STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM
AFDELING ZUIVERE WISKUNDE

ZW 1969 - 016

Voordracht in de serie

"Elementaire onderwerpen vanuit hoger standpunt belicht"

door

Prof.dr. J. de Groot

Topological Hilbert space and the drop-out effect

29 oktober 1969

Notations

$l_2 = \{(x_i)_{i=1}^{\infty} \mid x_i \in \mathbb{R}, \sum x_i^2 < \infty\}$ denotes the separable Hilbert space in its usual metric.

$s = \prod_{i=1}^{\infty} \mathbb{R}_i$, where $\mathbb{R}_i = \mathbb{R}$ for each i, denotes the topological product of countably many real lines with its product-topology. This s is clearly homeomorphic ($\approx$) to the product of countably many open real intervals.

"Can a small push send a mathematician out of society without leaving a vacancy?" [2], p.786.
Indeed, this should be the case, if we lived in Hilbert space (as might be correct): if K is a compact set in l_2:

$$l_2 \setminus K \approx l_2$$

In particular, for a point $p \in l_2$:

$$l_2 \setminus \{p\} \approx l_2$$

This last fact has been proved by the Polish (for references in general c.f. [2]) and has been one step in solving an old problem, raised by Fréchet (1928) whether or not

$$s \approx l_2 .$$

This has finally been proved in the affirmative in 1966 by R.D. Anderson (who spent '58-'59 at the Mathematical Centre). This highly interesting result gave new impetus and strength to the area of infinite-dimensional manifolds (and topology). Today we shall discuss one aspect, "the push-out effect".

<u>Definition</u>. A topological space X is called a <u>drop-out space</u>, or briefly a drop-out, if

$$(\exists\, x \in X) \qquad X \approx X \setminus \{x\}$$

A compact space without isolated points can <u>never</u> be a drop-out because of the topological invariance of compactness.
However the space of the rationals, $\mathbb{Q}$, or the space or the irrationals are drop-outs.
For $\mathbb{Q}$, because of the fact that every countable metrizable space without isolated points is homeomorphic to $\mathbb{Q}$. For the irrationals, because any zero-dimensional, separable, completely metrizable, nowhere locally compact space is homeomorphic to it.
But these zero-dimensional examples are no great help in understanding the drop-out property of s (or l_2). Therefore we should try to find nice examples of connected spaces of finite dimension which have the drop-out property. Such a space is e.g. the Euclidean plane minus a countable dense subset. Studying these (theorem 1) leads to a general class of

metrizable (not necessarily separable) spaces which have the drop-out property (theorem 2). We apply this to derive the drop-out effect for s.

Definition. A space X is locally homogeneous in $x \in X$ if $\exists$ arbitrarily small neighborhoods U of x such that $\forall y \in U$ $\exists$ an autohomeomorphism of X which is the identity outside X and which maps x onto y.

Theorem 1. (Blok - de Groot)
If M is a locally homogeneous, completely metrizable space, then for any two countable dense subsets A and B of M there exists an autohomeomorphism of M which maps A onto B.
Hence $M \setminus A$ and $M \setminus B$ are homeomorphic.

Corollary. $M \setminus A$ is a drop-out.

Remarks. Special cases of this theorem are known, e.g. $M = \mathbb{R}^n$, $M = l_2 \approx s$, but the proofs are not coordinate free. Our proof depends on the following Anderson-Bing lemma.

Lemma 1. (Anderson-Bing [1], p.777). Let M be a completely metrizable space and $(\phi_n)_n$ a sequence of autohomeomorphisms of M. Put

$$\psi_n = \phi_n \cdots \phi_2 \phi_1 .$$

Then

$$\psi = \lim_{n \to \infty} \psi_n$$

(defined by pointwise convergence) exists and is an autohomeomorphism of M if

$$\forall m \in M \quad \forall n \in \mathbb{N}$$

(i) $$\rho(\phi_n(m), \phi_{n+1}(m)) < 2^{-n} , \text{ and}$$

(ii) $$\rho(\psi_n^{-1}(m), \psi_{n+1}^{-1}(m)) < 2^{-n} .$$

Remarks. We obtain equivalent conditions by replacing (i) by

(i') $\rho(\psi_n(m),\psi_{n+1}(m)) < 2^{-n}$

Do not forget that

$$\psi_n^{-1}(m) = \phi_1^{-1} \dots \phi_{n-1}^{-1}\ \phi_n^{-1}(m) .$$

The topological completeness of a (metrizable) space (that is: $\exists$ a - the same topology inducing - metric in which the space is complete) is characterized intrinsicly by the topological property of cocompactness [1]. Here we only need a more superficial property, also characterizing topological completeness, namely that of subcompactness [3].

Lemma 2. (cf. [3], p.763). In a metric space (M,ρ), which is topologically complete (so not necessarily complete in the metric ρ !), exists an open base $\mathcal{U} = \{U \mid U \in \mathcal{U}\}$, which has the following property of subcompactness: for every sequence $(U_n)_{n\in\mathbb{N}}$ of open sets $U_n \in \mathcal{U}$ satisfying

(a) $\bar{U}_{n+1} \subset U_n \qquad (\forall n \in \mathbb{N})$,

(b) $\lim_{n\to\infty} \text{diameter } (U_n) = 0$,

the intersection of the elements of the sequence consists of precisely one point:

$$(\exists p \in M) \qquad \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} \bar{U}_n = \{p\} .$$

(The proof is about half a page, it can essentially be found in [3], p.764 and the only fact needed is a simple settheoretic lemma on p.763).

Remark. This property is at first sight rather surprising if e.g. one starts to visualize it for M = {irrationals} .

Lemma 3. (well-known; Hausdorff/Alexandroff). A metric space is topologically complete iff it is a G_δ subset of a complete metric space.

<u>Definition</u>. Let M be a topological space and

$$C \subset B \subset M \text{ , } C \text{ dense in } B \text{ .}$$

We say that M is <u>locally homogeneous relative to C</u> if every point $c \in C$ has arbitrarily small neighborhoods U in M, such that $\forall c' \in C \cap U$ $\exists$ an autohomeomorfism ϕ of M which maps c onto c' and is the identity outside U. If we can choose, for each c and c', a ϕ in such a way that $\phi B = B$, then we say that B is an <u>invariant set</u>.

<u>Lemma 4</u>. Let (M,ρ) be a <u>complete metric</u> space,

$$M = A \cup B$$

$$\emptyset = A \cap B$$

If

(i) A is a G_δ subset of M (so A is topologically complete: lemma 3), and

(ii) A is dense in M (i.e. $A^- = M$)
B is dense in M (i.e. $B^- = M$)

(iii) M is locally homogeneous relative to C (for a suitable dense subset C of B)

then $\forall c \in C$ $\exists$ an autohomeomorphism ψ of M (ψ depending on c) such that

$$\psi(c) \in A.$$

The proof depends heavily on lemma 1 (applied to M) and lemma 2 (applied to (A,ρ)).

<u>Theorem 2</u>. (<u>drop-out theorem</u>).
Suppose we have the situation of the preceding lemma and let, moreover, B be an invariant set, then $\exists$ an autohomeomorphism ψ of M and $\exists a \in A$ such that

$$\psi^{-1}(A \setminus \{a\}) = A$$

<u>Hence A is a drop-out space</u>.

Corollary. s is a drop-out.

Proof. Put

M = Hilbert cube = $\{x = (x_i)_{i=1}^{\infty} | x \in l_2, \forall i \ |x_i| < \frac{1}{i}\}$

B = pseudoboundary = $\{x \in M | \exists_i \ |x_i| = \frac{1}{i}\}$

$A = M \setminus B$

$C = \{x \in B | \exists_j \ \forall_{i>j} \ x_i = 0\}$

Corollary. For every "tamely embedded" $\mathbb{R}^n \subset s$ we have $s \setminus \mathbb{R}^n \approx s$.

Proof. We let drop out $\{p\}$ from s, so $\mathbb{R}^n \times \{p\}$ from $\mathbb{R}^n \times s$, so, because $s \approx \mathbb{R}^n \times s$, $\mathbb{R}^n$ from s.

Final remarks. There is reason to believe that the extension of the techniques used in the proofs will lead to a simplification of the still very complicated - though elementary - proof of Anderson's theorem (1966): $l_2 \approx s$.
This also should settle the following conjecture:

Conjecture. (de Groot '68, J. Keesling '69).
Hilbertspace of dimension m (where m is an infinite cardinal) is homeomorphic to the product of a countable infinite number of copies of an m-spider (an m-spider consists of a "body" of one point a and m "legs", each leg consisting of a closed interval $[a, b_\alpha]$ ($\alpha \in A$ and $|A| = m$ of length one, sticking out; the metric on the spider is determined by walking the shortest distance over the spider from one point to another).

REFERENCES

[1] J.M. Aarts - J. de Groot — Colloquium cotopology. 1964/1965 Mathematical Centre.

[2] R.D. Anderson - R.H. Bing — A complete elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines. Bull. Am. Math. Soc. 74 (1968), 771-792

(we refer also to this survey article for other references and additional results).

[3] J. de Groot — Subcompactness and the Baire category theorem. Ind. Math. 25 (1963), 761-767

7047 NL