STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

ZW 1955-015

Some simple properties of hyperspherical solid harmonics

Dr.ir. D.J. Hofsommer

1955

Some simple properties of hyperspherical solid harmonics

door

Dr. Ir. D.J. Hofsommer

Voordracht in de serie "Actualiteiten"

26 november 1955

1. Consider Laplace's equation in four dimensions

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_3^2} + \frac{\partial^2 v}{\partial x_4^2} = 0 \qquad 1.1)$$

and substitute

$$x_1 = \sqrt{y}\cos\varphi$$
$$x_2 = \sqrt{y}\sin\varphi$$
$$x_3 = \sqrt{z}\cos\psi$$
$$x_4 = \sqrt{z}\sin\psi$$

the transformed equation is

$$\frac{\partial}{\partial y}\left(y\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(z\frac{\partial v}{\partial z}\right) + \frac{1}{4y}\frac{\partial^2 v}{\partial \varphi^2} + \frac{1}{4z}\frac{\partial^2 v}{\partial \psi^2} = 0$$

Separate according to $V(y,z,\varphi,\psi) = U(y,z)\,\phi(\varphi)\Psi(\psi)$. We get the equations

$$\frac{d^2\phi}{d\varphi^2} + 4m^2\phi = 0 \qquad \frac{d^2\Psi}{d\psi^2} + 4n^2\Psi = 0$$

$$\frac{\partial}{\partial y}\left(y\frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial z}\left(z\frac{\partial U}{\partial z}\right) - \left(\frac{m^2}{y} + \frac{n^2}{z}\right)U = 0 \qquad 1.2)$$

Any homogeneous solution of (1.2) i.e. any solution satisfying

$$y\frac{\partial U}{\partial y} + z\frac{\partial U}{\partial z} = kU \qquad 1.3)$$

will be called a hyperspherical harmonic:

We will restrict ourselves to those values of k,m,n for which the four quantities $k \pm m \pm n$ are nonnegative integers. From (1.2) and (1.3) either the derivatives with respect to y or to z may be eliminated. The result is an ordinary second order differential equation. This shows that the pair of equations (2) and (3) has two linear independent solutions.

The Mathematical Centre at Amsterdam, founded the 11th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications, and is sponsored by the Netherlands Government through the Netherlands Organization for Pure Research (Z.W.O.) and the Central National Council for Applied Scientific Research in the Netherlands (T.N.O.), by the Municipality of Amsterdam and by several industries.

In order to solve (1.2) and (1.3) we substitute

$$U = y^{-m} z^{-n} W$$

Then $W(y,z)$ satisfies

$$\frac{\partial}{\partial y}\left(y\frac{\partial W}{\partial y}\right) + \frac{\partial}{\partial z}\left(z\frac{\partial W}{\partial z}\right) - 2m\frac{\partial W}{\partial y} - 2n\frac{\partial W}{\partial z} = 0 \qquad 1.4)$$

$$y\frac{\partial W}{\partial y} + z\frac{\partial W}{\partial z} = (k+m+n)W \qquad 1.5)$$

We now try to find a function $W(y,z)$ satisfying the initial conditions $W(0,0) = 0$. Applying a Laplace transformation we find, if $W(y,z) \doteqdot \overline{w}(p,q)$, that

$$p^2\frac{\partial \overline{w}}{\partial p} + q^2\frac{\partial \overline{w}}{\partial q} + (2mp + 2nq)\overline{w} = 0 \qquad 1.6)$$

which equation is solved by

$$\overline{w} = \frac{p^{-2m}q^{-2n}}{(k-m-n)!} f\left(\frac{1}{p} - \frac{1}{q}\right) \qquad 1.7)$$

and $$p\frac{\partial \overline{w}}{\partial p} + q\frac{\partial \overline{w}}{\partial q} = -(k+m+n)\overline{w} \qquad 1.8)$$

which equation specializes solution (1.7) into

$$\overline{w} = \frac{p^{-2m}q^{-2n}}{(k-m-n)!}\left(\frac{1}{p} - \frac{1}{q}\right)^{k-m-n}$$

$$= \sum_{s=0}^{k-m-n} \frac{(-1)^s}{s!(k-m-n-s)!}\left(\frac{1}{p}\right)^{k+m-n-s}\left(\frac{1}{q}\right)^{2n+s} \qquad 1.9)$$

Hence $$W(y,z) = \sum_{s=0}^{k-m-n} \frac{(-1)^s}{s!(k-m-n-s)!(k+m-n-s)!(2n+s)!} y^{k+m-n-s} z^{2n+s}$$

The corresponding $U(y,z)$ will be denoted by $P_k^{-m,-n}(y,z)$

$$P_k^{-m,-n}(y,z) = \sum_{s=0}^{k-m-n} \frac{(-1)^s}{s!(s+2n)!(k+m-n-s)!(k-m-n-s)!} y^{k-n-s} z^{s+n}$$

or $$P_k^{m,n}(y,z) = \sum_{s=0}^{k+m+n} \frac{(-1)^s}{s!(s-2n)!(k+m+n-s)!(k-m+n-s)!} y^{k+n-s} z^{s-n} \qquad 1.10)$$

Because of the factors $(s-2n)!(k-m+n-s)!$ in the denominator the terms for which $0 \leq s < 2n$ and $k-m+n < s \leq k+m+n$ vanish. Replacing s by $s+2n$ we find

$$P_k^{m,n}(y,z) = \sum_{s=0}^{k-m-n} \frac{(-1)^{s+2n}}{(s+2n)!s!(k+m-n-s)!(k-m-n-s)!} y^{k-n-s} z^{s+n}$$

$$= (-1)^{2n}\, P_k^{-m,-n}(y,z) \tag{1.11}$$

In the same way we find

$$P_k^{m,n}(y,z) = P_k^{-m,n}(y,z) = (-1)^{2n} P_k^{m,-n}(y,z) = (-1)^{2n} P_k^{-m,-n}(y,z) \tag{1.12}$$

If in (1.10) we replace s by k+m+n-s it is easily seen that

$$P_k^{m,n}(y,z) = (-1)^{k+m+n} P_k^{n,m}(z,y) \tag{1.13}$$

Formula (1.10) can be easily generalized to a form valid for arbitrary values of k,m,n. This generalization is

$$P_k^{m,n}(y,z) = \frac{y^k}{\prod(-2n)\prod(k+m+n)\prod(k-m+n)} \cdot \left(\frac{y}{z}\right)^n F(-k-m-n,-k+m-n;1-2n;-\frac{z}{y}) \tag{1.14}$$

We have found

$$P_k^{m,n}(y,z) = \sum_{s=0}^{k+m+n} \frac{(-1)^s}{s!(s-2n)!(k+m+n-s)!(k-m+n-s)!}\, y^{k+n-s} z^{s-n} \tag{1.10}$$

Hence

$$y^{-m} z^{-n} P_k^{m,n}(y,z) \doteqdot \sum_{s=0}^{k+m+n} \frac{(-1)^s}{s!(k+m+n-s)!} \left(\frac{1}{p}\right)^{k-m+n-s} \left(\frac{1}{q}\right)^{s-2n} \tag{1.15}$$

$$\doteqdot \frac{p^{2m} q^{2n}}{(k+m+n)!} \left(\frac{1}{p} - \frac{1}{q}\right)^{k+m+n} \tag{1.16}$$

$$\doteqdot \frac{p^{2m} q^{2n}}{(k+m+n)!} (q-p)^{k+m+n} (pq)^{-k-m-n} \tag{1.17}$$

and therefore

$$P_k^{m,n}(y,z) = y^m z^n \frac{\partial^{2(m+n)}}{\partial y^m \partial z^n} P_{k+m+n}^{0,0}(y,z) \tag{1.18}$$

since, from (1.16)

$$P_{k+m+n}^{0,0}(y,z) \doteqdot \frac{1}{(k+m+n)!} \left(\frac{1}{p} - \frac{1}{q}\right)^{k+m+n}$$

Again, from (1.17)

$$P_k^{m,n}(y,z) = \frac{1}{(k+m+n)!^3}\, y^m z^n \frac{\partial^{2(m+n)}}{\partial y^m \partial z^n} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\right)^{k+m+n} (yz)^{k+m+n} \tag{1.19}$$

We also have

$$y^m z^n P_k^{m,n}(y,z) = (-1)^{2n} y^m z^n P_k^{-m,-n}(y,z)$$

$$\doteqdot \frac{(-1)^{2n} p^{-2m} q^{-2n}}{(k-m-n)!} (q-p)^{k-m-n} (pq)^{-k+m+n}$$

$$\fallingdotseq \frac{(-1)^{2n}p^{2n}q^{2m}}{(k-m-n)!}(q-p)^{k-m-n}(pq)^{-k-m-n}$$

or $P_k^{m,n}(y,z) =$

$$\frac{(-1)^{2n}y^{-m}z^{-n}}{(k+m+n)!^2(k-m-n)!}\frac{\partial^{2(m+n)}}{\partial y^n \partial z^m}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial y}\right)^{k-m-n}(yz)^{k+m+n} \qquad 1.20)$$

Finally, if $m = n$ we find from (1.17)

$$y^{-m}z^{-m}P_k^{m,n}(y,z) \fallingdotseq \frac{1}{(k+2m)!}(q-p)^{k+2m}(pq)^{-k}$$

or $$P_k^{m,m}(y,z) = \frac{(yz)^m}{k!^2(k+2m)!}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial y}\right)^{k+2m}(yz)^k \qquad 1.21)$$

Having established the image of $y^m z^n P_k^{m,n}(y,z)$ it is easy to find numerous recurrent relations. We derive a few.

From

$$(k+m+n)\frac{p^{2m}q^{2n}}{(k+m+n)!}\left(\frac{1}{p}-\frac{1}{q}\right)^{k+m+n} =$$

$$\frac{p^{2m-1}q^{2n}}{(k+m+n-1)!}\left(\frac{1}{p}-\frac{1}{q}\right)^{k+m+n-1} - \frac{p^{2m}q^{2n-1}}{(k+m+n-1)!}\left(\frac{1}{p}-\frac{1}{q}\right)^{k+m+n-1}$$

follows

$$(k+m+n)P_k^{m,n}(y,z) = \sqrt{y}P_{k-\frac{1}{2}}^{m-\frac{1}{2},n}(y.z) - \sqrt{z}P_{k-\frac{1}{2}}^{m,n-\frac{1}{2}}(y,z) \qquad 1.22)$$

and by aid of (1.12)

$$(k-m+n)P_k^{m,n}(y,z) = \sqrt{y}P_{k-\frac{1}{2}}^{m+\frac{1}{2},n}(y,z) - \sqrt{z}P_{k-\frac{1}{2}}^{m,n-\frac{1}{2}}(y,z) \qquad 1.23)$$

$$(k+m-n)P_k^{m,n}(y,z) = \sqrt{y}P_{k-\frac{1}{2}}^{m-\frac{1}{2},n}(y,z) + \sqrt{z}P_{k-\frac{1}{2}}^{m,n+\frac{1}{2}}(y,z) \qquad 1.24)$$

$$(k-m-n)P_k^{m,n}(y,z) = \sqrt{y}P_{k-\frac{1}{2}}^{m+\frac{1}{2},n}(y,z) + \sqrt{z}P_{k-\frac{1}{2}}^{m,n+\frac{1}{2}}(y,z) \qquad 1,25)$$

Addition of (1.22) and (1.25) yields

$$2kP_k^{m,n}(y,z) = \sqrt{y}\left(P_{k-\frac{1}{2}}^{m+\frac{1}{2},n} + P_{k-\frac{1}{2}}^{m-\frac{1}{2},n}\right) + \sqrt{z}\left(P_{k-\frac{1}{2}}^{m,n+\frac{1}{2}} - P_{k-\frac{1}{2}}^{m,n-\frac{1}{2}}\right) \qquad 1.26)$$

2. The Laplce equation (1.1) has a solution

$$V = f(x_1 - ix_4)$$

Let

$$V = 2^{2k}(x_1 - ix_4)^{2k} = (\sqrt{y}e^{i\varphi} + \sqrt{y}e^{-i\varphi} - \sqrt{z}e^{i\psi} + \sqrt{z}e^{-i\psi})^{2k} \qquad 2.1)$$

Then

$$V = (2k)!\sum_{m,n} P_k^{m,n}(y,z)e^{2i(m\varphi + n\psi)} \qquad 2.2)$$

Assume this formula to be valid for k. Then

$$2^{2k+1}(x_1 - ix_4)^{2k+1} =$$

$$(2k)!(\sqrt{y}e^{i\varphi} + \sqrt{y}e^{-i\varphi} - \sqrt{z}e^{i\psi} + \sqrt{z}e^{-i\psi})\sum P_k^{m,n}(y,z)e^{2i(m\varphi + n\psi)}$$

$$= (2k)!\sum P_k^{m,n}\Big[\sqrt{y}\Big\{e^{2i\{(m+\frac{1}{2})\varphi + n\psi\}} + e^{-2i\{(m-\frac{1}{2})+n\psi\}}\Big\}$$

$$- \sqrt{z}\Big\{e^{2i\{m\varphi + (n+\frac{1}{2})\psi\}} - e^{-2i\{m\varphi + (n-\frac{1}{2})\psi\}}\Big\}\Big]$$

$$= (2k)!\sum\Big|\sqrt{y}(P_k^{m-\frac{1}{2},n} + P_k^{m+\frac{1}{2},n}) - \sqrt{z}(P_k^{m,n-\frac{1}{2}} - P_k^{m,n+\frac{1}{2}})\Big|\,e^{2i(m\varphi + n\psi)}$$

$$= (2k+1)!\sum P_{k+\frac{1}{2}}^{m,n}(y,z)e^{2i(m\varphi + n\psi)}$$

because of (1.26)

It is easily seen from (1.10) that $P_0^{0,0} = 1$. Hence our formula is valid for k = 0 and therefore for all values of k

3. Another approach is to replace (1.2) by a set of aequivalent differo-difference equations. Consider the set

$$\frac{\partial}{\partial y}\, y^m U_k^{m,n} = y^{m-\frac{1}{2}} U_{k-\frac{1}{2}}^{m-\frac{1}{2},n} \qquad \frac{\partial}{\partial z}\, z^n U_k^{m,n} = -z^{n-\frac{1}{2}} U_{k-\frac{1}{2}}^{m,n-\frac{1}{2}}$$

$$\frac{\partial}{\partial y}\, y^{-m} U_k^{m,n} = y^{-m-\frac{1}{2}} U_{k-\frac{1}{2}}^{m+\frac{1}{2},n} \qquad \frac{\partial}{\partial z}\, z^{-n} U_k^{m,n} = z^{-n-\frac{1}{2}} U_{k-\frac{1}{2}}^{m,n+\frac{1}{2}} \qquad 3.1)$$

Indeed we have

$$y^m \frac{\partial}{\partial y}\left(y^{1-2m} \frac{\partial}{\partial y}\, y^m U_k^{m,n}\right) = y^m \frac{\partial}{\partial y}\left(y^{1-m} \frac{\partial}{\partial y} U_k^{m,n} + m y^{-m} U_k^{m,n}\right)$$

$$= y\frac{\partial^2}{\partial y^2} U_k^{m,n} + \frac{\partial}{\partial y} U_k^{m,n} - \frac{m^2}{y} U_k^{m,n}$$

but also

$$y^m \frac{\partial}{\partial y}\left(y^{1-2m} \frac{\partial}{\partial y}\, y^m U_k^{m,n}\right) = y^m \frac{\partial}{\partial y}\, y^{-m+\frac{1}{2}} U_{k-\frac{1}{2}}^{m-\frac{1}{2},n} = U_{k-1}^{m,n}$$

In the same way we find

$$z^n \frac{\partial}{\partial z}\left(z^{1-2n} \frac{\partial}{\partial z}\, z^n U_k^{m,n}\right) = z\frac{\partial^2}{\partial z^2} U_k^{m,n} + \frac{\partial}{\partial z} U_k^{m,n} - \frac{n^2}{z} U_k^{m,n} = -U_{k-1}^{m,n}$$

which shows, that any function which satisfies the set (3.1) is a solution of equation (1.2)

Reading the lower formulae (3.1) the other way round we find

$$\begin{aligned} y^{-m} z^{-n} U_k^{m,n} &= z^{-n} \frac{\partial}{\partial y}\, y^{-m+\frac{1}{2}} U_{k+\frac{1}{2}}^{m-\frac{1}{2},n} \\ &= z^{-n} \frac{\partial^{2m}}{\partial y^{2m}} U_{k+m}^{0,n} \\ &= \frac{\partial^{2m}}{\partial y^{2m}} \frac{\partial}{\partial z}\, z^{-n+\frac{1}{2}} U_{k+m+\frac{1}{2}}^{0,n-\frac{1}{2}} \\ &= \frac{\partial^{2m+2n}}{\partial y^{2m}\partial z^{2n}} U_{k+m+n}^{0,0} \end{aligned} \qquad (3.2)$$

Hence every solution of (1.2) can be derived by differentiating the solutions of

$$\frac{\partial}{\partial y}\left(y \frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial z}\left(z \frac{\partial U}{\partial z}\right) = 0 \qquad 3.3)$$

The homogeneous solutions of degree k+m+n of this equation are

$$P_k^{0,0}(y,z) = \sum_{s=0}^{k} \frac{1}{s!^2 (k-s)!^2}\, y^{k-s} (-z)^s$$

$$Q_k^{0,0}(y,z) = \sum_{s=0}^{k} \frac{\ln z - \ln y + 2\psi(k-s) - 2\psi(s)}{s!^2 (k-s)!^2}\, y^{k-s} (-z)^s \qquad 3.4)$$

$$= \ln\frac{z}{y}\, P_k^{0,0} + 2\sum_{s=0}^{k} \frac{\psi(k-s) - \psi(s)}{s!^2 (k-s)!^2}\, y^{k-s} (-z)^s$$

The functions $P_k^{m,n}(y,z)$ follow from (3.4) by means of the differentiating process (3.2)

A special value is

$$Q_0^{0,0}(y,z) = \ln\frac{z}{y} \tag{3.5)}$$

4. Finally we establish some relations between hyperspherical harmonics and Bessel's functions. It is seen without difficulty that (1.2) is satisfied by a product of Bessel's ordinary and modified functions

The image of Bessel's functions is given by

$$y^m J_{2m}(2\sqrt{y}) \doteqdot p^{-2m} e^{-\frac{1}{p}} \qquad y^m I_{2m}(2\sqrt{y}) \doteqdot p^{-2m} e^{\frac{1}{p}}$$

Hence

$$y^m z^n I_{2m}(2\sqrt{y}) J_{2n}(2\sqrt{z}) \doteqdot p^{-2m} q^{-2n} e^{\frac{1}{p} - \frac{1}{q}}$$

$$\doteqdot \sum_{s=0}^{\infty} \frac{p^{-2m} q^{-2n}}{s!} \left(\frac{1}{p} - \frac{1}{q}\right)^s$$

$$= y^m z^n \sum_{s=0}^{\infty} (-1)^{2n} P^{m,n}_{s+m+n}(y,z)$$

or $$I_{2m}(2\sqrt{y}) J_{2n}(2\sqrt{z}) = (-1)^{2n} \sum_{s=0}^{\infty} P^{m,n}_{s+m+n}(y,z) \qquad 4.1)$$

In the same way

$$y^m z^n J_{2m}(2\sqrt{y}) I_{2n}(2\sqrt{z}) \doteqdot p^{-2m} q^{-2n} e^{-\frac{1}{p} + \frac{1}{q}}$$

$$\doteqdot \sum_{s=0}^{\infty} \frac{p^{-2m} q^{-2n}}{s!} (-1)^s \left(\frac{1}{p} - \frac{1}{q}\right)^s$$

$$= y^m z^n (-1)^{2n} \sum_{s=0}^{\infty} (-1)^s P^{m,n}_{s+m+n}(y,z)$$

or $$J_{2m}(2\sqrt{y}) I_{2n}(2\sqrt{z}) = (-1)^{2n} \sum_{s=0}^{\infty} (-1)^s P^{m,n}_{s+m+n}(y,z) \qquad 4.2)$$

According to (4.1) we have

$$I_{2m}(2\sqrt{y}) J_{2n}(2\sqrt{z}) = \sum_{s=0}^{\infty} \sum_{t=0}^{s} \frac{(-1)^t}{t!(s-t)!(t+2n)!(s+2m-t)!} y^{s+m-t} z^{t+n}$$

$$\frac{\partial}{\partial m} I_{2m}(2\sqrt{y}) J_{2n}(2\sqrt{z}) = \sum_{s=0}^{\infty} \sum_{t=0}^{s} \frac{(-1)^t [\ln y - 2\psi(s+2m-t)]}{t!(s-t)!(t+2n)!(s+2m-t)!} y^{s+m-t} z^{t+n} \qquad 4.3)$$

$$\frac{\partial}{\partial n} I_{2m}(2\sqrt{y}) J_{2n}(2\sqrt{z}) = \sum_{s=0}^{\infty} \sum_{t=0}^{s} \frac{(-1)^t [\ln z - 2\psi(t+2n)]}{t!(s-t)!(t+2n)!(s+2m-t)!} y^{s+m-t} z^{t+n} \qquad 4.4)$$

In (4.3) and (4.4) put m=n=0 and subtract (4.3) from (4.4). Then by use of the known relations

$$\left[\frac{\partial}{\partial n} J_n(z)\right]_{n=0} = \tfrac{1}{2} N_0(z), \quad \left[\frac{\partial}{\partial n} I_n(z)\right]_{n=0} = \tfrac{1}{2} K_0(z)$$

we find

$$I_0(2\sqrt{y})N_0(2\sqrt{z}) - K_0(2\sqrt{y})J_0(2\sqrt{z}) =$$

$$\sum_{s=0}^{\infty}\sum_{t=0}^{s}(-1)^t \frac{\ln z-\ln y+2\psi(s-t)-2\psi(t)}{t!^2(s-t)!^2} y^{s-t}z^t$$

$$=\sum_{s=0}^{\infty} Q_s^{0,0}(y,z)$$

Apply the operator $\frac{\partial^{2m+2n}}{\partial y^{2m}\partial z^{2n}}$ to both sides of this equation. Then by virtue of

$$\frac{\partial^{2n}}{\partial y^{2n}} J_0(2\sqrt{y}) = (-1)^{2n}y^{-n}J_{2n}(2\sqrt{y})$$

$$\frac{\partial^{2n}}{\partial y^{2n}} N_0(2\sqrt{y}) = (-1)^{2n}y^{-n}N_{2n}(2\sqrt{y})$$

$$\frac{\partial^{2n}}{\partial y^{2n}} I_0(2\sqrt{y}) = y^{-n}I_{2n}(2\sqrt{y})$$

$$\frac{\partial^{2n}}{\partial y^{2n}} K_0(2\sqrt{y}) = y^{-n}K_{2n}(2\sqrt{y})$$

We have

$$(-1)^{2n}y^{-m}z^{-n}\left[I_{2m}(2\sqrt{y})N_{2n}(2\sqrt{z}) - K_{2m}(2\sqrt{y})J_{2n}(2\sqrt{z})\right] =$$

$$= y^{-m}z^{-n}\sum_{s=0} Q_{s-m-n}^{m,n}(y,z)$$

or

$$I_{2m}(2\sqrt{y})N_{2n}(2\sqrt{z}) - K_{2m}(2\sqrt{y})J_{2n}(2\sqrt{z}) =$$

$$(-1)^{2n}\sum_{s=0}^{\infty} Q_{s+m+n}^{m,n}(y,z) \qquad 4.5)$$

because

$$Q_k^{m,n}(y,z) = 0 \text{ if } k < m+n$$

In the same way we derive

$$J_{2m}(2\sqrt{y})K_{2n}(2\sqrt{z}) - N_{2m}(2\sqrt{y})I_0(2\sqrt{z}) \doteq$$

$$(-1)^{2n}\sum_{s=0}^{\infty}(-1)^s Q_{s+m+n}^{m,n}(y,z) \qquad 4.6$$