P) + \sum_{i=1}^{k}\left\{V_h(S_i^*) - V_h(P_i)\right\}$$

(28)

$$< V_h(P) + \varepsilon \qquad (h=1,2).$$

The set S^* is the union of a finite number of rectangles with sides parallel to the coordinate axes. Each of these rectangles contains a point P; hence, since P is connected, S^* also is connected. Since S^* is contained in $\bigcup_{i=1}^{k} R_i = P^*$, we have

(29) $$V(S^*) \leqslant V(P^*) < V(P) + \varepsilon .$$

By enlarging slightly, if necessary, the rectangles of S^* we can ensure without disturbing (28) and (29), that the vertices of these rectangles have rational coordinates. Then for some rational $\rho_1 > 0$ the set S^* possesses properties 1º and 2º of section 1. By the same argument we can ensure that no two ρ_1-beams have an intersection which consists of a single point.

Similarly for each $\varepsilon' > 0$ we can find a set U^* which overlaps Q, for some rational ρ_2 possesses properties 1º, 2º, 3º and which satisfies the relations

(28') $$V_h(U^*) < V_h(Q) + \varepsilon \qquad (h=1,2)$$

(29') $$V(U^*) < V(Q) + \varepsilon$$

Let ρ^* be a submultiple of ρ_1 and ρ_2. Then S^* and U^* both possess properties 1º, 2º, 3º with $\rho = \rho^*$. Consequently we have, by the result of section 2,

$$V(S^* + U^*) \leqslant V(S^*) + V(U^*) + V_1(S^*)V_2(U^*) + V_2(S^*)V_1(U^*).$$

Obviously P+Q is contained in $S^* + U^*$, on account of $P \subset S^*$, $Q \subset U^*$. This gives

(30) $$V(P+Q) \leqslant V(S^*) + V(U^*) + V_1(S^*)V_2(U^*) + V_2(S^*)V_1(U^*).$$

In order to deduce (1) from (30) we distinguish three cases.

1) $V_h(P)$, $V_h(Q)$ $(h=1,2)$ all are finite. Then the required result follows at once from (28), (28'), (29), (29'), (30), if we let ε, ε' tend to zero.

2) exactly one of the quantities $V_h(P)$, $V_h(Q)$ is infinite; suppose $V_1(P) = \infty$. We remark that the quantities $V_h(S^*)$, $V_h(U^*)$ all are finite. If $V_2(Q) > 0$, nothing has to be proved. If $V_2(Q) = 0$, then we find by letting ε' tend to zero

$$V(P+Q) \leq V(S^*) + V(Q) + V_1(S^*)V_2(Q) + V_2(S^*)V_1(Q)$$

$$= V(S^*) + V(Q) + V_2(S^*)V_1(Q);$$

next letting ε tend to zero, the required result follows.

3) at least two of the numbers $V_h(P)$, $V_h(Q)$ are infinite. We may suppose that exactly two of these numbers are infinite and the two remaining numbers are equal to zero, since otherwise the right hand member of (1) is infinite and nothing has to be proved. If $V_1(P) = V_2(P) = \infty$, $V_1(Q) = V_2(Q) = 0$ then, since Q is connected, Q reduces to a single point; hence $V(P+Q) = V(P)$, from which the relation (1) is a trivial consequence. If $V_2(P) = V_2(Q) = 0$, then both P and Q are line-segments, so that the case $V_1(P) = V_1(Q) = \infty$, $V_2(P) = V_2(Q) = 0$ does not occur. If $V_1(P) = V_2(Q) = \infty$, then the right hand member of (1) is infinite. The other cases can be treated analoguously.

The assertion is now proved completely.

4. Proof of the theorem in the general case.

First suppose that $V_{12}(P)$ and $V_{12}(Q)$ are finite. Let $P^{(1)}, P^{(2)}, \ldots, P^{(k)}$ be the components of P and let $Q^{(1)}, Q^{(2)}, \ldots, Q^{(l)}$ be the components of Q.

A limit-point of a component $P^{(i)}$ belongs to P and is P-connected with the points of $P^{(i)}$. Hence each $P^{(i)}$, and similarly each $Q^{(j)}$, is closed.

Consider two different components $P^{(i)}$, $P^{(j)}$ of P. Suppose that the distance of these components is equal to zero. Then there exist two sequences of points $x^{(n)}$, $y^{(n)}$ $(n=1,2,\ldots)$ with $x^{(n)} \in P^{(i)}$, $y^{(n)} \in P^{(j)}$, $|x^{(n)} - y^{(n)}| \to 0$ as $n \to \infty$. Since P is bounded, there exist an increasing sequence of positive integers $n_1, n_2, \ldots$, such that $x^{(n_t)}$, $y^{(n_t)}$ converge if $t \to \infty$. But then the points of $P^{(i)}$ are P-connected with the points of $P^{(j)}$, which is a contradiction. Hence $P^{(i)}$ and $P^{(j)}$ have a positive distance. The same conclusion holds for the components of Q. In view of our definition of the volume of a bounded point set it follows from this fact that

$$(31) \qquad V(P) = \sum_{i=1}^{k} V(P_i), \quad V(Q) = \sum_{j=1}^{l} V(Q_j).$$

On account of the relation

$$m_h(P^{(i)} \cup P^{(j)};c) = m_h(P^{(i)};\ c) + m_h(P^{(j)};\ c)$$

($h=1,2;\ 1 \leqslant i < j \leqslant k$; c real)
and a similar formula for the components of Q we have

$$(32)\quad V_h(P) = \sum_{i=1}^{k} V_h(P^{(i)}),\ V_h(Q) = \sum_{j=1}^{l} V_h(Q^{(j)}) \qquad (h=1,2).$$

Clearly

$$(33)\quad V_{12}(P) = k = \sum_{i=1}^{k} V_{12}(P^{(i)}),\ V_{12}(Q) = l = \sum_{j=1}^{l} V_{12}(Q^{(j)}).$$

Write

$$F(P,Q) = V(P)V_{12}(Q) + V_1(P)V_2(Q) + V_2(P)V_1(Q) + V_{12}(P)V(Q),$$

$$F(P^{(i)},\ Q^{(j)}) = V(P^{(i)})V_{12}(Q^{(j)}) + V_1(P^{(i)})V_2(Q^{(j)}) +$$

$$+ V_2(P^{(i)})V_1(Q^{(j)}) + V_{12}(P^{(i)})V_{12}(Q^{(j)}) \quad (i=1,2,\dots,k;\ j=1,2,\dots,l).$$

Then it follows from (31), (32), (33) that

$$(34)\qquad F(P,Q) = \sum_{i=1}^{k} \sum_{j=1}^{l} F(P^{(i)},\ Q^{(j)}).$$

Clearly

$$P+Q = \bigcup_{\substack{i=1,2,\dots,k \\ j=1,2,\dots,l}} (P^{(i)} + Q^{(j)}),$$

hence

$$(35)\qquad V(P+Q) \leqslant \sum_{i=1}^{k} \sum_{j=1}^{l} V(P^{(i)} + Q^{(j)}).$$

To each pair of sets $P^{(i)}$, $Q^{(j)}$ we may apply the result of section 3. Then, in the case $V_{12}(P),\ V_{12}(Q) < \infty$, the relation (1) follows from (34) and (35).

Next let the number of components of P and Q be finite or enumerable Denoting by P_m the union of the first components of P and by Q_n the union of the first n components of Q we find by the above result

$$V(P_m + Q_n) \leqslant F(P_m,\ Q_n) \leqslant F(P,Q).$$

Letting m,n tend to $\infty(k)$, $\infty(l)$ we get

$$V(P+Q) = \lim V(P_m + Q_n) \leqslant F(P,Q),$$

which is the required result.

This completes the proof of the theorem.