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M.Ward [1] proved the following theorem.

Let $u_1, u_2, \ldots$ be elements of a ring R satisfying a linear recurring relation of order N of which the characteristic polynomial $f(x)$ is irreducible in $R[x]$. Let further two positive integers a en b satisfy

(1) $$u_{a+sb} = r \quad (s = 0,1,\ldots,N),$$

where r is an element $\neq 0$ of R.

Then $f(x)$ is a cyclotomic polynomial and every solution of the difference equation, of which $f(x)$ is the characteristic polynomial, is periodic.

In the below proof, contrary to Ward's proof, no use is made of the roots of $f(x) = 0$.

Introducing the operator E which transforms u_n into u_{n+1} $(n = 1,2,\ldots)$ one has

$$f(E)u_n = 0 \quad (n = 1,2,\ldots).$$

Let $T(F)$ denote the resultant belonging to $R[F]$ of $f(E)$ and E^b-F. Then there exist polynomials $P(E,F)$ and $Q(E,F)$ of $R[E,F]$ such that

(2) $$T(F) = P(E,F)f(E) + Q(E,F)(E^b-F),$$

where $Q(E,F)$ is of a degree $\leqslant N-1$ in E. Moreover from the Sylvester representation of $T(F)$ as a determinant one immediately finds that $T(F)$ is a monic polynomial of degree N in F.

From (2) one gets

$$T(E^b) = P(E,E^b)f(E),$$

hence

(3) $$T(E^b)u_n = 0.$$

Now put

$$v_n = u_{a+bn} \quad (n = 0,1,\ldots);$$

then, if the operator G transforms v_n into v_{n+1}, from $v_{n+1} = u_{a+bn+b} = E^b u_{a+bn}$ and from (3) one finds

$$T(G)v_n = 0 \quad (n = 0,1,\ldots).$$

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1) M.Ward, A property of recurring series, Proc. Nat. Acad. of Sci. 19 (1933), 914-916.

Taking $n = 0$ and using (1) one obtains

$$0 = T(G)v_o = T(1)r,$$

hence $T(1) = 0$ on account of $r \neq 0$.

Then (2) gives

$$0 = T(1) = P(E,1)f(E) + Q(E,1)\ (E^b-1)$$

hence

$$f(E) \mid Q(E,1)\ (E^b-1).$$

Since $Q(E,1)$ is of a degree $\leq N-1$ and $f(E)$ is irreducible and of degree N, one obtains $f(E) \mid E^b-1$, whence follow both the assertions immediately.