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A CASE OF COLOURATION IN THE FOUR COLOUR PROBLEM

BY

J. M. AARTS and J. DE GROOT

§ 1. *Introduction*

By a *map* (on the two-sphere S^2) we shall understand a covering of the two-sphere by means of a finite number of closed sets (called *countries*), such that

(i) each country is the closure of a region (= open connected set),
(ij) the boundary of each country is the union of a finite number of jordan-curves (= curve, homeomorphic to a circle),
(iij) the intersection of two countries is contained in the intersection of their respective boundaries.

Two different countries are *neighbours* (are *adjacent*) if their intersection contains a jordan-arc.

The *order of a country* is the number of its neighbours.

The *order of a map* is the maximum of the orders of its countries.

Colouring a map with a given number of colours is: to assign a colour to each country in such a way, that any two neighbours obtain different colours.

The *four colour problem* consists in proving or disproving *the four colour conjecture* which states that four colours are sufficient to colour any map.

A *regular map* is one in which not more than three countries have a non-empty intersection. It can easily be seen, that the four colour problem has a positive or negative answer, if this is the case for regular maps.

The four colour problem still stays unsolved in spite of many attemps to solve it, during more than seventy years.

Are there special classes of maps for which the answer is known to be in the affirmative? FRANKLIN [5] (see also ERRERA [4]) tells us what is known on this until 1940. Apart from a few trivial cases (see [5]) he mentions the cases (i) and (ij) below, to which we add (iij):

(i) The number of countries is < 36 (WINN, [14]),
(ij) *Regular* maps in which every country, with the exception of at most one, has order ≤ 6 (WINN, [13]),
(iij) Maps of order ≤ 4.

KEMPE [7] gave a wrong proof of the four colour conjecture, but the methods he developed enabled HEAWOOD [6] to prove that five colours suffice to colour any map. Also, the result (iij) follows easily from Kempe's method. However, we have not found it explicitly in the literature until DIRAC ([3], Th. 14) in which a slightly more general result is proved. (iij), and Dirac's result too, is contained in our lemma (§ 4) (after translation from the graph language to the map language; see § 2). This lemma serves us mainly in order to prove the main result of this paper, namely a fourth case in which the four colour conjecture holds, in fact strengthening (iij) to.

(iv) Maps of order ≤ 5. The proof is carried out in § 4.

We have not been able to find a proof for maps of order ≤ 6. This case becomes in any case much more complicated (if the result is correct at all). Neither seems there to be an obvious way to prove (iv) from (ij), or conversily.

During our work we have not obtained the impression that the four colour conjecture is necessarily correct.

In § 3 we recall a few propositions which may be known to the insiders, but which we have not seen in the literature. These emphasize the fact, that apart from a few trivialities the solution to the four colour conjecture wholly depends on Euler's characteristic -2 for the sphere. Indeed, changing to graph language, necessary and sufficient conditions are known for a graph to be a planar one (KURATOWSKI [9]; MAC LANE [10]). However, since in the four colour conjecture one can restrict oneself to triangular graphs (defined in § 2), a criterion for *these* graphs to be a planar graph is very simple; see § 3.

§ 2. *Definitions*

In this paper a *graph* G will be a pair (V, E), in which V is a set, and E a family of non-ordered pairs of distinct elements of V; the elements of V are called *vertices* of G, the members of E are called *edges* of G. Moreover, until the contrary is explicitly stated, we assume that G is a *finite* graph i.e. that V is a finite set. (For a general definition of a "graph" see [1], [8] or [12].) Vertices are denoted by $p, q, r, \ldots$, and edges by (p, q) etc., in which $p \neq q$ and $(p, q) = (q, p)$.

Two vertices p and q are *neighbours* in G, if (p, q) is an edge of G.

A *subgraph* $G' = (V', E')$ of a graph $G = (V, E)$ is a graph for which $V' \subset V$ and $E' \subset E$.

A *full subgraph* H of G is a subgraph of G, containing every edge of G the vertices of which are in H.

If $p_1, \ldots, p_k$ are vertices of G, $G \setminus (p_1, \ldots, p_k)$ will denote the full subgraph containing the vertices of G except for $p_1, \ldots, p_k$.

It is well known (e.g. see [11], p. 46) that a graph G can be *represented rectilinearly in* the Euclidean three-space E^3, i.e.:

(i) there is a one-to-one correspondence between the set of vertices of G and a set of points of E^3,

(ij) if two vertices are neighbours in G, corresponding points are joined in E^3 by a straight linesegment,

(iij) two such segments have at most an endpoint in common.

Henceforth, G will be identified with its representation in E^3.

A *path* is a subgraph homeomorphic to an interval.

A graph is *connected* if each pair of vertices can be joined by a path.

A *component* of a graph G is a maximal connected subgraph of G.

A *cycle* in G is a subgraph of G homeomorphic to a circle.

A *n-cycle* is a cycle containing n vertices.

A *triangle* of G is a cycle containing exactly three vertices, say p, q and r, such that, if C denotes the component of G which contains p, q and r, $C \setminus (p, q, r)$ is connected.

A graph is called *planar* if there is a homeomorphism from G into the two-sphere S^2. Henceforth, if G is a planar graph, we assume $G \subseteq S^2$.

Proposition: *A map can be represented by a planar graph i.e.*

(i) *there is an one-to-one correspondence between the countries of the map and the vertices of a planar graph.*

(ij) *two countries are adjacent if and only if the corresponding vertices are neighbours in this graph.*

(*c.f.* [1], *p.* 213).

Colouring a graph with a given number of colours is: to assign a colour to each vertex in such a way that any two neighbours obtain different colours.

In view of the proposition above the four colour problem for maps can be reduced to the four colour problem for planar graphs which consists in proving or disproving that four colours are sufficient to colour any planar graph. (Actually, these problems are equivalent.)

If G is a planar graph, a *face* of G is a region in S^2 the boundary of which consists of edges and vertices of G, and which does not contain any vertex or edge of G itself. A face of G which is bounded by a cycle containing n vertices of G is called an *n-gon*.

A graph G is called *triangular* if it is planar and if each face of G is a 3-gon.

It is easily seen that a regular map without rings consisting of one or two countries can be represented by a triangular graph. Conversely, given a triangular graph G, by forming its dual G^* (see [1], p. 213), and defining a country as the closure of a face of G^*, one obtains a regular map the representing graph of which is G.

§ 3. *Characterization of triangular graphs*

Theorem 1: *Let G be a connected graph, containing α_0 vertices, α_1 edges, and α_2 triangles. The following properties* I *and* II *are equivalent*:

I. *G is a triangular graph and G contains at least four vertices.*

II. (a) *Each edge of G belongs to exactly two triangles,*

(b) *The set of triangles of G is "connected", that is, it cannot be decomposed into two classes in such a way that each edge of G only belongs to triangles of the same class,*

(c) $-\alpha_0 + \alpha_1 - \alpha_2 = -2$.

Proof: I→II: If α_2^* denotes the number of 3-gons in G, we have $-\alpha_0 + \alpha_1 - \alpha_2^* = -2$ (Euler) (see e.g. [1], pag. 207). Because it is obvious that each triangle bounds one 3-gon in order to prove II (c), it is sufficient to show that the boundary of a 3-gon is a triangle. First, observe, that for each vertex of G there is a cycle the vertices of which are exactly the neighbours of this vertex. Now, let p, q and r denote the vertices of the boundary of a 3-gon. We prove that any two vertices s and t of $G\setminus(p, q, r)$ can be connected by a path in $G\setminus(p, q, r)$ (Because G contains at least four vertices $G\setminus(p, q, r)$ is non-void). s and t can be connected by a path in G and we may suppose that this path only contains the vertex p or the edge (p, q). Let us consider for example the second case. Let the vertices on the path be $s, \ldots, u, p, q, v, \ldots, t$. We find a path joining s and t in $G\setminus(p, q, r)$, (i) going from s to u, (ij), then going along a cycle the vertices of which are exactly the neighbours of p, and avoiding q and r, up to a neighbour of q, (iij), then going from this vertex along a cycle the vertices of which are exactly the neighbours of q, and avoiding p and r up to v, (iv), then joining v and t. Thus, we have proved II (c). On account of the property, that the boundary of a 3-gon is a triangle, II (a) is obvious. II (b) is trivial.

II→I: Set up a one-to-one correspondence between the set of triangles of G and a set of disjoint closed 2-cells in the euclidean plane. For each triangle take a homeomorphism between the triangle and the boundary of the corresponding closed 2-cell. In this way, using II (a) and II (b), one obtains a "Polygonsystem" as defined in chapter 6, p. 130, of [11]. Such a system gives rise to a complex of which the underlying polyhedron is the two-sphere II (c). The 1-skeleton of this complex is a planar graph satisfying I.

Theorem 2: *Let G be a connected graph, containing α_0 vertices, α_1 edges, and α_2 triangles. The following properties* I *and* II *are equivalent*:

I. (a) *G is a triangular graph,*
   (b) *If p, q and r are (not necessarily different) vertices of G, $G\setminus(p, q, r)$ is connected and non-void.*

II. (a) *The full subgraph the vertices of which are exactly the neighbours of a vertex of G is a cycle,*
   (b) $-\alpha_0 + \alpha_1 - \alpha_2 = -2.$

Proof: I→II: II (a) is obvious. By I (b) it follows that each 3-gon is bounded by a triangle. From this and from the identity of Euler II (b) follows.

II→I: If p, q and r are mutual neighbours, by the argument, employed in the first part of the proof of theorem 1, using II (a), we find that $G\setminus(p, q, r)$ is connected. Thus, p, q and r are vertices of a triangle. From this and from II (a) it follows that each edge is contained in at least two triangles. Further, an edge (p, q) is contained in at most two triangles. For, if not, the full subgraph the vertices of which are the neighbours of p contains a tripod. This however contradicts II (a). Thus (Th. 1, II (a)) holds. From II (a) we also obtain (Th. 1, II (b)). Property II (b) implies (Th. 1, II (c)). From theorem 1 it follows that G satisfies I (a). I (b) is a consequence of II (a).

Remark: Using the fact that $3\alpha_2 = 2\alpha_1$ in the graphs which are considered, property II (c) of theorem 1, and property II (b) of theorem 2 can be replaced by the property:

$$\alpha_0 = \tfrac{1}{3}\alpha_1 + 2.$$

§ 4. *A case of colouration*

The *order of a vertex p* of a graph G, denoted by $O(p, G)$, is the number of its neighbours in G.

Lemma: *Let G be a planar graph. Let H be a full subgraph of G which can be coloured using at most four colours. If for each vertex p of $G \setminus H$ the inequality $O(p, G) \leq 5$ holds and if in each component of $G \setminus H$ there is a vertex q with $O(q, G) \leq 4$, then G can be coloured using at most four colours.*

Proof: By induction with regard to the number of the vertices of G.

Let G be a graph having n vertices. We may suppose $H \neq G$. Let p be a vertex of $G \setminus H$ such that $O(p, G) \leq 4$ and let C denote the component of $G \setminus H$ to which p belongs.

We consider two cases:

(i) C contains only one vertex, namely p. It is trivial that $G \setminus p$ satisfies the induction hypothesis.

(ij) If C contains more than one vertex, let $C_1, \ldots, C_k$ $(k \geq 1)$ denote the components of $C \setminus p$. For each i $(1 \leq i \leq k)$, there is a vertex p_i, such that p_i belongs to C_i and p_i is a neighbour of p. Because $O(p_i, G) \leq 5$, we have $O(p_i, G \setminus p) \leq 4$.

From this it follows that $G \setminus p$ satisfies the induction hypothesis.

So in both cases $G \setminus p$ can be coloured using at most four colours. If for the neighbours of p at most three colours are used, one assigns to p a colour different from these ones. If not, let in "cyclic" order q, r, s and t denote the neighbours of p, coloured by a, b, c and d respectively. By definition, an *$(a-c)$-path* will be a path in G the vertices of which are coloured by a and c alternately.

First suppose q and s are not connected by an $(a-c)$-path. Then we interchange the colours a and c for all vertices which are connected with s by an $(a-c)$-path, and we colour s by a, and p by c. Secondly, suppose, q and s *are* connected by an $(a-c)$-path. By the Jordan curve theorem we know that in this case r and t are not connected by a $(b-d)$-path. Now, in the same way, we interchange the colours b and d for all vertices which are connected with t by a $(b-d)$-path, and we colour t by b, and p by d.

Theorem 3: *Any planar graph in which the order of each vertex is at most five can be coloured using at most four colours.*

Proof: By induction with regard to the number of vertices of G. Let G be a graph satisfying the conditions of the theorem and having n vertices.

If G is not connected, by the induction hypothesis each component of G – and thus G also – can be coloured using at most four colours.

(i) Thus we may suppose that G is connected.
Now, if G contains a vertex p with $O(p, G) \leq 4$, the lemma applies, using the induction hypothesis.

(ij) Thus, we may suppose, that the order of each vertex of G equals five.

If G contains a *cutpoint* i.e. a vertex p such that $G \setminus p$ is not connected, let $C_1, \ldots, C_k$ $(k \geq 2)$ denote the components of $G \setminus p$. Let C_i^* be the full subgraph of G the vertices of which are the vertices of C_i and p. Each C_i^* satisfying the induction hypothesis can be coloured and by a permutation of the colours we can attain that in each C_i^* p has a fixed colour. From this it follows, that we can colour G using at most four colours.

(iij) Thus we may suppose that G contains no cutpoint.

If G contains two vertices p and q such that $G \setminus (p, q)$ is not connected, let $C_1, \ldots, C_k$ $(k \geq 2)$ denote the components of $G \setminus (p, q)$.

Let C_i' be the full subgraph of G the vertices of which are the vertices of C_i and p and q $(1 \leq i \leq k)$. By (i) and (iij) we know that C_i' is connected. Thus p and q can be joined in C_i' by a path which is by definition a jordan-arc. For each j $(1 \leq j \leq k)$ we construct a *planar* graph C_j^* as follows:

If in C_j' p and q are joined by an edge we put $C_j' = C_j^*$. If not, we add a jordan-arc, joining p and q in a fixed C_i' $(i \neq j)$, to C_j', thus obtaining C_j^* from C_j'. For each j the order of p and q in C_j^* is at most five. Indeed, if p and q are neighbours in G, this fact is trivial, and, if not, we observe that both p and q are neighbours of at least one vertex of each C_i $(i \neq j)$ and such a vertex does not occur in C_j^*.

Thus, C_j^* containing less than n vertices satisfies the induction hypothesis and can be coloured using at most four colours.

From the construction of C_j^* it follows, that in each C_j^* p and q obtain different colours. Moreover, it is easily seen, that in each C_j^* by a permutation of the colours we can attain that both p and q obtain fixed (but different) colours. From this it follows that G can be coloured using at most four colours.

(iv) We may, therefore, suppose that G does not contain vertices p and q such that $G\setminus(p, q)$ is not connected.

In view of (iij) we know that the boundary of any face of G is a cycle. Now, suppose that there exists a face of G the boundary of which is a n-cycle with $n \geq 4$. Let in cyclic order the vertices on this n-cycle be $p, q, r, s, \ldots$. Observe, that by (iv) we know that p and r are not neighbours. Thus we can form a new graph G^* by contracting p and r to one point p^*. Moreover, using the fact that each edge of G can be supposed to be a polygonal-line, it is not difficult to prove that G^* is a *planar* graph. Because $d(q, G) \leq 5$, we have $d(q, G^*) \leq 4$. Because $G\setminus(r, p)$ is connected in view of (iv), $G^*\setminus p^*$ is connected. Taking $H = p^*$, the lemma applies and we can colour G^* and consequently G using at most four colours.

(v) Thus we may suppose that each face of G is bounded by a cycle which contains (at most) three vertices.

It is easily shown and well known that each graph which satisfies (i)–(v) is topologically equivalent to the 1-skeleton of the icosahedron. This graph, however, can be coloured using at most four colours. This completes the proof.

Corollary 1: *An infinite planar graph in which the order of each vertex is at most five can be coloured using at most four colours.*

Proof: Each finite subgraph of the given graph can be coloured in view of theorem 3, using at most four colours. By a theorem of De Bruijn and Erdös [2] stating, that an infinite graph can be coloured using at most n colours if and only if each finite subgraph can be coloured using at most n colours, the given graph can be coloured using at most four colours.

Corollary 2: *Any map of order* ≤ 5 *can be coloured using at most four colours.*

Proof: Use the proposition of § 2 and theorem 3.

Remark: In view of corollary 1 corollary 2 also holds for infinite maps which can be defined by a slight modification of the definition of a map.

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