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The three conjugates theorem.

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The Three Conjugates Theorem

by R.D. Anderson [1]

1. Introduction. In this paper we prove that for certain metric spaces (e.g. spheres) and naturally defined sets of homeomorphisms of such spaces onto themselves, each (non-identity) element of the set is the product of three conjugates of any other (non-identity) element of the set [2]. In fact, in Section 6 a slightly stronger version of such a proposition is proved. The arguments are elementary. In Section 8, it is proved that for many of the spaces and sets of homeomorphisms considered "three" is the best possible number, i.e. there exist homeomorphisms f and g such that f is not the product of two conjugates of g.

In Section 2, properties of spaces and sets of homeomorphisms sufficient for the Three Conjugates Theorem to be true are listed. The spaces concerned all have a form of "invertibility", i.e. for some set of neighborhoods forming a basis (with respect to a subset), the closure of each neighborhood is homeomorphic to the closure of its complement under a space homeomorphism. Thus the proposition in its form in this paper is not applicable to closed manifolds other than spheres (or cells) nor is it applicable to Euclidean spaces as such.

Examples of spaces and sets of homeomorphisms for which the "Three Conjugates Theorem" is true are [3]

(a) The Cantor Set C and the set of all homeomorphisms of C onto itself. [4]

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(1) Alfred P. Sloan Research Fellow.

(2) A conjugate of h is a homeomorphism of the form $\psi^{-1}h\psi$ where ψ is a homeomorphism. In this paper all the conjugating homeomorphisms (like ψ) will be of a particular simple type.

(3) In Section 7, a more detailed discussion of the examples is given.

(4) We could also cite the universal curve M and the set of all homeomorphisms of M onto itself. But this example requires a somewhat more detailed structure than that given in Section 2.

(b) the n-sphere, S^n, $n \geq 1$ and the set of all those homeomorphisms of S^n onto itself having the cell homeomorphism extension property (5), CHEP. For n=1,2,3, all orientation-preserving homeomorphisms of S^n have the CHEP. For $n > 3$, it is not known whether such is the case, the CHEP for all orientation-preserving homeomorphisms being equivalent to the affirmative annulus problem for spheres [2] and [4]. The conditions of Section 2 are only applicable for $n > 1$ but the broad outlines of the argument given are valid for n=1.

(c) S^n, $n \geq 1$, and the set of all those orientation-reversing homeomorphisms which are subject to a condition like the CHEP.

(d) I^n (the closed n-cell), $n > 1$, and the set of all those orientation-preserving homeomorphisms which are not the identity on the boundary of I^n and satisfy a version of the CHEP. As in the case for S^1, the specific conditions of Section (2) are not applicable to I^2 but the general argument is valid.

(e) I^n, $n > 1$, and the set of all those orientation-reversing homeomorphisms which are subject to a condition like the CHEP.

(f) the space of all rationals (or irrationals) on the line and the set of all homeomorphisms of such space.

(g) the spaces of (a) - (f) above and sets of homeomorphisms with the added restriction that all homeomorphisms carry an appropriate dense subset onto itself.

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(5) A homeomorphism h of S^n onto itself has the CHEP provided $h = \alpha\beta$ where α and β are homeomorphisms of S^n onto itself and each of α and β is the identity on some open set. The name of this property comes from the alternative formulation that on any open cell D on which β is the identity, α restricted to D is h restricted to D and thus α extends h restricted to D to a homeomorphism supported on a cell. In [3], Brown and Gluck study "stable" homeomorphisms of S^n and give several important properties of the set of stable homeomorphisms including the fact that such set is the set of homeomorphisms with the CHEP. Earlier in [4], Gordon Fisher studied such homeomorphisms in a slightly different context.

In 1947, Ulam and von Neumann asserted [5] that for the sphere, S^2, there is an $N > 0$ such that any orientation-preserving homeomorphism is the product of not more than N conjugates of any other (not the identity). In a letter, Ulam stated that N could be taken as 23. As far as the author knows, the proof of their proposition has not been published. In [1], it was shown by methods considerably different from those of this paper, that, for instance, every orientation-preserving homeomorphism of S^2 or S^3 is the product of six conjugates of an arbitrary (non-identity) homeomorphism <u>and its</u> inverse. In [4] the methods of [1] were extended to S^n for the group of all homeomorphisms isotopic to the identity (equivalent to the group of all homeomorphisms with the CHEP).

In these earlier papers, algebraic methods employing commutators were strongly used. This resulted in conjugates of the inverse as well as of the original homeomorphism being necessary in the arguments given.

2, Description of General Structures

It will be understood throughout that all homeomorphisms are of the space under consideration onto itself. For any space X, e denotes the identity homeomorphism. For any homeomorphism f of X and any $Z \subset X$, $f|Z$ denotes the homeomorphism f restricted to (the domain) Z. For $Y \subset X$, $Y^{\sim}$ denotes the complement of Y and Int Y denotes the interior of Y in X. If X is a space and $Y \subset C$, a homeomorphism f of X is said to be supported on Y provided $f|Y^{\sim} = e|Y^{\sim}$.

In the definition below and in the remainder of this paper the following notation is adopted:

(a) X is a metric space and X' a subset of X containing no isolated points (of X'),

(b) K is the set of closures of some open basis of X' in X (with each element of the open basis containing a point of X'),

(c) $G(X,X')$ is a non-null set of homeomorphisms of X onto itself, each carrying X' onto X',

(d) G_K is the set of all homeomorphisms supported on elements of K and carrying X' onto X', and

(e) G_K^* is the set (and thus the group) of all finite products of elements of G_K.

The set $(X,X',K,G(X,X'))$ is called an A-quadruple provided
(1) for any $k \in K$, $Cl(k^{\sim}) \in K$,
(2) for any ordered sextuple $(k_1,k_2,k_3,k_4,k_5,k_6)$ of disjoint elements of K, there exists $k \in K$ such that $(k_1 \cup k_2 \cup k_3) \subset \text{Int } k$ and $k \cap (k_4 \cup k_5 \cup k_6) = \emptyset$,
(3) for any $k \in K$ and $g \in G(X,X')$, $g(k) \in K$,
(4) for any $g_1, g_2 \in G(X,X')$ and $g \in G_K$, $gg_1, g_1g \in G(X,X')$ and g_1g_2, $g_2g_1 \in G_K^*$,
(5) for any $k_1,k_2,k \in K$ with $(k_1 \cup k_2) \subset \text{Int } k$ and $k \neq k_1,k_2$, there exists $g \in G_K$ with support on k such that $g(k_1) = k_2$, and
(6) for any $k \in K$, open set $U \supset k$ and $g \in G_K^*$ with $g(k) = k$, there exists $h \in G_K$ with support on U such that $h|k = g|k$.

We henceforth assume the existence of an A-quadruple and shall refer to Conditions (1) - (6) above.

Remark. The set X' may be thought of as the boundary of X if X is an n-cell. In the examples (a), (b), (c), and (f) cited in the introduction X' = X. With this understanding it is possible to see that, except for S^1 and I^2, the A-quadruple can be interpreted in terms of these examples. A fuller discussion is given in Section 7. In general , the structure and the theorems are not true for diffeomorphisms or piecewise-linear homeomorphisms of S^n (with all homeomorphisms concerned so restricted). The difficulty is that G_K is the set of all homeomorphisms of a certain type and the convergence criterion of Section (3) yields homeomorphisms not necessarily differentiable or piece-wise-linear.

In fact, the Milnor example of non differentiably-related diffeomorphisms of S^7 shows that the result cannot be true for diffeomorphisms in S^7 even though orientation-preserving diffeomorphisms of S^7 are known to have the CHEP.

Remark: From condition (3) it follows that every conjugate of an element of G(X,X') by an element of G_K^* is an element of G(X,X').

Remark: From conditions (3) and (4) it follows that for $k \in K$ and $g \in G_K^*$, $g(k) \in K$ and from condition (4) that any product of an element of G(X,X') and an element of G_K^* is an element of G(X,X').

Remark: Condition (1) in the presence of Conditions (2) and (5) and the fact that K is the set of closures of an open basis of X' in X is a condition for invertibility of the space X. We could use a somewhat weaker form of (1) asserting that $Cl(k^{\sim})$ is non-null and a subset of some element of K. For convenience we use condition (1) as written. We note that manifolds other than spheres (or cells) do not, in general, satisfy the condition of invertibility.

Remark: It follows, from the definitions and conventions, that all homeomorphisms under consideration must be of X onto X and must carry X' onto X'. Any homeomorphism of X onto X constructed by a convergence process must be checked for this second condition.

Remark: Condition (2) in light of the other conditions cannot be achieved for $X = S^1 = X'$ or $X = I^2$ and $X' = S^1$. Alternative conditions are possible to give in such cases and the theorems are true as stated. Condition (2) is intended primarily for use in Section 5 and for proving Lemma 32. In both instances the results are easily seen true for $X = S^1$ or $X = I^2$. We stated the conditions for an A-quadruple in as simple and intuitive a way as we could so that it would be easy to verify that the conditions are achieved in the higher-dimensional cases.

3. Convergence of a sequence of homeomorphisms

In Lemma 3.2 and in Section 6, sequences of homeomorphisms will be set up with the intention that such sequences converge to homeomorphisms. The nature of the constructions require only a rather weak convergence lemma. We state it and regard the proof as obvious.

Lemma 3.1: *Let X and Y be spaces. Let* $\{A_i\}$ *and* $\{B_i\}$ *be nested sequences of closed sets in X and Y respectively such that* dia $A_i \to 0$, dia $B_i \to 0$, *and* $\cap A_i \neq \emptyset \neq \cap B_i$. *Let* $\{h_i\}$ *be a sequence of homeomorphisms of X onto Y such that* $h_{i+1}^{-1} h_i$ *and* $h_{i+1} h_i^{-1}$ *are supported on subsets of* A_i *and* B_i *respectively. Then* $\{h_i\}$ *converges to a homeomorphism h of X onto Y with* $h(\cap A_i) = \cap B_i$ *and for each* $x \in X$ *with* $x \neq \cap A_i$, $h(x) = h_i(x)$ *for all sufficiently large* i.

In the applications of this lemma, X and Y will be the same space X and $\cap A_i$ and $\cap B_i$ will be points of X'.

Suppose g and ψ are homeomorphisms of X onto X. Suppose for some set W, $\psi g(W) \subset W$. Then if dia $(\psi g)^i(W) \to 0$ and $\bigcap_{i>0} (\psi g)^i(W)$ is a point we say that g *telescopes* W *with respect to* ψ.

Lemma 3.2: *Let* $\psi \in G_K$ *be such that for some* $A_o, W_o, Z_o \in K$ *with* $A_o \cap W_o = \emptyset$, *and* $Z_o \subset W_o$, $\psi(A_o) \subset \mathrm{Int}\, W_o \setminus Z_o$. *Let* $k, k_o \in K$ *be such that* $k \supset k_o$ *and* $A \cup W_o \subset \mathrm{Int}\, k \setminus k_o$. *Let* g_o *be any homeomorphism with support on* k *and with* $g_o(W_o) = A_o$. *Then there exists an element* $g \in G_K^*$ *with* g *supported on* k, $g(W_o) = A_o$ *and* $g|\tilde{W}_o = g_o|\tilde{W}_o$ *such that* g *telescopes* W_o *with respect to* ψ *and* $\cap(\psi g)^i(W_o)$ *and* $\cap(g\psi)^i(A_o)$ *are elements of* X'.

Proof. By hypothesis, $g(W_o) = A_o$. We introduce notation as follows:

$$\psi(A_o) = W_1$$

$$g_o(W_1) = A_1$$

$$\psi(A_1) = W_2$$

$$g_o(W_2) = A_2'.$$

Int W_o properly contains W_1 and Int A_o properly contains A_1. The analogous statements for i and i+1 will be assumed in what follows.

Let $p \in X' \cap \text{Int } A_1$. Let $A_2 \in K$ such that $p \in \text{Int } A_2$, A_2 is properly contained in Int A_1, dia $A_2 < \frac{1}{2}$ and dia $\psi(A_2) < \frac{1}{2}$. Condition (5) implies the existence of $g_1 \in G_K$ with support on A_1 such that $g_1(A_2') = A_2$. Hence for $g_1^* = g_1 g_o$, $g_1^*(W_2) = A_2$. We note that since g_1 is supported on A_1, then g_1^* may differ from g_o only in range A_1 and domain W_1. Thus $g_1^{*-1} \cdot g_o$ and $g_1 g_o^{-1}$ are supported on W_1 and A_1 respectively.

We now set up an induction in an analogous manner. Let $\psi(A_i) = W_{i+1}$ and $g_{i-1}^*(W_{i+1}) = A_{i+1}'$. Let $A_{i+1} \in K$ such that $p \in \text{Int } A_{i+1}$, A_{i+1} is properly contained in Int A_i, dia $A_{i+1} < \frac{1}{2^i}$ and dia $\psi(A_{i+1}) < \frac{1}{2^i}$. As before Condition (5) implies the existence of $g_i \in G_K^*$ with support on A_i such that $g_i(A_{i+1}') = A_{i+1}$. Hence for $g_i^* = g_i \cdot g_{i-1}^*$, $g_i^*(W_{i+1}) = A_{i+1}$. Since g_i is supported on A_i, then g_i^* may differ from g_{i-1}^* only in range A_i and domain W_i. Thus $g_i^{*-1} \circ g_{i-1}^*$ and $g_i^* \circ g_{i-1}^{*-1}$ are supported on W_i and A_i respectively. But dia $W_i < \frac{1}{2^{i-2}}$ and dia $A_i < \frac{1}{2^{i-1}}$ and thus except possibly for $\cap W_i$ and $\cap A_i$ as elements of X' the conditions of Lemma 3.1 are met. But, for each i, $p \in A_i$ and $p \in X'$. Hence $\cap A_i \in X'$ and $\cap A_i = p$. Also $\psi(p) = \cap W_i$ and by hypothesis ψ carries X' onto X'. Thus Lemma 3.1 implies Lemma 3.2.

4. The Conjugation Modification Procedure

In this section we have the following standing hypotheses. Let $\alpha, \beta \in G(X,X')$ and let k be an element of K such that (6)

(1) $[(\alpha)(\beta)](k) = k$ and

(2) $[(\beta)^{-1}(k) \cup (\alpha)(k)] \cap k = \emptyset$ and $((\beta)^{-1}(k) \cup (\alpha)(k) \cup k)$ is contained in some element of K.

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(6) Here, as later, for $\alpha \in G(X,X')$, (α) denotes a conjugate of α by an element of G_K^* and $(\alpha)(\beta)$ denotes the composition homeomorphism of (β) followed by (α).

Lemma 4.1: Let ω be an element of G_K supported on a subset A of k. Then $\omega(\alpha)\omega^{-1}(\beta)$ is a (twofold) product of conjugates of α and β by elements of G_K^* and

(1) $\omega(\alpha)\omega^{-1}(\beta)|k$ is $\omega(\alpha)(\beta)|k$,

(2) $\omega(\alpha)\omega^{-1}(\beta)|\tilde{k}$ is $(\alpha)\omega^{-1}(\beta)|\tilde{k}$

and (3) $\omega(\alpha)\omega^{-1}(\beta)|\tilde{k}$ differs from $(\alpha)(\beta)|\tilde{k}$ only on the domain $(\beta)^{-1}(A)$ to the range $(\alpha)(A)$.

Proof of Lemma

We first note that $\omega(\alpha)\omega^{-1}(\beta) = [\omega(\alpha)\omega^{-1}](\beta)$ and thus is a (twofold) product of suitable conjugates of α and β. Also, for any $x \notin (\beta)^{-1}(A)$, $(\beta)(x) \notin A$ and thus as ω^{-1} is supported on A, $\omega^{-1}(\beta)(x) = (\beta)(x)$ and for any $\gamma \in G(X,X')$ $\gamma\omega^{-1}(\beta)(x) = \gamma(\beta)(x)$ from which (1) follows. But (2) alsofollows since the remark above implies that $(\alpha)\omega^{-1}(\beta)$ carries k onto k and thus $\tilde{k}$ onto $\tilde{k}$ and since ω is supported on k then $\omega[(\alpha)\omega^{-1}(\beta)]|\tilde{k} =$ $= (\alpha)\omega^{-1}(\beta)|\tilde{k}$. Finally the same remark implies (3) with respect to the domain $\beta^{-1}(A)$ and since $\alpha(A)$ is the image of $\beta^{-1}(A)$ under $(\alpha)(\beta)$, the lemma follows.

Lemma 4.2: Let $\omega \in G_K$ be supported on an open set $U \supset k$ such that $\omega(k) = k$ and $U \cap (\beta^{-1}(U) \cup \alpha(U)) = \emptyset$. Then $\omega(\alpha)\omega^{-1}(\beta)$ is a (twofold) product of conjugates of α and β by elements of G_K^* and

$$\omega(\alpha)\omega^{-1}(\beta)|k \text{ is } \omega(\alpha)(\beta)|k.$$

Proof: Similar to a part of the proof of Lemma 4.1.

5. The Cell-Support Lemma

Lemma 5.1. Let $\alpha,\beta \in G(X,X')$ be such that $\alpha|X'$, $\beta|X' \neq e|X'$. Then for any $k_o, k_o' \in K$ with $k_o \cap k_o' = \emptyset$ and $(k_o \cup k_o') \cap X' \neq X'$, there exist elements $f_1, f_2 \in G_K^*$ such that $(f_1^{-1}\alpha f_1)(f_2^{-1}\beta f_2)$ is supported on k_o' and $(f_1^{-1}\alpha f_1)(k_o) \cup (f_2^{-1}\beta^{-1}f_2)(k_o) \subset k_o'$.

Proof: Let $p,q \in X'$ be such that $p,q\beta(p)$ and $\alpha(q)$ are all distinct. By considering small enough neighborhoods of these points, it follows that there exist disjoint elements k_1, k_2, k_3, k_4, k_5, k_6 of K with $\alpha(q) \in k_2$, $\alpha^{-1}(k_2) = k_4$, $p \in k$, and $\beta(k_1) = k_3$. By Condition 2 of Section 2, there exists an element $Z_o \in K$ with Int $Z_o \supset k_1 \cup k_2 \cup k_5$ and with Z_o not intersecting $k_3 \cup k_4 \cup k_6$. But as k_6 contains three disjoint elements k_7 , k_8 and k_9 of K, then Condition 2 implies the existence of an element $Z_o' \in K$ with Int $Z_o' \supset k_3 \cup k_4 \cup k_9$ and Z_o' not intersecting $Z_o \cup k_7 \cup k_8$.

Condition 5 asserts the existence of homeomorphisms ψ_o and ψ_o' of G_K supported on Z_o and Z_o' respectively and with $\psi_o(k_1) = k_2$ and $\psi_o'(k_3) = k_4$. Then if $\psi = \psi_1\psi_2$, ψ is an element of G_K (it is supported on $Cl(\tilde{k_7})$) and $\psi(k_1) = k_2$, $\psi(k_3) = k_4$. But $\psi^{-1}\alpha\psi\beta = (\psi^{-1}\alpha\psi)\beta$ carries k_1 onto itself and is a product of a conjugate of β by one of α (in that order of action on X).

We note that $\beta(k_1)$ and k_1 are disjoint and hence k_1 and $\beta^{-1}(k_1)$ must be disjoint as they are the images of disjoint sets under β^{-1}. Also

$$\psi(k_1) = k_2, \quad \alpha^{-1}(k_2) = k_4 \text{ and } \psi^{-1}(k_4) = k_3.$$

Also $(\psi^{-1}\alpha^{-1}\psi)(k_1)[= k_3]$ and k_1 were chosen as disjoint which implies, as above, that $(\psi^{-1}\alpha\psi)(k_1)$ and k_1 are also disjoint.

Now we shall use the modification procedure of Lemma 4.2 of Section 4 for some sufficiently small neighborhood U of k_1. We note that $(\psi^{-1}\alpha\psi)\beta$ is the product of two elements of $G(X,X')$ and thus by Condition (4) is an element of G_K^*. Also it carries k_1 onto k_1 and thus condition (6) of the definition of A-quadruple applies. Let μ be an element of G_K such that μ is supported on U and $\mu|k_1 = (\psi^{-1}\alpha\psi)\beta|k_1$. Then by Lemma 4.2 $[\mu^{-1}(\psi^{-1}\alpha\psi)\mu]\beta$ is a homeomorphism which (obviously) is the product of a conjugate of β by one of α (conjugations being by elements of G_K^*) with $(\mu^{-1}\psi^{-1}\alpha\psi\mu)\beta$ being the identity on k_1 and with $[\beta^{-1}(k_1) \cup (\mu^{-1}\psi^{-1}\alpha\psi\mu)(k_1)] \cap k_1 = \emptyset$. Let k_1^* be an element of K with $k_1^* \subset$ Int k_1 and $k_1^* \neq k_1$.

To establish the lemma it suffices to exhibit an element $\lambda \in G_K$ such that λ carries k_o onto k_1^* and k_o' onto $Cl(\tilde{k}_1)$. Then

$$\lambda^{-1}[(\mu^{-1}\psi^{-1}\alpha\psi\mu)\beta]\lambda = (\lambda^{-1}\mu^{-1}\psi^{-1}\alpha\psi\mu\lambda)(\lambda^{-1}\beta\lambda)$$

is, on the left side of the equation, the homeomorphism desired and is, on the right, expressed as a product of a conjugate of β by one of α. Since K is a basis for X' and X' has no isolated points, there exist four sufficiently small elements of K $(\tilde{k}_o, \tilde{k}_o', \hat{k}, \hat{k}')$ such that the sets $(k_o, k_o', \tilde{k}_o, \tilde{k}_o')$, $(\tilde{k}_o, \tilde{k}_o', \hat{k}, \hat{k}')$ and $(\hat{k}, \hat{k}', k_1^*, Cl(\tilde{k}_1))$ are quadruples of elements of K with all elements of each quadruple disjoint from each other and in some element of K. Thus for each quadruple there exist two more disjoint elements of K also disjoint from the elements of the quadruple. Now the hypotheses for Condition 2 are set up and we may by the use of Condition 2 and then Condition 5 as above assert the existence of $\lambda_1, \lambda_2, \lambda_3 \in G_K$ such that $\lambda_1(k_o) = \tilde{k}_o$, $\lambda_1(k_o') = \tilde{k}_o'$, $\lambda_2(\tilde{k}_o) = \hat{k}$, $\lambda_2(\tilde{k}_o') = \hat{k}'$, $\lambda_3(\hat{k}) = k_1^*$, and $\lambda_3(\hat{k}') = Cl(\tilde{k}_1)$. Then $\lambda = \lambda_3 . \lambda_2 . \lambda_1$ carries k_o onto k_1^* and k_o' onto $Cl(\tilde{k}_1)$ and is an element of G_K^* as was to be shown.

6. The Main Theorems and Their Corollaries

Theorem I:

Let $\alpha_o, \beta_o, \gamma_o, \delta_o \in G(X,X')$ with each different from e on X'. Then there exist $\lambda, \rho, \sigma, \tau \in G_K^*$ such that $(\lambda^{-1}\alpha_o\lambda)(\rho^{-1}\beta_o\rho) = (\sigma^{-1}\gamma_o\sigma)(\tau^{-1}\delta_o\tau)$.

Proof: By Lemma 5.1 and the properties of K, there exist $\alpha, \beta, \gamma, \delta \in G(X,X')$ and $A, B, V, W, Z_1, Z_2 \in K$ such that

(1) α, β, γ, and δ are conjugates of $\alpha_o, \beta_o, \gamma_o, \delta_o$ respectively by elements of G_K^*,

(2) A, B, V, W, Z_1 and Z_2 are disjoint,

(3) $\eta_o = \alpha\beta$ and $\mu_o = \gamma\delta$ are supported on V and W respectively
and (4) $(\alpha(A) \cup \beta^{-1}(A)) \subset \text{Int } V$ and $(\gamma(B) \cup \delta^{-1}(B)) \subset \text{Int } W$.

Either for every $A_o' \subset \text{Int } A$ with $A_o' \in K$, $\alpha(A_o') = \beta^{-1}(A_o')$ or for some $A_o \in K$ with $A_o \subset \text{Int } A$, $\alpha(A_o) \cap \beta^{-1}(A_o) = \emptyset$. We may make a similar remark about B_o, $\gamma(B_o)$ and $\delta^{-1}(B_o)$.

We let $A_o, B_o \in K$ with $A_o \subset \text{Int } A$ and $B_o \subset \text{Int } B$ and suppose, without loss of generality, that either Case I: $\gamma(B_o) \cap \delta^{-1}(B_o) = \emptyset$ or Case II: for all A_o', $B_o' \in K$, $A_o' \subset \text{Int } A_o$ and $B_o' \subset \text{Int } B_o$, $\alpha(A_o') = \beta^{-1}(A_o')$ and $\gamma(B_o') = \delta^{-1}(B_o')$. Case II would occur if α, β, γ and δ were all involutions.

Let, by Condition 2, $k_1, k_2 \in K$ such that $(V \cup B \cup Z_1) \subset \text{Int } k_1$ and $(A \cup W \cup Z_2) \cap k_1 = \emptyset$ and also $(A \cup W \cup Z_3) \subset \text{Int } k_2$ and $(k_1 \cup Z_4 \cup Z_5) \cap k_2 = \emptyset$ where $Z_3, Z_4, Z_5 \subset Z_2$ are disjoint elements of K. Let, by Condition (5), π_1 be an element of G_K supported on k_1 with $\pi_1(B_o) = V$. Also let g_o be an element of G_K supported on k_2 with $g_o(W) = A_o$. Consider $\gamma\pi_1^{-1}\beta^{-1}$ as ψ of Lemma 3.2 and consider $\psi g_o(W)$ as W_o of that Lemma. Let g be the telescoping homeomorphism promised by Lemma 3.2. Then $g|\tilde{W}_o = g_o|\tilde{W}_o$.
For Case II we consider $g = g'$ and omit the next step. For Case I, we consider, using primed notation for that of Lemma 3.2, g as a new g_o' and $\delta^{-1}\pi_1^{-1}\alpha$ as a new ψ'. Consider $\psi' g_o'(W)$ as W_o' for the application of Lemma 3.2. We note that $W_o' \cap W_o = \emptyset$. Let g' be the telescoping homeomorphism promised by the Lemma. Then we let $\pi = \pi_1 g' = g'\pi_1$ and note that π is an element of G_K as π_1 and g' are elements of G_K supported on k_1 and k_2 respectively and are both the identity on Z_4.

We now consider the final step of the construction. Let $\omega_o = \pi\mu_o\pi^{-1}$. Clearly ω_o is supported on A_o. Then $\alpha\omega_o^{-1}\beta$ is supported on V and $\alpha\omega_o^{-1}\beta|V = \omega_o\alpha\,\omega_o^{-1}\beta|V$. Let $\alpha\omega_o^{-1}\beta = \eta_1$ and let $\phi_o = \pi^{-1}\eta_1\pi$. Then ϕ_o is supported on B_o and $\gamma\,\phi_o^{-1}\delta = \mu_1$ is supported on W. Also $\gamma\phi_o^{-1}\delta|\,W$ is $\phi_o\gamma\,\phi_o^{-1}\delta|\,W$ and may differ from μ_o only on domain $\delta^{-1}(B_o)$ to range $\gamma(B_o)$.

Let $\omega_1 = \pi\mu_1\pi^{-1}$ and ω_1 is supported on A_o and may differ from ω_o only on domain $\pi\delta^{-1}(B_o)$ to range $\pi\gamma(B_o)$. Thus ω_1^{-1} may differ from ω_o^{-1} only on domain $\pi\gamma(B_o)$ to range $\pi\delta^{-1}(B_o)$. Then $\alpha\omega_1^{-1}\beta = \eta_2$ is supported on V and may differ from η_1 only on domain $\beta^{-1}\pi\gamma(B_o)$ to range $\alpha\pi\delta^{-1}(B_o)$. Continuing we let $\phi_1 = \pi^{-1}\eta_2\pi$ and ϕ_1 may differ from ϕ_o only on domain $\pi^{-1}\beta^{-1}\pi\gamma(B_o)$ to range $\pi^{-1}\alpha\pi\delta^{-1}(B_o)$. Thus ϕ_1^{-1} may differ from ϕ_o^{-1} only on domain $\pi^{-1}\alpha\pi\delta^{-1}(B_o)$ to range $\pi^{-1}\beta^{-1}\pi\gamma(B_o)$ and $\mu_2 = \gamma\phi_1^{-1}\delta$ may differ from μ_1 only on domain $\delta^{-1}\pi^{-1}\alpha\pi\delta^{-1}(B_o)$ to range $\gamma\pi^{-1}\beta^{-1}\pi\gamma(B_o)$.

It is clear that the procedure can be iterated, μ_i producing ω_i producing η_{i+1} producing ϕ_i producing μ_{i+1}.

Consider the sequence $\{\mu_i\}$. For each $i \geq 0$, $\mu_{i+1}^{-1}\mu_i$ is supported on $(\delta^{-1}\pi^{-1}\alpha\pi)^i\delta^{-1}(B_o)$ and $\mu_{i+1}\mu_i^{-1}$ is supported on $(\gamma\pi^{-1}\beta\pi)^i\ \gamma(B_o)$. From the definition of π, $\cap(\gamma\pi^{-1}\beta^{-1}\pi)^i(W_o) \subset X'$ and $\cap(\delta^{-1}\pi^{-1}\alpha\pi)^i(W_o') \subset X'$. For $i > 1$, we may write $(\delta^{-1}\pi^{-1}\alpha\pi)^i\delta^{-1}(B_o)$ as $(\delta^{-1}\pi^{-1}\alpha\pi)^{i-1}(\delta^{-1}\pi^{-1}\alpha\pi\delta^{-1})(B_o)$*F* Also $\delta^{-1}\pi^{-1}\alpha\pi\delta^{-1}(B_o) \subset W_o$ and $\gamma\pi^{-1}\beta^{-1}\pi\gamma(B_o) \subset W_o'$. Thus the conditions of Lemma 3.1 are satisfied and $\{\mu_i\}$ converges to a homeomorphism μ carrying $\cap(\delta^{-1}\pi^{-1}\alpha\pi)^i(W_o')$ onto $\cap(\gamma\pi^{-1}\beta^{-1}\pi)^i(W_o)$ and with for each other point w of W $\mu(w) = \mu_i(w)$ for all sufficiently large i. But ω, η and ϕ may be similarly defined.

From the (almost) pointwise agreement of ω, η, and ϕ with ω_i, η_i, and ϕ_i (respectively) for sufficiently large i it follows immediately that $\alpha\omega^{-1}\beta = \eta$ and $\gamma\phi^{-1}\delta = \mu$. Also $\omega = \pi\mu\pi^{-1}$, $\phi = \pi^{-1}\eta\pi$ and μ is supported on W, η is supported on V, ω is supported on A_o and ϕ is supported on B_o. Thus

$$\omega\alpha\omega^{-1}\beta = \begin{cases} \eta & \text{on } V \\ \omega & \text{on } A_o \\ e & \text{otherwise} \end{cases} \quad \text{and}$$

$$\phi\gamma\phi^{-1}\delta = \begin{cases} \mu & \text{on } W \\ \phi & \text{on } B_o \\ e & \text{otherwise} \end{cases} \ .$$

F and $(\gamma\pi^{-1}\beta^{-1}\pi)^i\gamma(B_o)$ as $(\gamma\pi^{-1}\beta^{-1}\pi)^{i-1}(\gamma\pi^{-1}\beta^{-1}\pi\gamma)(B_o)$.

Finally $\pi^{-1}\omega\alpha\omega^{-1}\beta\pi = \phi\gamma\phi^{-1}\delta$ for $\pi^{-1}\omega\alpha\omega^{-1}\beta\pi = (\pi^{-1}\omega\pi)(\pi^{-1}\alpha\omega^{-1}\beta\pi)$ and

$$\pi^{-1}\omega\pi=\eta=\gamma\phi^{-1}\delta \text{ and } \pi^{-1}\alpha\omega^{-1}\beta\pi = \pi^{-1}\mu\pi = \phi$$

and $\eta\phi=\phi\eta$ as η and ϕ are supported on disjoint sets. Also

$\pi^{-1}\omega\alpha\omega^{-1}\beta\pi$ may be written as $(\pi^{-1}\omega\alpha\omega^{-1}\pi)(\pi^{-1}\beta\pi)$ and $\phi\gamma\phi^{-1}\delta$ may be written as $(\phi\gamma\phi^{-1})(\delta)$. Since $\phi,\omega,\pi \in G_K$; the conjugating homeomorphisms are all elements of G_K^* and the theorem is established.

Theorem II:

Let $\alpha,\beta,\gamma,\delta \in G(X,X')$ with none of $\alpha,\beta,\gamma,\delta$ being the identity on X'. Then α is the product of a conjugate of δ by a conjugate of γ by a conjugate of β.

Proof: Let $\beta^{-1} = \alpha_o$, $\alpha = \beta_o$, $\gamma = \gamma_o$ and $\delta = \delta_o$ of Theorem I. Then for some $\lambda,\rho,\sigma,\tau \in G_K^*$, $(\gamma^{-1}\beta^{-1}\gamma)(\rho^{-1}\alpha\rho) = (\sigma^{-1}\gamma\sigma)(\tau^{-1}\delta\tau)$ and

$$\alpha = \rho(\lambda^{-1}\beta\lambda)\rho^{-1}\rho(\sigma^{-1}\gamma\sigma)\rho^{-1}\rho(\tau^{-1}\delta\tau)\rho^{-1}$$

$$= (\rho\lambda^{-1}\beta\lambda\rho^{-1})(\rho\sigma^{-1}\gamma\sigma\rho^{-1})(\rho\tau^{-1}\delta\tau\rho^{-1}) \text{ with}$$

$\lambda\rho^{-1}, \sigma\rho^{-1}, \tau\rho^{-1}$ and their inverses all in G_K^* since λ,σ,τ and ρ are.

Theorem III:

Let $\alpha_1, \beta_1 \in G(X,X')$ with $\alpha_1|X', \beta_1|X' \neq e|X'$. Then α_1 is the product of three conjugates of β_1.

Proof: Theorem III is an immediate corollary of Theorem II letting $\beta = \gamma = \delta = \beta_1$ and $\alpha = \alpha_1$.

It is interesting to note that for $G(X,X') = G_K^*$, it is not known, either in general or in the cases of the specific examples, whether the identity e is the product of three conjugates of β. It seems possible that e is a true exception. In a slightly different situation, suggested by Theorem II, in many of the examples e is the only exception. It will be shown in Section 8 that for many of the examples, there exist

homeomorphisms h and γ such that h is not the product of two conjugates of γ. Then e is not the product of a conjugate of h^{-1} by a conjugate of γ by a conjugate of γ, for if so, h would be the product of two conjugates of γ.
The following corollary is already known for most of the examples mentioned [1], [2], and [4].

Corollary I.

<u>The group</u> G_K^* <u>is algebraically simple if</u> $X = X'$.

Proof. Let $\beta \neq e$ be an element of a normal subgroup N. Then every element of G_K^* is in N for each is the product of three conjugates of β where the conjugating homeomorphisms are elements of G_K^*.

The following corollary is also known [4].

Corollary II.

<u>If</u> $X = S^n$, <u>every element of</u> G_K^* <u>is isotopic to the identity.</u>

We may note that the nature of the isotopy can be particularly simple. Let $h \in G_K^*$ be any homeomorphism with a nice geometric isotopy to the identity (and there exist many such). Then an arbitrary element of G_K^* can be isotopied back to the identity by dealing with the three conjugates of h consecutively and thus having only two levels of "singularities", namely as one goes from the coordinate system of one conjugate to the coordinate system of the next.

7. The Cases of the Examples of Section 1

a) $X = X' = C$, the Cantor Set. K is the set of all non-null open and closed sets whose complements are non-null. $G(X,X')$ is the set of all homeomorphisms of C onto itself. Since the elements of K are all open and closed and freely admit homeomorphisms onto each other, there is no difficulty in verifying that $G_K^* = G(X,X')$ and G_K^* is the set of all homeomorphisms with the CHEP. Also the conditions (1) - (6) are all satisfied, e.g. (6) is immediate since $h|k^{\sim}$ can be taken to be $e|k^{\sim}$. The condition in (5) that $k_1, k_2 \neq k$ is designed for examples of this type.

(b) Let $X = X' = S^n$, the n-sphere, $n > 1$. K is the set of all images of a canonical hemisphere under homeomorphisms each of which is the finite product of homeomorphisms supported on cells. G_K and G_K^* are thus well-defined and we let $G(X,X') = G_K^*$. In [3] in their study of stable homeomorphisms, Brown and Gluck give their Theorem 3.2 and its corollary. The theorem implies immediately that the Conditions (5) and (6) are satisfied. The Corollary implies that all elements of G(X,X ')= { the set of all stable homeomorphisms } have the CHEP. Conditions 1,3 and 4 are obvious from our definition. For Condition 2 we note that as the cells $k_1, \ldots, k_6$ are all disjoint there exist disjoint neighborhoods $U_1, \ldots, U_6$ with $k_i \subset U_i$, $1 \leq i \leq 6$, such that, for each i, k_i can be contracted in U_i toward a point of k_i. (Note that each k_i is a topological hemisphere in S^n). Then there exists a thin tame (or flat) cell containing the contracted k_1, k_2 and k_3 and missing the contracted k_4, k_5 and k_6. Then under the anti-contractions this cell becomes the desired cell for Condition 2.

(c) Let X, X', K, G_K and G_K^* be as in (b) but let G(X,X') be the set of all homeomorphisms which are the product of some geometric orientation-reversing involution h and an element of G_K^*. Then Condition (4) is satisfied and the other conditions follow as in example (b). If g is any orientation-reversing homeomorphism we note that g·h will be orientation-preserving and g·h either is an element of G_K^* or it is not. If so, then obviously $g \in G(X,X')$ and if not then $g \notin G(X,X')$. But the second alternative can only occur if some orientation-preserving homeomorphism does not have the CHEP (or, equivalently, if the affirmative annulus conjecture is not true). Conversely, if some orientation-preserving homeomorphism does not have the CHEP then its product by h produces such a g.

(d) $X = I^n$, $n > 2$, $X' = \text{Bdry } I^n = S^{n-1}$, K is the set of all images of a canonical half-cell of I^n under homeomorphisms which are the finite product of homeomorphisms each of which is the identity in some neighborhood of some point of Bdry I^n. Finally, $G(X,X') = G_K^*$.

Conditions (1), (3) and (4) are obvious from the definitions given. Condition (2) follows from an argument like that given for Condition (2) in Example (b) above. Conditions (5) and (6) follow by arguments like those leading to the Brown-Gluck Theorem for S^n (referred to in (b)). The arguments (not given here) are non-trivial but routine and depend on the basic structure of "stable" homeomorphisms.

(e) Let X, X', K, G_K and G_K^* be as in (d) but let $G(X,X')$ be the set of all homeomorphisms each of which is the product of some geometric orientation-reversing involution h and an element of G_K^*. Further considerations are like those of example (c).

(f) Let $X = X'$ be the space of rationals (or irrationals) on the line. Let K be the set of closed and open proper subsets of X. Then considerations like those of example (a) lead to verification of Conditions (1) to (6).

(g) Let X be a space of (a) - (f) and let X" be a countable dense subset of the original X' such that "enough" homeomorphisms exist carrying X onto X and X" onto X". We are to treat X" as if it were X'. The convergence lemmas imply that the set X' (or X") must be homogeneous in X. It is not difficult to verify that Conditions 1 - 6 can be satisfied in such cases nor is it difficult to give some additional examples in the same spirit. We could, for instance, let $X = S^2$ or S^3 and let X' be a tame (or geometric) Cantor Set C in such space, with $G(X,X')$ being the set of orientation-preserving homeomorphisms of X which carry X' onto X'.

8. The Necessity for the Use of at Least Three Conjugates

In this section we establish that there exist homeomorphisms of C, S^n, I^n, the rationals, the irrationals, etc. which are not the product of two involutions. Thus, in general, three is the least number of conjugates of an arbitrary homeomorphism which can be used to produce another arbitrary homeomorphism.

Let Y be an abstract set (no topology) and let h be a 1-1 transformation of Y onto itself. For each $y \in Y$, the orbit $O_h(y)$ of y under h is the set of all images of y under e, iterates of h, or iterates of h^{-1}. If $O_h(y)$ is finite, then the elements of $O_h(y)$ are cyclically permuted by h. If $O_h(y)$ is infinite, then on $O_h(y)$, h is equivalent to a translation on the set of integers on the line.

Lemma 8.1: Suppose $h = \lambda_1 \lambda_2$ with λ_1 and λ_2 both involutions. Then

(1) each of λ_1 and λ_2 carries each orbit under h onto an orbit of the same cardinality and carries such orbit back to the original one,

(2) for any orbit $O_h(y)$, $\lambda_1 [O_h(y)] = \lambda_2 [O_h(y)]$, and

(3) each of λ_1 and λ_2 reverses the order of orbits under h and

(4) if either carries an orbit onto itself then one of λ_1 and λ_2 has a fixed point on that orbit.

Sketch of Proof: Let $y_o \in Y$ and let y_i denote $h^i(y_o)$. Suppose $\lambda_2(y_o) = z_o$. Let z_i denote $h^i(z_o)$. Since $\lambda_2(y_o) = z_o$, then $\lambda_1(z_o) = y_1$ for $\lambda_1\lambda_2 = h$. Since λ_2 is an involution, $\lambda_2(z_o) = y_o$ and then $\lambda_1(y_o) = z_1$ also. But also since $\lambda_1(z_1) = y_o$, then $\lambda_2(y_{-1}) = z_1$, and since $\lambda_1(y_1) = z_o$ then $\lambda_2(z_{-1}) = y_1$. Also since $\lambda_2(z_1) = y_{-1}$ then $\lambda_1(y_{-1}) = z_2$ etc. The above pattern is valid even if $z_o \in O_h(y_o)$. The Lemma follows directly from these considerations.

Remark: If Y is a metric space, then to show that there exists a homeomorphism h of Y such that h is not the product of two involutions, it suffices to exhibit an h with one or more distinctive orbits each dense in a closed set such that no involution can reverse the sense of such orbits or carry such orbits onto others.

Let α be an irrational rotation of the circle Σ_1. Let $O_1,\ldots,O_j$, $j \geq 3$ be j orbits under α such that no α-orbit preserving involution of Σ_1 carries $\cup O_i$ onto itself. Since all the orbits under α are geometric copies of each other and are rigid in both directions, any involution permuting orbits must be an isometry. Hence it is not difficult to exhibit such $O_1,\ldots,O_j$ which are "irrationally related".

Let ψ be a map of a circle Σ_2 onto Σ_1 such that for each point p of $\cup O_i$, $\psi^{-1}(p)$ is an arc and for each other point q, $\psi^{-1}(q)$ is degenerate. Let β_j be a homeomorphism of Σ_2 onto itself uniquely induced by α on Σ_1 except on the interiors of the non-degenerate inverses under ψ. But an extension is clearly possible to such sets. A necessary and sufficient condition that two points be distal (7) under β_j is that the two points do not belong to the same point-inverse under ψ. But then any involution λ of Σ_2 which carries orbits under β_j onto orbits under β_j cannot carry an orbit projecting under ψ onto $O_1,O_2,\ldots$ or O_j onto an orbit not projecting under ψ onto $O_1,O_2,\ldots$ or O_j. Hence λ induces an α-orbit preserving involution of $\Sigma_1 \setminus \cup O_j$ which in turn induces one on Σ_1 contrary to the selection of $O_1,O_2,\ldots,O_j$. Hence β_j is not the product of two involutions on Σ_2.

For any orbit O^* under β_j which comes from an orbit under α other than $O_1,O_2,\ldots,O_j$, Cl O^* is a Cantor Set C. The homeomorphism β_j cut down to C also admits no involutions permuting orbits for such would induce one on Σ_2 and then on Σ_1. Thus we have examples of homeomorphisms on C and S^1 which are not products of two involutions. Also the homeomorphisms on C can be cut down to non-closed subsets of C homeomorphic to the rationals or irrationals on the line with such restricted homeomorphisms also not the product of two involutions.

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(7) x,y are distal under β_j provided that for some $\varepsilon > 0$ and each i, $-\infty < i < \infty$, $d(\beta^i(x),\beta^i(y)) > \varepsilon$.

To get a homeomorphism of S^n which is not a product of two involutions, we may use the homeomorphisms β_j on copies of Σ_2 in S^n. Suppose β_j acts on a geometric circle $S^1(j)$ in S^n. Then β_j can be extended to any small neighborhood U_j of $S^1(j)$ in such a way that (each point of Brdry U_j is fixed and) each point of U_j is either fixed or moves under the extension toward a point near or on the boundary of U_j. Under the inverse of the extension such points move toward (a subset of) $S^1(j)$. Thus the orbits on $S^1(j)$ are distinctive. Further for $j_1 \neq j_2$, no involution can carry $S^1(j_1)$, to $S^1(j_2)$ since β_{j_1} and β_{j_2} are essentially different homeomorphisms. Consider a countable null-collection $\{S^1(j)\}$ of disjoint isolated circles the closure of whose union contains two (n-1)-spheres S_1^{n-1} and S_2^{n-1} disjoint from the circles. Define β_j on $S^1(j)$ and extend as above to a small neighborhood of $S^1(j)$ disjoint from S_1^{n-1}, S_2^{n-1} and the other small neighborhoods. Then the composite of these extended homeomorphisms (extended itself by the identity elsewhere) is a homeomorphism h of S^n which as we shall verify is not the product of two involutions. First if $h = \lambda_1\lambda_2$ is such product then each of λ_1 and λ_2 must carry each $S^1(j)$ onto itself and must therefore be the identity on each $S^1(j)$ and thus be the identity on S_1^{n-1} and S_2^{n-1}. But then as the fixed point set contains two (n-1)-spheres each of the involutions must be the identity on S^n, a contradiction.

From such an h on the (n-1)-sphere regarded as the boundary of I^n there may be constructed (by inward projection of h toward the center of I^n) an h on I^n which is not the product of two involutions.

Thus in the cases of the examples, the number "three" of the Three Conjugates Theorems is the lowest possible number.

9. Questions and Comments

Does there exist any (nice) space admitting many homeomorphisms such that for any α,β having the CHEP, $\alpha,\beta \neq e$, α is the product of two conjugates of β? For C or S^n and α having the CHEP, is e the product of three conjugates of α? (It is not difficult to find some β such that e is the product of 3 conjugates of β). In research now continuing, Ellard Nunnally has shown by methods originally suggested by those of this paper but without using homeomorphisms which are the identity in any open set, that a dilation is the product of conjugates of two arbitrary (non-identity) "stable" homeomorphisms of S^n (or C), and hence that every "stable" homeomorphism of S^n (or C) is the product of two conjugates of a dilation. What homeomorphisms have one or both of these properties of dilations?

While the apparatus of this paper is set up in terms of metric spaces, which include the more interesting examples, we really only need to assume that our space X is first-countable Hausdorff. The additional restrictions imposed by the conditions of the A-quadruple, of course, limit the type of space very substantially. The theorems are true for certain non-metric zero-dimensional spaces.

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