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ON A FORMULA OF VAN DER POL AND A PROBLEM CONCERNING THE ORDINATES OF THE NON-TRIVIAL ZERO'S OF RIEMANN's ZETA FUNCTION

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On a formula of Van der Pol and a problem concerning the ordinates of the non-trivial zeros of Riemann's zeta function

by

J. van de Lune

ABSTRACT

This report mainly deals with Van der Pol's identity

$$\sum_{n=-\infty}^{\infty} \frac{2^n}{a^{2^n}+1} = \frac{1}{\log a}, \qquad (a > 1)$$

and some of its generalizations. One of the proofs of the above identity leads quite naturally to the question whether the set of non-trivial zeros of Riemann's zeta function contains any arithmetical sequence $\{\sigma + n\lambda i\}_{n=1}^{\infty}$, where σ and λ are fixed positive numbers. Some evidence will be given that such a sequence does not exist.

INTRODUCTION. In 1950, VAN DER POL [9] stated

$$(*)\qquad \sum_{n=-\infty}^{\infty} \frac{2^n}{a^{2^n}+1} = \frac{1}{\log a}, \qquad (a > 1).$$

In this report two proofs will be given, one of which leads quite naturally to the question whether the set of non-trivial zeros of $\zeta(s)$ contains any arithmetical sequence $\{\sigma + n\lambda i\}_{n=1}^{\infty}$, where σ and λ are fixed positive numbers. Some evidence will be given that such a sequence does not exist.

1. VAN DER POL's IDENTITY

Because of the identity

$$(1 - e^{-\frac{x}{2^n}}) \cdot \prod_{k=1}^{n} (1 + e^{-\frac{x}{2^k}}) = 1 - e^{-x}$$

we have for $x > 0$

$$(1.1)\qquad \sum_{k=1}^{n} \log (1 + e^{-\frac{x}{2^k}}) = \log (1 - e^{-x}) - \log (1 - e^{-\frac{x}{2^n}}).$$

Differentiating both sides of (1.1) we obtain

$$\sum_{k=1}^{n} \frac{-2^{-k}\, e^{-x2^{-k}}}{1 + e^{-x2^{-k}}} = \frac{e^{-x}}{1 - e^{-x}} - \frac{2^{-n}\, e^{-x2^{-n}}}{1 - e^{-x2^{-n}}}$$

or

$$\sum_{k=1}^{n} \frac{2^{-k}}{e^{x2^{-k}} + 1} = \frac{1}{x} \cdot \frac{x2^{-n}}{e^{x2^{-n}} - 1} - \frac{1}{e^{x} - 1}.$$

Taking limits $(n \to \infty)$ we find

$$(1.2)\qquad \sum_{k=1}^{\infty} \frac{2^{-k}}{e^{2^{-k}x} + 1} = \frac{1}{x} - \frac{1}{e^{x} - 1}, \qquad (x > 0).$$

In a simular manner one obtains

$$(1.3)\qquad \sum_{n=0}^{\infty} \frac{2^{n}}{e^{2^{n}x} + 1} = \frac{1}{e^{x} - 1}, \qquad (x > 0),$$

from the identity

$$(1 - e^{-x}) \prod_{k=0}^{n-1} (1 + e^{-2^{k}x}) = 1 - e^{-2^{n}x}.$$

Combining (1.2) and (1.3) we obtain

$$(1.4)\qquad \sum_{n=-\infty}^{+\infty} \frac{2^{n}}{e^{2^{n}x} + 1} = \frac{1}{x}, \qquad (x > 0),$$

which is equivalent to (*). Replacing x, in (1.4), by 2^{t} one finds

$$(1.5)\qquad \sum_{n=-\infty}^{+\infty} \frac{2^{n+t}}{e^{2^{n+t}} + 1} = 1, \text{ for all real } t,$$

which is also equivalent to (*).

Remark. Formula (1.2), which may serve to make the well known formula [8 , p.23]

$$(1.6)\qquad \Gamma(s)\zeta(s) = \int_{0}^{\infty} \left(\frac{1}{e^{x} - 1} - \frac{1}{x}\right)x^{s-1}ds, \qquad (0 < \mathrm{Re}\ s < 1),$$

almost trivial, may also be proved as follows. From

$$(1.7)\qquad \cot z = \frac{1}{2} \{\cot \frac{z}{2} - \tan \frac{z}{2}\}$$

one obtains

$$\cot z = - \sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{z}{2^n} + \frac{1}{z} ,$$

and from this, using Euler's formulas for sin z and cos z, it is easily seen that (1.2) holds.

It is well known that (1.6) may be used to prove the functional equation for $\zeta(s)$, by observing that the function

$$f(x) = \frac{1}{e^{x\sqrt{2\pi}} - 1} - \frac{1}{x\sqrt{2\pi}}$$

is self-reciprocal for sine transforms, i.e.

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin tx \, dx.$$

This well known result, often proved by means of complex integration methods, may be proved in an elementary way as follows:

$$(1.8) \qquad \int_0^{\infty} f(x)\sin tx\,dx = \int_0^{\infty} \left(\frac{1}{e^{x\sqrt{2\pi}} - 1} - \frac{1}{x\sqrt{2\pi}}\right)\sin tx\,dx =$$

$$= - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{\sin tx}{x}\,dx + \int_0^{\infty} \frac{\sin tx}{e^{x\sqrt{2\pi}} - 1}\,dx =$$

$$= - \frac{1}{\sqrt{2\pi}} \cdot \frac{\pi}{2} + \int_0^{\infty} \frac{e^{itx} - e^{-itx}}{2i} \left(\sum_{k=1}^{\infty} e^{-kx\sqrt{2\pi}}\right)dx =$$

$$= - \frac{\sqrt{2\pi}}{4} + \sum_{k=1}^{\infty} \frac{t}{k^2 \cdot 2\pi + t^2} = - \frac{\sqrt{2\pi}}{4} \sum_{k=1}^{\infty} \frac{2\pi t}{4k^2\pi^2 + (t\sqrt{2\pi})^2} =$$

$$= - \frac{\sqrt{2\pi}}{4} + \frac{\sqrt{2\pi}}{2} \left\{\frac{1}{e^{t\sqrt{2\pi}} - 1} - \frac{1}{t\sqrt{2\pi}} + \frac{1}{2}\right\} = \sqrt{\frac{\pi}{2}}\, f(t).$$

The relation

$$\frac{1}{e^u - 1} - \frac{1}{u} + \frac{1}{2} = 2u \sum_{k=1}^{\infty} \frac{1}{4k^2\pi^2 + u^2}$$

which is crucial in (1.8) may be obtained directly from (1.7) as is shown by SCHRÖTER [3 , p. 204]. Putting things together we get, following TITCHMARSH [8 , p. 23], that the functional equation for $\zeta(s)$ may be based almost entirely upon (1.7).

2. ANOTHER PROOF

In order to prove (1.5) directly, define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{2^{n + \frac{t}{2\pi}}}{e^{2^{n + \frac{t}{2\pi}}} + 1}$$

Clearly f is periodic with period 2π. It is easily seen that f is continuously differentiable on $\mathbb{R}$ so that f may be represented (pointwise) by its Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}.$$

In order to find the coefficients in this expansion we compute

$$\int_0^{2\pi} f(t)e^{\nu ti}dt = \int_0^{2\pi} e^{\nu ti} \cdot \left(\sum_{n=-\infty}^{+\infty} \frac{2^{n + \frac{t}{2\pi}}}{e^{2^{n + \frac{t}{2\pi}}} + 1} \right) dt =$$

$$= \sum_{n=-\infty}^{+\infty} \int_0^{2\pi} \frac{e^{(n + \frac{t}{2\pi})\log 2 + \nu ti}}{e^{e^{(n + \frac{t}{2\pi})\log 2}} + 1} dt =$$

$$= 2\pi \cdot \sum_{n=-\infty}^{+\infty} \int_n^{n+1} \frac{e^{u \log 2 + \nu(u-n)2\pi i}}{e^{e^{u \log 2}} + 1} du =$$

$$= 2\pi \cdot \sum_{n=-\infty}^{+\infty} \int_n^{n+1} \frac{e^{u \log 2 + \nu u 2\pi i}}{e^{e^{u \log 2}} + 1} du =$$

$$= 2\pi \cdot \int_{-\infty}^{+\infty} \frac{e^{u \log 2 + \nu u 2\pi i}}{e^{e^{u \log 2}} + 1} du =$$

$$= \frac{2\pi}{\log 2} \int_0^{\infty} \frac{w^{s-1}}{e^w + 1} dw =$$

$$= \frac{2\pi}{\log 2} \Gamma(s)\eta(s) = \frac{2\pi}{\log 2} \Gamma(s)(1-2^{1-s})\zeta(s),$$

where

$$s = 1 + \frac{2\pi\nu}{\log 2} i \quad \text{and} \quad \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} .$$

Since

$$1 - 2^{1-s} = 0 \text{ for } s = 1 + \frac{2\pi\nu}{\log 2} i \qquad (\pm\ \nu = 1,2,3,\ldots)$$

and

$$\int_0^{\infty} \frac{dw}{e^w + 1} = \int_1^{\infty} \frac{du}{u(u+1)} = \log 2,$$

we find that the Fourier coefficients of f are all zero except the one giving the constant term 1.

Since the Fourier series of f represents f on $\mathbb{R}$, we obtain

$$f(t) = 1, \text{ for all } t \in \mathbb{R},$$

which is equivalent to (1.5).

3. A GENERALIZATION

In the previous section it turned out that the periodic function

$$f(t) = \sum_{n=-\infty}^{+\infty} \frac{2^{n + \frac{t}{2\pi}}}{e^{2^{n + \frac{t}{2\pi}}} + 1}, \qquad (t \in \mathbb{R})$$

is actually a constant. The question arises if there are any other series of a similar type with a constant sum.

In this section we will only consider the functions $f_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$, defined by

$$f_{\alpha,\beta}(t) = \sum_{n=-\infty}^{+\infty} \frac{e^{\alpha(n + \frac{t}{2\pi})}}{e^{e^{\beta(n + \frac{t}{2\pi})}} + 1},$$

where $t \in \mathbb{R}$ and $\alpha, \beta > 0$.
It is easily seen that the functions $f_{\alpha,\beta}$

* are well defined on $\mathbb{R}$
* are periodic with period 2π
* may be represented on $\mathbb{R}$ by their Fourier series.

Before carrying out any Fourier analysis on the functions $f_{\alpha,\beta}$ we will show directly that there are indeed some constant functions $f_{\alpha,\beta}$ different from the case $\alpha = \beta = \log 2$.

Proposition 3.1. If $f_{\alpha,\beta}$ is constant then the functions $\phi_{k,\alpha,\beta} : \mathbb{R} \to \mathbb{R}$, $(k = 1,2,3,\ldots)$, such that

$$\phi_{k,\alpha,\beta}(t) = \sum_{n=-\infty}^{+\infty} \frac{e^{\frac{\alpha}{k}(n + \frac{t}{2\pi})}}{e^{e^{\frac{\beta}{k}(n + \frac{t}{2\pi})}} + 1}, \qquad (t \in \mathbb{R}),$$

where

$$c_{m+k} = c_m \text{ for all } m \in \mathbb{Z},$$

are also constant.

Proof.
$$\phi_{k,\alpha,\beta}(t) = \sum_{r=0}^{k-1} \sum_{n=-\infty}^{+\infty} c_{nk+r} \cdot \frac{e^{\frac{\alpha}{k}(nk + r + \frac{t}{2\pi})}}{e^{e^{\frac{\beta}{k}(nk + r + \frac{t}{2\pi})}} + 1} =$$

$$= \sum_{r=0}^{k-1} c_r \sum_{n=-\infty}^{+\infty} \frac{e^{\alpha(n + \frac{r}{k} + \frac{t}{2\pi k})}}{e^{e^{\beta(n + \frac{r}{k} + \frac{t}{2\pi k})}} + 1} =$$

$$= \sum_{r=0}^{k-1} c_r \cdot f_{\alpha,\beta}\left(\frac{r}{k} + \frac{t}{2\pi k}\right) = f_{\alpha,\beta}(t) \cdot \sum_{r=0}^{k-1} c_r,$$

which proves the proposition. □

Hence $f_{\alpha,\beta}$ with $\alpha = \beta = \frac{\log 2}{k}$, $(k = 1,2,3,\ldots)$ is constant and has the value k.

THEOREM 3.1. If $\frac{\alpha}{\beta} \geq 1$ and $f_{\alpha,\beta}$ is constant then $\alpha = \beta = \frac{\log 2}{k}$ for some $k \in \{1,2,3,\ldots\}$.

Proof. If $f_{\alpha,\beta}$ is constant then "all" Fourier coefficients of $f_{\alpha,\beta}$ must be zero. Consequently the integrals

$$\int_0^{2\pi} f(t)e^{\nu ti}dt$$

must vanish for $\nu \in Z\backslash\{0\}$. Since

$$\int_0^{2\pi} f(t)e^{\nu ti}dt = \int_0^{2\pi} e^{\nu ti} \cdot \left(\sum_{n=-\infty}^{+\infty} \frac{e^{\alpha(n + \frac{t}{2\pi})}}{e^{e^{\beta(n + \frac{t}{2\pi})}} + 1} \right) =$$

$$= \sum_{n=-\infty}^{+\infty} \int_0^{2\pi} \frac{e^{\alpha(n + \frac{t}{2\pi}) + \nu ti}}{e^{e^{\beta(n + \frac{t}{2\pi})}} + 1} dt =$$

$$= 2\pi \sum_{n=-\infty}^{+\infty} \int_n^{n+1} \frac{e^{\alpha u + \nu(u-n)2\pi i}}{e^{e^{\beta u}} + 1} du =$$

$$= 2\pi \int_{-\infty}^{+\infty} \frac{e^{\alpha u + \nu u 2\pi i}}{e^{e^{\beta u}} + 1} du =$$

$$= \frac{2\pi}{\beta} \int_0^{\infty} \frac{w^{(\frac{\alpha}{\beta} + \frac{2\pi\nu}{\beta} i) - 1}}{e^w + 1} dw =$$

$$= \frac{2\pi}{\beta} \Gamma(s)(1-2^{1-s})\zeta(s), \text{ where } s = \frac{\alpha}{\beta} + \frac{2\pi\nu}{\beta} i,$$

one must have $(1-2^{1-s})\zeta(s) = 0$, because of the well known fact that the Γ-function has no zero's at all.

It is also well known that $(1-2^{1-z})\zeta(z)$ (on Re $z \geq 1$) is zero only in the points $z = 1 \pm \frac{2\pi k}{\log 2} i$, where $k = 1,2,3,\ldots$. Thus, $s = 1 + \frac{2\pi\nu}{\beta} i$ must (for $\nu = 1,2,3,\ldots$) be equal to one of these zero's. From this it easily follows that $\alpha = \beta = \frac{\log 2}{k}$ for some positive integer k, which proves the theorem. □

4. THE CASE $0 < \frac{\alpha}{\beta} < 1$

Let us consider the case $0 < \sigma \overset{\text{def}}{=} \frac{\alpha}{\beta} < 1$, σ fixed. Define

$$\phi(t) = \phi_{\alpha,\beta}(t) = \sum_{n=-\infty}^{+\infty} \frac{e^{\alpha(n+t)}}{e^{e^{\beta(n+t)}} + 1}, \quad (t \in \mathbb{R}).$$

The average value of ϕ is

$$\int_0^1 \phi(t)dt = \int_0^1 \left(\sum_{n=-\infty}^{+\infty} \frac{e^{\alpha(n+t)}}{e^{e^{\beta(n+t)}} + 1} \right) dt =$$

$$= \sum_{n=-\infty}^{+\infty} \int_n^{n+1} \frac{e^{\alpha u}}{e^{e^{\beta u}} + 1} du = \int_{-\infty}^{+\infty} \frac{e^{\alpha u}}{e^{e^{\beta u}} + 1} du =$$

$$= \frac{1}{\beta} \int_0^{\infty} \frac{w^{\sigma-1}}{e^w+1} dw = \frac{A_1}{\beta} ,$$

where $A_1 = \Gamma(\sigma)\eta(\sigma)$. Hence, if ϕ is constant, as a function of t, then

$$\phi(t) = \frac{A_1}{\beta} \text{ for all } t \in \mathbb{R}.$$

Let us first consider the simpler case

$$\psi(t) = \psi_{\alpha,\beta}(t) = \sum_{n=-\infty}^{+\infty} e^{\alpha(n+t)} e^{-e^{\beta(n+t)}} .$$

The average value of ψ is

$$\frac{A_2}{\beta} , \text{ where } A_2 = \Gamma(\sigma).$$

We will show by means of a complex function argument that ψ is not constant for any choice of σ between 0 and 1.

Suppose ψ is a constant. Then we must have

$$\sum_{n=-\infty}^{+\infty} e^{\alpha(n+t)} \cdot e^{-e^{\beta(n+t)}} = \frac{A_2}{\beta} \text{ for all } t \in \mathbb{R}.$$

Replacing $e^{\beta t}$ by x (> 0) we get

$$\sum_{n=-\infty}^{+\infty} e^{\alpha n} x^{\sigma} e^{-e^{\beta n}x} = \frac{A_2}{\beta}, \qquad (x > 0),$$

or

$$\sum_{n=-\infty}^{+\infty} e^{\alpha n} e^{-e^{\beta n}x} = \frac{A_2}{\beta} x^{-\sigma}, \qquad (x > 0).$$

Taking Laplace transforms we arrive at

$$\int_0^{\infty} e^{-sx}\Big(\sum_{n=-\infty}^{+\infty} e^{\alpha n} e^{-e^{\beta n}x}\Big)dx = \frac{A_2}{\beta} \int_0^{\infty} e^{-sx}x^{-\sigma}dx$$

or

$$\sum_{n=-\infty}^{+\infty} \frac{e^{\alpha n}}{s + e^{\beta n}} = \frac{A_2}{\beta} s^{\sigma-1}\Gamma(1-\sigma), \qquad (s > 0).$$

Since $0 < \sigma < 1$ the analytic continuation of the right-hand side yields a multi-valued analytic function whereas the left-hand side is a single valued analytic function with simple poles in $s = -e^{\beta n}$, $(n \in \mathbb{Z})$ and a non-isolated singularity in $s = 0$. From this we conclude that the left- and the right-hand side cannot be identical on $s > 0$, showing that $\psi_{\alpha,\beta}(t)$ is not constant.

Remark. Actually all Fourier coefficients of $\psi_{\alpha,\beta}(t)$ are different from zero.

We now return to the function ϕ and suppose that ϕ is constant

$$\phi(t) = \sum_{n=-\infty}^{+\infty} \frac{e^{\alpha(n+t)}}{e^{e^{\beta(n+t)}} + 1} = \frac{A_1}{\beta}, \text{ for all } t \in \mathbb{R}.$$

Again, replacing $e^{\beta t}$ by x, we obtain

$$\sum_{n=-\infty}^{+\infty} \frac{e^{\alpha n} x^{\sigma}}{e^{e^{\beta n} x} + 1} = \frac{A_1}{\beta}, \qquad (x > 0)$$

or

$$\sum_{n=-\infty}^{+\infty} \frac{e^{\alpha n}}{e^{e^{\beta n} x} + 1} = \frac{A_1}{\beta} x^{-\sigma}, \qquad (x > 0).$$

Taking Laplace transforms, we arrive at

$$\frac{A_1}{\beta} s^{\sigma-1} \Gamma(1-\sigma) = \sum_{n=-\infty}^{+\infty} \int_0^{\infty} \frac{e^{-s} \cdot e^{\alpha n}}{e^{e^{\beta n} x} + 1} dx =$$

$$= \sum_{n=-\infty}^{+\infty} e^{\alpha n} \int_0^{\infty} e^{-sx} \left\{ \sum_{m=1}^{\infty} (-1)^{m+1} e^{-me^{\beta n} x} \right\} dx =$$

$$= \sum_{n=-\infty}^{+\infty} e^{\alpha n} \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{s + me^{\beta n}} \right), \qquad (s > 0).$$

The analytic continuation of the left-hand side is again a multi-valued analytic function with only one singularity in s = 0. The right-hand side seems to have no analytic continuation at all across the negative real axis. However, we are not able to prove this. We therefore state

CONJECTURE 1. The set of non-trivial zeros of $\zeta(s)$ does not contain a sequence of the form $\{\sigma + n\lambda i\}_{n=1}^{\infty}$, $(\sigma > 0, \lambda > 0)$.

This conjecture is the first step in the direction of

CONJECTURE 2. The ordinates of the non-trivial zeros of $\zeta(s)$, lying in the upper halfplane, are linearly independent over Q.

Remark. In [1, pp. 18-25] BOAS and POLLARD discuss functions $\psi(t)$ for which

$$(\#) \qquad \sum_{n=-\infty}^{\infty} \psi(n) = \int_{-\infty}^{\infty} \psi(t)dt.$$

Defining

$$\psi_x(t) = \frac{2^t}{e^{2^t x} + 1}, \qquad (t \in \mathbb{R},\ x > 0),$$

we have

$$\int_{-\infty}^{\infty} \psi_x(t)dt = \int_{-\infty}^{\infty} \frac{2^t}{e^{2^t x} + 1}\, dt =$$

$$= \frac{1}{x \log 2} \int_0^{\infty} \frac{du}{e^u + 1} = \frac{1}{x}.$$

Hence, because of (1.4), we obtain

$$\sum_{n=-\infty}^{\infty} \psi_x(n) = \int_{-\infty}^{\infty} \psi_x(t)dt, \text{ for all } x > 0.$$

In case the function $\phi_{\alpha,\beta}(t)$ is constant, we have

$$\phi_{\alpha,\beta}(t) = \frac{\Gamma(\sigma)\eta(\sigma)}{\beta}.$$

Since

$$\int_{-\infty}^{\infty} \frac{e^{\alpha(u+t)}}{e^{e^{\beta(u+t)}} + 1}\, du = \frac{1}{\beta} \int_0^{\infty} \frac{w^{\sigma-1}}{e^w + 1}\, dw = \frac{\Gamma(\sigma)\eta(\sigma)}{\beta}$$

we would have another example of (#), namely

$$\sum_{n=-\infty}^{\infty} \frac{e^{\alpha(n+t)}}{e^{e^{\beta(n+t)}}+1} = \int_{-\infty}^{\infty} \frac{e^{\alpha(u+t)}}{e^{e^{\beta(u+t)}}+1}\, du.$$

Also compare [5].

5. In section 4 we proved by a complex function argument that if $\alpha,\beta > 0$ and $0 < \frac{\alpha}{\beta} < 1$, then

$$\psi_{\alpha,\beta}(t) = \sum_{n=-\infty}^{\infty} e^{\alpha(n+t)}\, e^{-e^{\beta(n+t)}}$$

is not a constant function of t.

It seems to be worth the effort to prove this fact in as many different ways as possible in order to discover a method to prove conjecture 1.

Therefore, in this section we will prove once more that $\psi_{\alpha,\beta}$ is not constant (for all α, $\beta > 0$).

Let α,β and δ be positive constants and assume that

$$\sum_{n=-\infty}^{\infty} e^{\alpha(n+x)}\, e^{-\delta e^{\beta(n+x)}} = C$$

for all $x \in \mathbb{R}$, where C is a positive constant. Replace $e^{\beta x}$ by t (>0) in order to obtain

$$\sum_{n=-\infty}^{\infty} e^{\alpha n}\, e^{-\delta e^{\beta n} t} = C t^{-\frac{\alpha}{\beta}}.$$

Differentiating both sides k times we get

$$\sum_{n=-\infty}^{\infty} e^{\alpha n+k\beta n}\, \delta^{k}\, e^{-\delta e^{\beta n} t} = C\, \frac{\alpha}{\beta}\, \frac{\alpha+\beta}{\beta} \cdots \frac{\alpha+(k-1)\beta}{\beta}\, t^{-\frac{\alpha}{\beta}-k}.$$

Hence, only considering the term corresponding to n = 0, we see that $(\sigma = \frac{\alpha}{\beta})$

$$\delta^k e^{-\delta t} < C\sigma(1+\sigma)(2+\sigma) \ \ldots \ (k-1+\sigma)t^{-\sigma-k}$$

or

$$\delta^k e^{-\delta t} t^{\sigma+k} < C\sigma(1+\sigma)(2+\sigma) \ \ldots \ (k-1+\sigma).$$

Now let $t = \frac{k+\sigma}{\delta}$. Then we obtain

$$\frac{k}{k!}\,\delta^k\, e^{-(k+\sigma)}\left(\frac{k+\sigma}{\delta}\right)^{k+\sigma} < C\sigma(1+\tfrac{\sigma}{1})(1+\tfrac{\sigma}{2}) \ \ldots \ (1+\frac{\sigma}{k-1}) <$$

$$< C\sigma k^{\sigma}\, e^{\sigma(1+\frac{1}{2}+\ldots+\frac{1}{k-1}) - \sigma\log k}.$$

Now observe that

$$n! \sim n^n e^{-n}\sqrt{2\pi n}, \ [7, \text{ p. } 200],$$

and

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n \leq 1.$$

It follows that

$$\sqrt{k}\,\frac{k^k e^{-k}\sqrt{2\pi k}}{k!}\, e^{-\sigma}(1+\tfrac{\sigma}{k})^k (1+\tfrac{\sigma}{k})^{\sigma} < C\sigma e^{\sigma}\,\delta^{\sigma}\sqrt{2\pi}.$$

However, it is easily seen that the last inequality is false if k is large enough.

The above method of repeated differentiation, applied to the function $\phi_{\alpha,\beta}$ (see p.8), did not yet lead to any proof of conjecture 1. However, trying to do so we obtained some results which are interesting in themselves. They will be described in the next section.

6. SOME REMARKS ON THE FUNCTION $\frac{1}{e^x+1}$

Let $f(x) = \frac{1}{e^x+1}$. Then we have

Proposition 6.1. $f^{(k)}(x) = \sum_{r=1}^{k+1} \frac{c_{k,r}}{(e^x+1)^r}$, where the coefficients $c_{k,r}$ $(k = 0,1,2,\ldots;\ 1 \le r \le k+1)$ are integers satisfying

$$(6.1) \qquad c_{k,1} = (-1)^k, \qquad c_{k,k+1} = k!$$

and

$$(6.2) \qquad c_{k+1,r} = (r-1)c_{k,r-1} - rc_{k,r}.$$

Proof. Mathematical induction. □

Definition. $P_k(z) = \sum_{r=1}^{k+1} c_{k,r} z^r$.

Proposition 6.2. $P_{k+1}(z) = z(z-1)P_k'(z)$, $(k = 0,1,2,\ldots)$.

Proof. This follows easily from (6.2). □

Proposition 6.3. $\sum_{r=1}^{k+1} c_{k,r} = 0$, $(k = 1,2,3,\ldots)$.

Proof. Mathematical induction, using (6.2). □

Proposition 6.4. $\sum_{r=1}^{k+1} rc_{k,r} = 1$, $(k = 0,1,2,\ldots)$.

Proof. Since $P_{k+1}(z) = z(z-1)P_k'(z)$, $(k \ge 0)$ we have

$$P_{k+1}(0) = P_{k+1}(1) = 0.$$

Hence

$$P_k'(z) = \frac{P_{k+1}(z) - P_{k+1}(1)}{z - 1} \cdot \frac{1}{z}$$

and it follows that $(z \to 1), P_k'(1) = P_{k+1}'(1)$ or $P_k'(1) = 1$. Hence

$$\sum_{r=1}^{k+1} rc_{k,r} = 1, \quad (k \geq 0). \quad \Box$$

Proposition 6.5. $P_k(z) = (-1)^{k+1} P_k(1-z), \quad (k = 1,2,3,\ldots).$

Proof. The proposition is true for $k = 1$ because $P_1(z) = z(z-1)$ and $P_1(1-z) = (1-z)(-z) = z(z-1)$. Now suppose that the proposition is true for $k \leq m$. Then we have

$$P_{m+1}(z) = z(z-1)P_m'(z) = z(z-1)(-1)^{m+1}P_m'(1-z) \ . \ (-1) =$$

$$= z(z-1)(-1)^{m+2}P_m'(1-z) = (-1)^{m+2}P_{m+1}(1-z)$$

and the proposition follows by induction. $\Box$

Proposition 6.6. $P_{2k}(\frac{1}{2}) = 0, \quad (k = 1,2,3,\ldots).$

Proof. This follows immediately from proposition 6.5. $\Box$

Proposition 6.7. $P_{2k-1}(\frac{1}{2}) = (-1)^k \frac{2^{2k} - 1}{2k} B_k$

where B_k is the k-th Bernoulli number, defined by

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2} z + B_1 \frac{z^2}{2!} - B_2 \frac{z^4}{4!} + \ldots , \quad (|z| < \pi).$$

Proof. For $x > 0$ we have

$$f(x) = \frac{1}{e^{x}+1} = \frac{e^{-x}}{1+e^{-x}} = e^{-x} - e^{-2x} + e^{-3x} - + \ldots$$

Hence

$$f^{(k)}(x) = (-1)^{k}\,\{e^{-x} - 2^{k}e^{-2x} + 3^{k}e^{-3x} - + \ldots\},$$

and it follows that (compare [3 , p. 492] and [4 , p. 294])

$$P_{2k-1}(\tfrac{1}{2}) = \sum_{r=1}^{2k} \frac{c_{2k-1,r}}{2^{r}} = \sum_{r=1}^{2k} \frac{c_{2k-1,r}}{(e^{0}+1)^{r}} =$$

$$= f^{(2k-1)}(0) = \lim_{x\to 0} f^{(2k-1)}(x) =$$

$$= (-1)^{2k-1} \lim_{x\to 0} \{e^{-x} - 2^{2k-1}e^{-2x} + 3^{2k-1}e^{-3x} - + \ldots\} =$$

$$= -\eta(1-2k) = -(1-2^{2k})\zeta(1-2k) = (2^{2k}-1)\,\frac{(-1)^{k}B_{k}}{2k}\,. \quad \square$$

Proposition 6.8.

$$B_{k} = (-1)^{k}\,\frac{2k}{2^{2k}-1} \sum_{r=1}^{2k} \frac{c_{2k-1,r}}{2^{r}}\,.$$

Proof. This is a restatement of proposition 6.7. □

Remark. The numbers $c_{k,r}$ may be computed very quickly by means of formula (6.2).

Table of $c_{k,r}$

k \ r	1	2	3	4	5	6
0	1					
1	-1	1				
2	1	-3	2			
3	-1	7	-12	6		
4	1	-15	50	-60	24	
5	-1	31	-180	390	-360	120

Therefore, Proposition 6.8 furnishes a direct method to compute the Bernoulli numbers. For example

$$B_1 = \frac{2}{2^2 - 1} \{\frac{-1}{2} + \frac{1}{2^2}\} = \frac{1}{6} ,$$

$$B_2 = \frac{4}{2^4 - 1} \{\frac{-1}{2^1} + \frac{7}{2^2} + \frac{-12}{2^3} + \frac{6}{2^4}\} = \frac{1}{30} .$$

Definition. The (Stirling) numbers $b_{k,r}$ ($k = 1,2,3,\ldots;\ 1 \le r \le k$), are defined by means of the following factorial expansion of the function x^k (compare [6 , p. 90])

$$x^k = b_{k,1}x + b_{k,2}x(x-1) + \ldots + b_{k,k}x(x-1)(x-2) \ldots (x-k+1).$$

This expansion is possible and is unique.

Proposition 6.9. $b_{k,1} = b_{k,k} = 1$ and $b_{k+1,r} = b_{k,r-1} + rb_{k,r}$,

($k = 1,2,3,\ldots;\ 2 \le r \le k$)

Proof. Use mathematical induction. □

Definition.

$$a_{k,r} = \frac{1}{(r-1)!} c_{k,r}, \quad (k = 0,1,2,\ldots;\ 1 \le r \le k+1).$$

Proposition 6.10. $a_{k+1,r} = a_{k,r-1} - ra_{k,r}$

Proof. This follows easily from (6.2) □

Definition. $\alpha_{k,r} = (-1)^{k+r+1}\ a_{k,r}$.

Proposition 6.11. $\alpha_{k+1,r} = \alpha_{k,r-1} + r\alpha_{k,r}$.

Proof. This is an immediate consequence of proposition 6.10. □

Proposition 6.12. $\alpha_{k,r} = b_{k+1,r}$

Proof. This follows from the observation that

$$\alpha_{0,1} = a_{0,1} = c_{0,1} = 1,\ b_{1,1} = 1$$

and the fact that $\alpha_{k,r}$ and $b_{k,r}$ satisfy the same recurrence relation (see propositions 6.9 and 6.11). □

Proposition 6.13

$$c_{k,r} = \frac{(-1)^{k+1}}{r} \sum_{n=1}^{r} (-1)^n \binom{r}{n} n^{k+1} .$$

Proof. Mathematical induction, using proposition 6.1. □

Proposition 6.14. (compare [6 , p. 226])

$$b_{k,r} = \frac{(-1)^r}{r!} \sum_{n=1}^{r} (-1)^n \binom{r}{n} n^k.$$

Proof. $b_{k,r} = \alpha_{k-1,r} = (-1)^{k+r} a_{k-1,r} = \frac{(-1)^{k+r}}{(r-1)!} c_{k-1,r}$

and the proposition follows from proposition 6.13. □

Proposition 6.15.

$$B_k = (-1)^k \frac{2k}{2^{2k}-1} \sum_{r=1}^{2k} \sum_{n=1}^{r} \frac{(-1)^n \binom{r}{n} n^{2k}}{r2^r}.$$

Proof. This is an immediate consequence of propositions 6.8 and 6.13. □

Finally we prove

Proposition 6.16. If p is a prime larger than 2 and if $2 \le r \le p-1$ then $c_{p-1,r}$ is divisible by p.

Proof. Since $c_{p-1,r} = (-1)^{r-1}(r-1)!\, b_{p,r}$ it suffices to show that $p \mid r!b_{p,r}$ or, using proposition 6.14, that

$$p \;\Big|\; \sum_{n=1}^{r} (-1)^n \binom{r}{n} n^p.$$

Now observe that

$$\sum_{n=1}^{r} (-1)^n \binom{r}{n} n^p = \sum_{n=1}^{r} (-1)^n \binom{r}{n} (n^p - n),$$

because of

$$\sum_{n=1}^{r} (-1)^n \binom{r}{n} n = 0, \quad (r = 2,3,4,\dots).$$

Now, applying FERMAT's theorem [2 , p.63], the proposition follows. □

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