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Prof. Dr. B.L.J. Braaksma

Jacobi and Gegenbauer Polynomials as Spherical Harmonics.

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"Jacobi and Gegenbauer Polynomials as spherical harmonics."

We shall define a point in q -dimensional space by a vector $\underline{x} = (x_1, x_2, \ldots, x_q)$ and shall write $u(\underline{x})$ for a funtion u of $x_1, x_2, \ldots, x_q$. The length of $\underline{x}$ will be denoted by r, explicitly we have $r = (x_1^2 + \ldots + x_q^2)^{\frac{1}{2}}$. If $\underline{y} = (y_1, \ldots, y_q)$ is a second vector, the inner product of $\underline{x}$ and $\underline{y}$ is denoted by $\underline{x}.\underline{y} = x_1y_1 + x_2y_2 + \ldots + x_qy_q$.
If A is an orthogonal transformation, then it is well known that $A\underline{x}.A\underline{y} = \underline{x}.\underline{y}$.

We proceed to the consideration of solutions of Laplace's equation in q dimensions:

$$\Delta_q V \equiv \frac{\partial^2 V}{\partial x_1^2} + \ldots + \frac{\partial^2 V}{\partial x_q^2} = 0$$

These solutions are called harmonic functions. If these solutions are homogeneous functions of degree n in $x_1, \ldots, x_q$ i.e. $V(\lambda\underline{x}) = \lambda^n V(\underline{x})$ for every $\lambda \neq 0$, they are called solid spherical harmonics or simply spherical harmonics.

Every point $\underline{x} \neq 0$ has a projection $\underline{\xi}$ on the unit sphere: $\underline{\xi} = r^{-1}\underline{x} = (\xi_1, \ldots, \xi_q)$.
So a spherical harmonic $H_n(\underline{x})$ of order n can be expressed in the form:

$$H_n(\underline{x}) = r^n H_n(\underline{\xi}).$$

The function $H_n(\underline{\xi})$ on the unit sphere is called a spherical surface harmonic of degree n.
It is well known that the operator Δ_q is invariant under orthogonal transformations i.e. if $H(\underline{x})$ is harmonic then also $H(A\underline{x})$ is harmonic for every orthogonal A.

Now we want to consider spherical harmonics $H(\underline{x})$ which are invariant under special classes of orthogonal transformations. First let C_1 be the class of A's such that $A\underline{\varepsilon}_q = \underline{\varepsilon}_q$ where $\underline{\varepsilon}_q = (0,0, \ldots,0,1)$. In fact the A's are all orthogonal transformations of the first q-1 coordinates.
Let $H_n(\underline{x})$ be a homogeneous polynomial of degree n such that $H_n(A\underline{x}) = H_n(\underline{x})$ for all $A \in C_1$. It is well known that if $\underline{x}$ and $\underline{x}'$ are points with
$t^2 = x_1^2 + \ldots x_{q-1}^2 = x_1'^2 + \ldots x_{q-1}'^2,\ x_q = x_q'$,
then there exists an A in C_1 such that $A\underline{x} = x'$. This implies that $H_n(\underline{x})$ only depends on x_q and r. So

$$H_n(\underline{x}) = H_n(\sqrt{r^2-x_q^2}, 0, 0, \ldots, 0, x_q).$$

Since $H_n(\underline{x})$ is a homogeneous polynomial of degree n, we have

$$H_n(\sqrt{r^2-x_q^2}, 0, \ldots, 0, x_q) = \sum_{j=0}^{n} a_j (r^2-x_q^2)^{\frac{1}{2}j}\ x_q^{n-j}$$

or

$$H_n(\underline{x}) = \sum_{j=0}^{n} a_j (x_1^2 + \ldots + x_{q-1}^2)^{\frac{1}{2}j}\ x_q^{n-j}$$

As $H_n(\underline{x})$ is a polynomial in $x_1, \ldots, x_q$, we have $a_1 = a_3 = \ldots = 0$.
If we restrict $\underline{x}$ to lie on Ω_q then $H_n(\underline{x})$ becomes a polynomial in x_q of degree n, say $G_n(x_q)$.
Now we require that these $H_n(\underline{x})$ are spherical harmonics i.e. $H_n(\underline{x})$ satisfy Laplace's equation.

By Green's theorem we have

$$\int_{|x| \leq 1} \{H_\lambda(\underline{x})\Delta_q H_\mu(\underline{x}) - H_\mu(\underline{x})\Delta_q H_\lambda(\underline{x})\}dV = \tag{1}$$
$$= \int_{\Omega_q} H_\lambda(\underline{x}) \frac{\partial}{\partial r} H_\mu(\underline{x}) - H_\mu(\underline{x}) \frac{\partial}{\partial r} H_\lambda(\underline{x})\, d\omega_q.$$

Since $H_\mu(\underline{x})$ is homogeneous, we have

$$\frac{\partial}{\partial r} H_\mu(\underline{x}) = \frac{\partial}{\partial r} r^\mu H_\mu(\underline{\xi}) = \mu r^{\mu-1} H_\mu(\underline{\xi}).$$

Furthermore $\Delta_q H(\underline{x}) = 0$. Hence (1) becomes

$$0 = (\mu-\lambda) \int_{\Omega_q} H_\lambda(\underline{\xi})H_\mu(\underline{\xi})d\omega_q, \text{ or}$$

$$\int_{\Omega_q} G_\lambda(\xi_q)G_\mu(\xi_q)d\omega_q = 0, \quad \text{if } \lambda \neq \mu \tag{2}$$

It can be shown that (2) is also sufficient condition for $H_n(\underline{x})$ to be spherical harmonics.

Now we reduce the integral in (2) to an integral with respect to ξ_q. We denote $G_\lambda(\xi_q)G_\mu(\xi_q)$ by $\Phi(\xi_q)$. It is well known that

$d\omega_q = \frac{1}{|\xi_q|} d\xi_1 \dots d\xi_{q-1}$, where $|\xi_q| = (1-\xi_1^2 \dots \xi_{q-1}^2)^{\frac{1}{2}}$ and the domain of integration is $\xi_1^2 + \dots + \xi_{q-1}^2 \leq 1$.

Instead of the variables $\xi_1, \dots, \xi_{q-1}$ we introduce the variables by means of $\xi_q = t$, $\xi_1 = \sqrt{1-t^2}\, \eta_1$, $\xi_{q-2} = \sqrt{1-t^2}\, \eta_{q-2}$. The new domain D of integration is $-1 \leq t \leq 1, \eta_1^2 + \dots + \eta_{q-2}^2 \leq 1$. The Jacobian of this transformation is equal to $(1-\eta_1^2- \dots -\eta_{q-2}^2)^{-\frac{1}{2}}(1-t^2)^{\frac{1}{2}(q-3)}|t|$. Hence the integral is transformed into

$$\int_{-1}^{1} dt \int\limits_{\eta_1^2+\dots+\eta_{q-2}^2 \le 1} \Phi(t)(1-t^2)^{\frac{1}{2}(q-3)}(1-\eta_1^2-\dots-\eta_{q-2}^2)^{-\frac{1}{2}} d\eta_1 \dots d\eta_{q-2} =$$

$$= \int\limits_{\Omega_{q-1}} d\omega_{q-1} \int_{-1}^{1} \Phi(t)(1-t^2)^{\frac{1}{2}q-\frac{3}{2}} dt$$

Hence from (2) we have

$$\int_{-1}^{1} G_\lambda(t)G_\mu(t)(1-t^2)^{\frac{1}{2}q-\frac{3}{2}} dt = 0 \text{ if } \lambda \neq \mu$$

and the $G_n(t)$ are orthogonal polynomials of degree n with respect to the weight-function $(1-t^2)^{\frac{1}{2}q-\frac{3}{2}}$ on the interval $[-1,1]$ and are apart from a multiplicative constant the Gegenbauer polynomials.

This is founded on the theory of the classical polynomials. One of the important theorems is:

Let $P_0(x)$, $P_1(x)$, ... be a system of orthogonal polynomials with respect to a weight function $w(x)$ in the interval (a,b) and let the degree of $P_n(x)$ be equal to $n, n = 0,1,2, \dots$ then every $P_n(x)$ is apart from a constant factor $\neq 0$ uniquely determined.

Proof with mathematical induction to the degree (=index) of the polynomials. For $n = 0$ the theorem is trivial. Suppose the theorem holds to a degree $n-1$, i.e. the polynomials $P_0(x), \dots, P_{n-1}(x)$ are uniquely apart from a constant factor. $P_n(x)$ is of degree n, and can be represented in the form $P_n(x) = \pi_{n-1}(x) + a_n x^n = c_0P_0(x) + \dots + c_{n-1}P_{n-1}(x) + a_n x^n$, where $\pi_{n-1}(x)$ is a polynomial of degree $\leq n-1$. Multiplying with $w(x)P_i(x)$, $(i=0,1,2, \dots, n-1)$ and integrating we find:

$$\int_a^b w(x)P_i(x)P_n(x)dx = C_i \int_a^b w(x)\{P_i(x)\}^2 dx + a_n \int_a^b w(x)P_i(x)x^n dx = 0$$

Hence

$$C_i = -a_n \frac{\int_a^b w(x)P_i(x)x^n\,dx}{\int_a^b w(x)\{P_i(x)\}^2 dx} \qquad i=0, \ldots, n-1$$

This means that the constants $C_0, \ldots, C_{n-1}$ apart from the common factor a_n depend not on $P_n(x)$ but only on $P_0(x)$. This completenes the proof.

Is the interval $[-1,1]$ and $w(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha > -1, \beta > -1$ then the corresponding orthogonal polynomials are apart from a constant factor, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

These polynomials are normalised by the condition $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$.

If $\alpha=\beta$, the polynomials $P_n^{(\alpha\alpha)}(x)$ are called ultraspherical polynomials.

Hence our polynomials $G_n^\lambda(t)$ are ultraspherical polynomials. There is a close relation between the ultraspherical and the Gegenbauer polynomials $C_n^\lambda(x)$:

$$C_n^\lambda(x) = \frac{(2\lambda)n}{(\lambda+\frac{1}{2})n} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x).$$

For $\alpha=\beta=0$ or $\lambda=\frac{1}{2}$ we have the Legendre polynomials.

Above we have defined the Jacobi polynomials by the orthogonality relations. These polynomials can also be characterized by the generalised Rodrigues'formula: The functions $P_n(x)$ defined by

$$P_n(x) = \frac{1}{K_n w(x)} \frac{d^n}{dx^n} \left[w(x)(1-x^2)^n\right]$$

are polynomials only if $w(x)=(1-x)^\alpha(1+x)^\beta$. These polynomials are Jacobi polynomials and orthogonal with respect to $w(x)$. The K_n are determined by the normalisation condition.

In a simular way it is possible to characterize the Jacobi polynomials as certain spherical harmonics. Let q_1 be an integer with $1 \leqq q_1 < q$. Consider now the class C of all orthogonal transformations A such that the subspace $x_{q_1+1}=\ldots=x_q=0$ is invariant with respect to A.

The last condition is equivalent to the invariance of the subspace $x_1=\ldots=x_{q_1}=0$. Consider the homogeneous polynomials $H_n(\underline{x})$ such that $H_n(A\underline{x}) = H_n(\underline{x})$ for all A in C. If $\underline{x}$ and $\underline{x}'$ are points with $x_1^2 + \ldots + x_{q_1}^2 = x_1'^2 + \ldots + x_{q_1}'^2$ and $x_{q_1+1}^2 + \ldots + x_q^2 = x_{q_1+1}'^2 + \ldots + x_q'^2$, then there exists an A in C such that $A\underline{x} = \underline{x}'$. This implies that $H_n(\underline{x})$ only depends on

$$t = (x_1^2 + \ldots + x_{q_1}^2)^{\frac{1}{2}} \text{ and } t_1 = (x_{q_1+1}^2 + \ldots + x_q^2)^{\frac{1}{2}}.$$

So $H_n(\underline{x}) = H_n(t,0, \ldots,0,t_1)$. Since $H_n(\underline{x})$ is a homogeneous polynomial of degree n we have

$$H_n(t,0, \ldots,0,t_1) = \sum_{j=0}^{n} a_j t^j t_1^{n-j}, \text{ or}$$

$$H_n(\underline{x}) = \sum_{j=0}^{n} a_n (x_1^2 + \ldots + x_{q_1}^2)^{\frac{1}{2}j} (x_{q_1+1}^2 + \ldots + x_q^2)^{\frac{1}{2}(n-j)}.$$

As $H_n(\underline{x})$ is a polynomial in $x_1, \ldots, x_q$ we have $a_1=a_3=\ldots=0$. If n is odd, then moreover $a_0=a_2=\ldots=0$, so $H_n(\underline{x})=0$. Therefore we only have to consider the case n is even, say n=2m. If we restrict $\underline{x}$ to lie on the unit sphere then $H_{2m}(\underline{x})$ becomes a polynomial in t^2 of degree m, say $G_m(t^2)$.
One should expect that the case $q_1 = q-1$ is the same as the first treated case. This is not true because the classes C with $q_1 = q-1$ and C_1 are different. In class C $A\underline{\varepsilon}_q = \pm \underline{\varepsilon}_q$ and in class C_1 $A\underline{\varepsilon}_q = \underline{\varepsilon}_q$. The consequence of this difference is that in the first case $H_n(\underline{x})$ depends on $t_1 = (x_1^2 + \ldots + x_{q-1}^2)^2$ and $|x_q|$ whereas in the second case $H_n(\underline{x})$ depends on t_1 and x_q.
A further consequence of this is that the polynomials of odd degree identically vanish in the first case.

If we require again that these $H_{2m}(\underline{x})$ are spherical harmonics a necessary and sufficient condition is that the corresponding surface spherical harmonics satisfy

$$\int_{\Omega_q} G_\lambda(t^2) G_\mu(t^2) d\omega_q = 0 \text{ if } \lambda \neq \mu$$

The reduction of this integral proceeds in the same manner as before and we obtain

$$\int_0^1 G_\lambda(t^2)G_\mu(t^2)t^{q_1-1}(1-t^2)^{\frac{1}{2}q-\frac{1}{2}q_1-1}dt = 0 \quad \text{if } \lambda \neq \mu$$

or putting $t^2 = \frac{1-x}{2}$, $q_2 = q-q_1$:

$$\int_{-1}^{1} (1-x)^{\frac{1}{2}q_1-1}(1+x)^{\frac{1}{2}q_2-1}\ G_\lambda(\tfrac{1-x}{2})G_\mu(\tfrac{1-x}{2})dx = 0$$

So $G_n(\frac{1-x}{2}) = c_nP_n^{(\frac{1}{2}q_1-1,\frac{1}{2}q_2-1)}(x)$; or $G_n(t^2) = c_nP_n^{(\frac{1}{2}q_1-1,\frac{1}{2}q_2-1)}(1-t^2)$.

Remark. From

$$H_{2m}(\underline{x}) = \sum_{j=0}^{m} a_{2j}(x_1^2 + \ldots + x_{q_1}^2)^j(x_{q_1+1}^2 + \ldots + x_q^2)^{m-j}$$

we may evaluate $\Delta_qH_{2m}(\underline{x})$ directly. Putting $\Delta_qH_{2m}(\underline{x}) = 0$ we obtain a recurrence relation between the coefficients a_{2j}. From this an expression of $G_m(t^2)$ in terms of a hypergeometric function can be obtained which corresponds to the same Jacobi polynomial as above.

As an application of this theory we will derive an integral representation for the Jacobi polynomials. Let $q=q_1 + q_2$, $q_1 > 0$, $q_2 > 0$. Let R_{q_1} be the space spanned by $e_{q_1+1}, \ldots, \underline{e}_q$. Then every vector $\underline{\eta}$ in R_q can be represented by $\underline{\eta} = s\underline{\eta}' + \sqrt{1-s^2}\,\underline{\eta}''$, $0 \leq s \leq 1$, where $\underline{\eta}'$ and $\underline{\eta}''$ are unit vectors in R_{q_1} and R_{q_2} respectively.

For every transformation A in class S we have $A\underline{\eta} = sA(\underline{\eta}') + \sqrt{1-s^2}\,A(\underline{\eta}'')$. Since $A(\underline{\eta}')\in R_{q_1}$ and $A(\underline{\eta}'')\in R_{q_2}$ are unit vectors, we have $(A\underline{\eta})' = A(\underline{\eta}')$ and $(A\underline{\eta})'' = A(\underline{\eta}'')$. Now we consider the integral

$$\int_{\Omega_q}(\underline{x}\cdot\underline{\eta}' \pm i\underline{x}\cdot\underline{\eta}'')^n d\omega_q(\underline{\eta}) = F(\underline{x}).$$

for every $\underline{x}$ in R_q. Calculating $\Delta_qF(\underline{x})$ one finds that $F(\underline{x})$ is a spherical harmonic of degree n.

We will show that $F(\underline{x})$ is invariant with respect to every orthogonal transformation A of class S:

$$F(A\underline{x}) = \int_{\Omega_q} (A\underline{x}.\underline{n}' \pm iA\underline{x}.\underline{n}'')^n d\omega_q(\underline{n}) =$$

$$= \int_{\Omega_q} (\underline{x}.A^{-1}\underline{n}' \pm i\underline{x}.A^{-1}\underline{n}'')^n \, d\omega_q(\underline{n}) =$$

$$= \int_{\Omega_q} \{\underline{x}.(A^{-1}\underline{n})' \pm i\underline{x}.(A^{-1}\underline{n})''\}^n \, d\omega_q(A^{-1}\underline{n}) =$$

$$= \int_{\Omega_q} (\underline{x}.\underline{n} \pm i\underline{x}.\underline{n}'')^n \, d\omega_q(\underline{n}) = F(\underline{x}),$$

since the orthogonal transformation leaves the surface-element unaltered. Therefore, if n is even, say n=2m, and $|\underline{x}| = 1$ then $F(\underline{x})$ is apart from a multiplicative constant equal to:

$$P_m^{(\frac{1}{2}q_1-1,\frac{1}{2}q_2-1)}(1-2t^2), \text{ where } t = (x_1^2 + \ldots + x_{q_1}^2)^{\frac{1}{2}}.$$

If n is odd then $F(\underline{x}) = 0$. Hence

$$\int_{\Omega_q} (\underline{x}.\underline{n}') \pm i\underline{x}.\underline{n}'')^{2m} \, d\omega_q(\underline{n}) = c_{\pm} P_m^{(\frac{1}{2}q_1-1,\frac{1}{2}q_2-1)}(1-t^2).$$

Choose $\underline{x} = (t, 0, \ldots, 0, 1-t^2)$, then we obtain

$$F(\underline{x}) = \int_{\Omega_q} (t n_1' \pm i\sqrt{1-t^2}\, n_q'')^n \, d\omega_q(\underline{n}),$$

where n_1' is the first component of $\underline{n}'$ and n_q'' the last component of $\underline{n}''$. Reducing this integral as in the first part of the lecture we have the result

$$P_m^{(\frac{1}{2}q_1-1,\frac{1}{2}q_2-1)}(1-t^2) = c_{\pm} \int_{-1}^{1}\int_{-1}^{1} (tu \pm i\sqrt{1-t^2}\, v)^{2m}(1-u^2)^{\frac{1}{2}q_1-\frac{3}{2}}(1-v^2)^{\frac{1}{2}q_2-\frac{3}{2}} \, dudv,$$

and

$$\int_{-1}^{1}\int_{-1}^{1} (tu \pm i\sqrt{1-t^2}\, v)^{2m+1}(1-u^2)^{\frac{1}{2}q_1-\frac{3}{2}}(1-v^2)^{\frac{1}{2}q_2-\frac{3}{2}}\, dudv = 0$$

The constant $c_{\pm}$ may be obtained by choosing $t=0$. For $q_1=3$ (or $q_2=3$) the double integral can be reduced to a single integral and then we obtain Laplace's definite integral expression for the Gegenbauer polynomial. For $q_1=3$, $q_2=2$ the Laplace representation for the Legendre polynomial occurs

Finally we remark that the last two formulas may be verified directly by application of the binomial formula. Then one sees that these formulas also hold for arbitrary real numbers q_1 and q_2 with $q_1 > 1$, $q_2 > 1$.