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SUPEREXTENSIONS

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CONTENTS

INTRODUCTION

For practical reasons, it has appeared useful (c.f. [1], [2]) to generalize the concept of a filter as follows: Instead of the requirement that every finite set of elements of the filter has non-empty intersection, one only requires that every pair of elements intersects. A collection satisfying this weaker requirement is called a linked system. Thus we have linked systems as compared with centered systems and maximal linked systems as compared with maximal centered systems or ultrafilters. All kinds of extensions of spaces which are defined by using ultrafilters as points can now be generalized by using maximal linked systems. The extensions obtained in this way are called superextensions (actually, there are many superextensions of a space). This report is the first in which we try to study this concept in some generality and although it is communis opinio among the authors that we have only scratched the surface of the subject, we hope to make it clear that the results obtained so far are satisfactory and not pathological.
Also, the superextensions seem to supply us with a new method by which we can create new (bigger) spaces from old ones (c.f. hyperspaces, products, unions, cones, suspensions, inverse limits).

The superextension of a discrete space of 1, 2, 3, 4 and 5 points is a discrete space of 1, 2, 4, 12 and 81 points, respectively. The superextension of a compact Hausdorff space is a compact Hausdorff space of the same weight. In particular, the superextension of a compact metrizable space is a compact metrizable space. Superextensions are even supercompact (i.e. there exists an open subbase such that every cover by subbase elements has a subcover consisting ot two elements). The interest in this strengthening of compactness is enhanced by a recent proof (by J. O'Conner [5] at the Univ. of Florida) that every compact metrizable space is supercompact.

Extensions of continuous mappings over the superextensions are - as almost always - important, and a necessary tool. The results obtained here are as good as can be expected. This part of the work has mainly been carried out by the second author who also proved the weight theorem.

There is an external (though seemingly not internal) relationship between the superextension of a space and the hyperspace of a space. Just as the finite point sets of the space play an important role in the theory of the hyperspace, the more sophisticated finitely determined maximal linked systems (see section 3) play an important role in the theory of superextensions. The introduction of this notion and the development of techniques are due to the third author. His main result in this section (Theorem 3) states that under very general conditions, the superextension of a connected space is both connected and locally connected.

In the fourth section, we analyse a wide range of exemples. Much time and energy have been spent on this section by the second and third authors (well yes, the first author wrote the introduction and served as stand-by and fortune teller).

From the unsolved problems mentioned at the end of the paper, we emphasize one. Is the superextension of the unit interval homeomorphic to the Hilbert cube? The answer is certainly yes, but it might be very difficult to prove this because the corresponding problem for the hyperspace is also unsolved (although the superextension is essentially bigger, there seems to be no natural "mapping relation" between them). The Hausdorff metric is a natural metric for the hyperspace in the compact metric case, and there is a similar metric for the superextension (given by the third author) in section 2.

1. Definitions and direct consequences.

This section contains most of the definitions and preliminary results which are needed throughout the remainder of this report. We also include a number of related results which help to give insight into the theory and background of the subject.

"(SUB)BASE" will always mean "(sub)base for the CLOSED sets".

DEFINITION 1.1. Let $\mathcal{S}$ be a subbase for a space X. $\mathcal{S}$ is said to be a T_1-subbase in case for each $x \in X$, $\{x\} = \bigcap \{S \in \mathcal{S} \mid x \in S\}$, and for each $x \in X$ with $x \notin S$, there exists $T \in \mathcal{S}$ with $x \in T$ and $S \cap T = \emptyset$.

DEFINITION 1.2. Two subsets A and B of a set X are said to be screened by a family $\mathcal{C}$ of subsets of X if $\mathcal{C}$ covers X and each element of $\mathcal{C}$ meets at most one of A and B.

DEFINITION 1.3. Let $\mathcal{S}$ be a subbase for a space X. $\mathcal{S}$ is said to be normal in case each $S,T \in \mathcal{S}$ with $S \cap T = \emptyset$ be screened by a pair of elements of $\mathcal{S}$.

$\mathcal{S}$ is said to be weakly normal in case each $S,T \in \mathcal{S}$ with $S \cap T = \emptyset$ can be screened by a finite family of elements of $\mathcal{S}$.

ad def.1.1. Any closed subbase containing all singletons is a T_1-subbase.
For any space X:
(X is T_1)
$\Longleftrightarrow$ (X has a T_1-subbase)
$\Longleftrightarrow$ (the closed sets of X constitute a T_1-subbase)

ad def.1.2. Usually $\mathcal{C}$ is a finite (two element) subfamily of a subbase $\mathcal{S}$ of X. Now "$\mathcal{C}$ screens A and B" is equivalent to "A and B have disjoint open neighbourhoods which are elements of the open subbase (base) that corresponds with $\mathcal{S}$".

ad def.1.3. The collection of all closed sets of a topological space X forms a normal subbase iff X is normal.
Any normal subbase is weakly normal.

If $f,g: X \to [0,1]$ are two continuous functions such that $f^{-1}(0) \cap g^{-1}(0) = \emptyset$, then $(\frac{f}{f+g})^{-1}[0,\frac{1}{2}]$ and $(\frac{f}{f+g})^{-1}[\frac{1}{2},1]$ form a pair of zerosets screening $f^{-1}(0)$ and $g^{-1}(0)$.

The theory of superextensions originates from the following theorem: A T_1-space X is completely regular if and only if X has a (weakly) normal T_1-subbase.
The first proof has appeared in [2]. In this report it is a corollary to proposition 1.5.

DEFINITION 1.4. If $\mathcal{T}$ is a collection of subsets of a space X, then a <u>linked system</u> of $\mathcal{T}$ is a subcollection of $\mathcal{T}$ with the property that every pair of elements of the subcollection has nonempty intersection.

It is easy to see, with the aid or Zorn's lemma, that every linked system of a collection $\mathcal{T}$ of subsets of a space X is contained in a <u>maximal linked system</u> (m.l.s.) of $\mathcal{T}$ (i.e. maximal in $\mathcal{T}$ with respect to the property of being linked). We will use the script letters $\mathcal{K},\mathcal{M},\mathcal{N},\mathcal{P}$ to denote maximal linked systems of a given collection of subsets of a space X.

EXAMPLE 1.1. Let D_n be a discrete space of n elements with all nonempty subsets as a normal T_1-subbase for D_n.
For $D_3 = \{1,2,3\}$, the maximal linked systems of the base of all closed sets are $\mathcal{M}_1 = \{\{1\},\{1,2\},\{1,3\},\{1,2,3\}\}$, $\mathcal{M}_2 = \{\{2\},\{1,2\},\{2,3\},\{1,2,3\}\}$, $\mathcal{M}_3 = \{\{3\},\{1,3\},\{2,3\},\{1,2,3\}\}$, and $\mathcal{M}_4 = \{\{1,2\},\{2,3\},\{1,3\},\{1,2,3\}\}$.

EXAMPLE 1.2. Let S^1 be the unit circle (i.e. circle of radius 1) and as a normal T_1-subbase for S^1, let $\mathcal{S}$ be the collection of all intervals of S^1. One m.l.s. of $\mathcal{S}$ is the set of all elements of $\mathcal{S}$ whose length is at least Π.

For other examples of maximal linked systems, see section 4.

PROPOSITION 1.1. Let $\mathcal{S}$ be a subbase for a T_1-space in X. Then $\mathcal{S}$ is a T_1-subbase for X iff for each $x \in X$, $\{S \in \mathcal{S} \mid x \in S\}$ is an m.l.s. of $\mathcal{S}$.

DEFINITION 1.5. If $\mathcal{S}$ is a T_1-subbase for a space X, then we let $\lambda_{\mathcal{S}}X$ denote the collection of all maximal linked systems of $\mathcal{S}$. If X is any T_1-space, then we let λX denote the collection of all maximal linked systems of the base of all closed sets of X.

For each $A \subset X$, define

$$A^+ = \{\mathcal{M} \in \lambda_{\mathcal{S}}X \mid \exists S \in \mathcal{M} \text{ with } S \subset A\}.$$

For each $S \in \mathcal{S}$, it is easy to see that

$$S^+ = \{\mathcal{M} \in \lambda_{\mathcal{S}}X \mid S \in \mathcal{M}\}.$$

DEFINITION 1.6. If $\mathcal{S}$ is a T_1-subbase for X, then $\{S^+ \mid S \in \mathcal{S}\}$ is a subbase for a topology on $\lambda_{\mathcal{S}}X$ and $\lambda_{\mathcal{S}}X$ equipped with this topology is called *the superextension of X with respect to $\mathcal{S}$*. In case $\mathcal{S}$ is all of the closed sets of a T_1-subbase X, then $\lambda_{\mathcal{S}}X = \lambda X$ is called *the superextension of X*.

In example 1.1 above, $\lambda D_3 = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4\}$. Moreover, $\{1\}^+ = \{\mathcal{M}_1\}$, $\{2\}^+ = \{\mathcal{M}_2\}$, $\{3\}^+ = \{\mathcal{M}_3\}$, $\{1,2\}^+ = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_4\}$, $\{1,3\}^+ = \{\mathcal{M}_1, \mathcal{M}_3, \mathcal{M}_4\}$, $\{2,3\}^+ = \{\mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4\}$, and $\{1,2,3\}^+ = \lambda D_3$. It is easy to see that λD_3 is discrete with 4 points. It is also the case that λD_4 is discrete with 12 points and λD_5 is discrete with 81 points.

The following proposition contains some immediate consequences of the preceding definitions. We omit the proof.

PROPOSITION 1.2. Let $\mathcal{S}$ be a T_1-subbase for a space X.

(1) If $A, B \subset X$ and $A \cap B = \emptyset$, then $A^+ \cap B^+ = \emptyset$.

(2) If $S, T \in \mathcal{S}$, then $S \cap T = \emptyset$ iff $S^+ \cap T^+ = \emptyset$.

(3) If $S, T \in \mathcal{S}$ with $S \subset T$ and $S \in \mathcal{M} \in \lambda_{\mathcal{S}}X$, then $T \in \mathcal{M}$.

(4) If $A \subset B \subset X$, then $A^+ \subset B^+$.

(5) If $\mathcal{M} \in \lambda_{\mathcal{S}}X$ and $S \in \mathcal{S}$ with $\mathcal{M} \cup \{S\}$ linked then $S \in \mathcal{M}$.

(6) If $S \in \mathcal{S}$, then $S^+ \cup (X \setminus S)^+ = \lambda_{\mathcal{S}}X$ (and by (1): $S^+ \cap (X \setminus S)^+ = \emptyset$).

(7) If $\mathcal{M} \in \lambda_{\mathcal{S}}X$ and $\bigcap \mathcal{M} \neq \emptyset$, then by proposition 1 there exists a unique $x \in X$ with $\bigcap \mathcal{M} = \{x\}$. Then $\mathcal{M} = \{S \in \mathcal{S} \mid x \in S\}$.

(8) If $S, T \in \mathcal{S}$ with $S \cup T = X$, then $S^+ \cup T^+ = \lambda_{\mathcal{S}}X$.

(9) Let $\mathcal{N}$ be a linked system of $\mathcal{S}$, and let $\mathcal{M} = \{S \in \mathcal{S} \mid \exists T \in \mathcal{N} \text{ with } T \subset S\}$ be an m.l.s. of $\mathcal{S}$. Then $\mathcal{M} = \{S \in \mathcal{S} \mid \mathcal{N} \cup \{S\} \text{ is linked}\}$. Moreover, if $A \subset X$, then $\mathcal{M} \in A^+$ iff there exists $T \in \mathcal{N}$ such that $T \subset A$.

(10) Let $i:X \to \lambda_{\mathcal{S}}X$ be the mapping $i(x) = \{S \in \mathcal{S} \mid x \in S\}$. Then i is 1-1 and for each $S \in \mathcal{S}$:

$$i(S) = S^{+} \cap i(X)$$

COROLLARY 1. If $\mathcal{S}$ is a T_1-subbase, then the mapping $x \mapsto \{S \in \mathcal{S} \mid x \in S\}$ is an embedding of X in $\lambda_{\mathcal{S}}X$. (cf. (10)).
<u>From now on we will identify</u> x <u>with</u> $\{S \in \mathcal{S} \mid x \in S\}$ <u>for each</u> $x \in X$.
COROLLARY 2. If $\mathcal{S}$ is a T_1-subbase, then $\{(X \setminus S)^{+} \mid S \in \mathcal{S}\}$ forms an <u>open</u> subbase for $\lambda_{\mathcal{S}}X$ (cf. (6)).

We remark that generalizations of (2) (and hence (1)) and (8) are not necessarily possible, in example 1 above,

$$\{1,2\}^{+} \cap \{2,3\}^{+} \cap \{1,3\}^{+} = \{\mathcal{M}_4\}, \text{ while } (\{1,2\} \cap \{2,3\} \cap \{1,3\})^{+} = \emptyset$$

and $\{1\}^{+} \cup \{2\}^{+} \cup \{3\}^{+} = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\}$, while $(\{1\} \cup \{2\} \cup \{3\})^{+} = \lambda D_3$.

PROPOSITION 1.3. If $\mathcal{S}$ is a T_1-subbase for X, then $\lambda_{\mathcal{S}}X$ is a compact T_1-space; indeed, $\lambda_{\mathcal{S}}X$ is supercompact [1].

<u>Proof</u>. To see that $\lambda_{\mathcal{S}}X$ is a T_1-space, note that for each $\mathcal{M} \in \lambda_{\mathcal{S}}X$, $\{\mathcal{M}\} = \bigcap \{S^{+} \mid S \in \mathcal{M}\}$.
Since supercompactness implies compactness, we show that $\lambda_{\mathcal{S}}X$ is supercompact with respect to the subbase $\{S^{+} \mid S \in \mathcal{S}\}$; i.e., every linked system of $\{S^{+} \mid S \in \mathcal{S}\}$ has nonempty intersection. (Compare this with Alexander's lemma which states that compactness is equivalent to "every centered system of $\{S^{+} \mid S \in \mathcal{S}\}$ has nonempty intersection".)
Let $\mathcal{S}_1 \subset \mathcal{S}$ so that $\{S^{+} \mid S \in \mathcal{S}_1\}$ is a linked system. By proposition 1.2 (2), $\mathcal{S}_1$ is a linked system of $\mathcal{S}$ and so is contained in a maximal linked system $\mathcal{M}$ of $\mathcal{S}$. It follows that $\mathcal{M} \in \bigcap \{S^{+} \mid S \in \mathcal{S}\}$ and so $\lambda_{\mathcal{S}}X$ is supercompact.

In general, it is not the case that $\lambda_{\mathcal{S}}X$ is Hausdorff, even when X is a compact Hausdorff space (c.f. example 4.1).

However, if $\mathcal{S}$ is a "nice" subbase, then $\lambda_{\mathcal{S}}X$ is Hausdorff. To make this more precise, we introduce the following definition. Though this is the weakest condition we did find, it is not satisfactory.

DEFINITION 1.7. Let $\mathcal{S}$ be a subbase for X. We say that a pair $S,T\in\mathcal{S}$ is nicely screened by $\{S_1,\ldots,S_n\}\subset\mathcal{S}$ in case $\{S_1,\ldots,S_n\}$ screens S and T and if $\{T_1,\ldots,T_n\}\subset\mathcal{S}$ with $T_i\cap S_i = \emptyset$, $i = 1,\ldots,n$, then $\{T_1,\ldots,T_n\}$ is not linked.

This means that a pair $S,T\in\mathcal{S}$ is nicely screened by $\{S_1,\ldots,S_n\}$ iff $\{S_1,\ldots,S_n\}$ screens S,T and $S_1^+ \;\ldots\; S_n^+ = \lambda_{\mathcal{S}}X$. This is equivalent to: $\{S_1^+,\ldots,S_n^+\}$ screens S^+ and T^+. C.f. the proof of the following proposition.

PROPOSITION 1.4. If $\mathcal{S}$ is a T_1-subbase for X such that every pair $S,T\in\mathcal{S}$ with $S\cap T = \emptyset$ is nicely screened by a finite family of $\mathcal{S}$, then $\lambda_{\mathcal{S}}X$ is Hausdorff.

<u>Proof</u>. Let $\mathcal{M},\mathcal{N}\in\lambda_{\mathcal{S}}X$ with $\mathcal{M}\neq\mathcal{N}$. Then there exist $S,T\in\mathcal{S}$ with $S\in\mathcal{M}$, $T\in\mathcal{N}$ and $S\cap T = \emptyset$. By assumption, there is a set $\{S_1,\ldots,S_n\}\subset\mathcal{S}$ which nicely screens S and T. If $\lambda_{\mathcal{S}}X \neq \bigcup_{i=1}^{n} S_i^+$, then there exists $\mathcal{P}\in\lambda_{\mathcal{S}}X$ such that $\mathcal{P}\notin\bigcup_{i=1}^{n} S_i^+$. This implies that there exist $T_1,\ldots,T_n\in\mathcal{P}$ with $T_i\cap S_i = \emptyset$, $i = 1,\ldots,n$.
Since $\{T_1,\ldots,T_n\}$ is linked, this contradicts the fact that $\{S_1,\ldots,S_n\}$ nicely screens S and T. Thus $\lambda_{\mathcal{S}}X = \bigcup_{i=1}^{n} S_i^+$ and the Hausdorffness follows from the fact that $\{S_1^+,\ldots,S_n^+\}$ screens S^+ and T^+, and hence screens $\mathcal{M}$ and $\mathcal{N}$.

Since every normal T_1-subbase for a space X satisfies the hypothesis of proposition 1.4, we have:

THEOREM 1.1. If $\mathcal{S}$ is a normal T_1-subbase for X, then $\lambda_{\mathcal{S}}X$ is Hausdorff.

The remainder of this section is devoted to some results on compactifications. If $\mathcal{S}$ is a T_1-subbase for X, then the closure of X in $\lambda_{\mathcal{S}}X$, denoted by <u>$\beta_{\mathcal{S}}X$</u>, is a compactification of X with subbase $\{S^+\cap\beta_{\mathcal{S}}X \mid S\in\mathcal{S}\}$. Since X need only be a T_1-space, it is clear that $\beta_{\mathcal{S}}X$ need not be a Hausdorff compactification of X.

However, we have the following:

PROPOSITION 1.5. If $\mathcal{S}$ is a weakly normal T_1-subbase for X, then $\beta_{\mathcal{S}}X$ is Hausdorff.

The proof is similar to that of proposition 1.4 except that only needs $(S_1^+ \cup \ldots \cup S_n^+) \cap \beta_{\mathcal{S}}X = \beta_{\mathcal{S}}X$ if $S_1 \cup \ldots \cup S_n = X$. For then $S_1^+ \cup \ldots \cup S_n^+$ is a closed set containing X and hence $\beta_{\mathcal{S}}X$.

Since the zerosets of a $T_{3\frac{1}{2}}$-space constitute a (weakly) normal (sub)base we immediately obtain the mentioned result of de Groot-Aarts [2].

COROLLARY. A topological space is $T_{3\frac{1}{2}}$ iff it has a weakly normal T_1-subbase.

Proposition 1.6 below yields a characterization of the elements of $\beta_{\mathcal{S}}X$ in the general situation and proposition 1.9 below yields a nicer characterization in case $\mathcal{S}$ is a weakly normal T_1-subbase for X. Before obtaining these characterizations, we need the following definition.

DEFINITION 1.8. A subset $\mathcal{L}$ of a collection $\mathcal{S}$ of subsets of a set X is said to be a <u>prime system</u> of $\mathcal{S}$ in case for any $S_1, \ldots, S_n \in \mathcal{S}$ with $\bigcup_{i=1}^{n} S_i = X$, at least one of the S_i is a member of $\mathcal{L}$.

PROPOSITION 1.6. Let $\mathcal{S}$ be a T_1-subbase for X and let $\mathcal{M} \in \lambda_{\mathcal{S}}X$. Then $\mathcal{M} \in \beta_{\mathcal{S}}X$ iff $\mathcal{M}$ is a prime system of $\mathcal{S}$.

<u>Proof</u>. Since $\beta_{\mathcal{S}}X$ is the closure of X in $\lambda_{\mathcal{S}}X$, we have that $\mathcal{M} \notin \beta_{\mathcal{S}}X$ iff there exists $S_1, \ldots, S_n \in \mathcal{S}$ with $\beta_{\mathcal{S}}X \subset \bigcup_{i=1}^{n} S_i^+$ and $\mathcal{M} \notin \bigcup_{i=1}^{n} S_i^+$. This is equivalent to the condition that there exists $S_1, \ldots, S_n \in \mathcal{S}$ with $X = \bigcup_{i=1}^{n} S_i$ and $S_1, \ldots, S_n \notin \mathcal{M}$; i.e. $\mathcal{M}$ is not prime.

PROPOSITION 1.7. Let $\mathcal{S}$ be a weakly normal T_1-subbase for X. Then each prime centered system $\mathcal{L}$ of $\mathcal{S}$ is contained in a unique maximal linked system of $\mathcal{S}$ (which is prime and hence belongs to $\beta_{\mathcal{S}}(X)$).

<u>Proof</u>. Since $\mathcal{L}$ is linked, it is contained in some mls. As $\mathcal{L}$ is prime, this mls is prime and, by proposition 6, belongs to $\beta_{\mathcal{S}}X$.

Now let $\mathcal{M}$ and $\mathcal{N}$ be two different mls-s containing $\mathcal{L}$. Then there exist $S \in \mathcal{M}$ and $T \in \mathcal{N}$ with $S \cap T = \emptyset$. By the weak normality of $\mathcal{S}$, there exist $S_1, \ldots, S_n \in \mathcal{S}$ screening S and T. By the primeness of $\mathcal{L}$, there is and $i \in \{1, \ldots, n\}$ such that $S_i \in \mathcal{L} \subset \mathcal{M} \cap \mathcal{N}$ so that this S_i meets both S and T, a contradiction.

The following proposition contains some well-known set-theoretical results on prime and centered systems which are useful for the proof of proposition 1.9.

PROPOSITION 1.8. Let $\mathcal{S}$ be a collection of subsets of a set X.
(1) Any centered system of $\mathcal{S}$ is contained in a maximal centered system of $\mathcal{S}$.
(2) Any prime system of $\mathcal{S}$ contains a minimal prime system of $\mathcal{S}$.
(3) Any maximal centered system of $\mathcal{S}$ is a prime system of $\mathcal{S}$.
(4) Any minimal prime system of $\mathcal{S}$ is centered.
Proof. We first remark that (1) and (3) are well known and (2) and (4) follow from (1) and (3) using the following two observations: Let $\mathcal{L} \subset \mathcal{S}$.
(i) $\mathcal{L}$ is a prime system of $\mathcal{S}$ iff $\{X \setminus S \mid S \in \mathcal{S} \setminus \mathcal{L}\}$ is a centered system of $\{X \setminus S \mid S \in \mathcal{S}\}$.
(ii) $\mathcal{L}$ is a minimal prime system of $\mathcal{S}$ iff $\{X \setminus S \mid S \in \mathcal{S} \setminus \mathcal{L}\}$ is a maximal centered system of $\{X \setminus S \mid S \in \mathcal{S}\}$.

PROPOSITION 1.9. Let $\mathcal{S}$ be a (weakly) normal T_1-subbase for X and let $\mathcal{M} \in \lambda_{\mathcal{S}}X$. Then $\mathcal{M} \in \beta_{\mathcal{S}}X$ iff $\mathcal{M}$ contains a maximal centered system of $\mathcal{S}$.
Proof. If $\mathcal{M}$ contains a maximal centered system of $\mathcal{S}$, then the proposition 1.8 (3), that centered system is prime and hence $\mathcal{M}$ is prime. Thus $\mathcal{M} \in \beta_{\mathcal{S}}X$ by proposition 1.6.

If $\mathcal{M} \in \beta_{\mathcal{S}}X$, then $\mathcal{M}$ is prime by proposition 1.6. Using proposition 1.8 (2) and 1.8 (4), $\mathcal{M}$ contains a prime centered system $\mathcal{C}$ of $\mathcal{S}$. By 1.8 (1), $\mathcal{C}$ is contained in a maximal centered ystem $\mathcal{C}'$ of $\mathcal{S}$. It follows from proposition 1.7 that $\mathcal{C}' \subset \mathcal{M}$.

REMARK. We observed earlier that the collection of all zerosets $\mathcal{Z}$ of a Tychonoff space X forms a normal T_1-(sub)base for X. It can easily be shown (c.f.[2]) that $\beta_{\mathcal{Z}}(X)$ is the Wallman-Shanin compactification of X with respect to $\mathcal{Z}$ and thus $\beta_{\mathcal{Z}}(X) = \beta(X)$, the Čech-Stone-compactification. In section 2 we make some comments on the extension of continuous functions over $\lambda_{\mathcal{S}}X$ and $\lambda_{\mathcal{S}}X$ for certain subbases $\mathcal{S}$, and prove that $\beta_{\mathcal{Z}}(X) = \beta(X)$ by these means. (Corollary 1 to theorem 2.1).

2. The invariance of some properties.

This section is primarily concerned with the question: If X has a certain property, does $\lambda_{\mathcal{S}}X$ or λX have this property? We have already seen that if X is compact Hausdorff and if $\mathcal{S}$ is a normal T_1-subbase for X, then $\lambda_{\mathcal{S}}X$ is compact Hausdorff. In this section we will discuss such properties as weight, zero-dimensionality, metrizability and super-connectedness.

Since, continuous functions play a role in the invariance of topological properties, we begin this section with a number of results concerning the extension of certain continuous functions.

THEOREM 2.1. *Let $\mathcal{S}$ be a T_1-subbase for X, let $\mathcal{T}$ be a normal T_1-subbase for Y, and let f be a continuous function of X into Y such that $f^{-1}[\mathcal{T}] \subset \mathcal{S}$. Then f has a continuous extension $\bar{f}$ from $\lambda_{\mathcal{S}}X$ into $\lambda_{\mathcal{T}}Y$. Moreover, if f is onto, then $\bar{f}$ is onto.*

Proof. We not first that if $\mathcal{M} \in \lambda_{\mathcal{S}}X$, $\{T \in \mathcal{T} \mid f^{-1}T \in \mathcal{M}\}$ is a linked system of $\mathcal{T}$ and hence is contained in an m.l.s. of $\mathcal{T}$. Suppose that $\{T \in \mathcal{T} \mid f^{-1}T \in \mathcal{M}\}$ is contained in two distinct m.l.s.'s $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{T}$. Then there exist $T_1, T_2 \in \mathcal{T}$ with $T_1 \in \mathcal{N}_1$, $T_2 \in \mathcal{N}_2$ and $T_1 \cap T_2 = \emptyset$. By the normality of $\mathcal{T}$, there exist $T_3, T_4 \in \mathcal{T}$ with $T_3 \cup T_4 = Y$, $T_3 \cap T_2 = \emptyset$ and $T_4 \cap T_1 = \emptyset$. This implies that $f^{-1}[T_3] \cup f^{-1}[T_4] = X$ and hence $f^{-1}[T_3]$ or $f^{-1}[T_4] \in \mathcal{M}$. If $f^{-1}[T_3] \in \mathcal{M}$, then $T_3 \in \{T \in \mathcal{T} \mid f^{-1}T \in \mathcal{M}\}$ and hence $T_3 \in \mathcal{N}_1 \cap \mathcal{N}_2 \subset \mathcal{N}_2$, contrary to $T_3 \cap T_2 = \emptyset$. Similarly, $f^{-1}[T_4] \notin \mathcal{M}$ and so $\{T \in \mathcal{T} \mid f^{-1}T \in \mathcal{M}\}$ must be contained in a unique m.l.s. of $\mathcal{T}$; denote this unique m.l.s. by $\bar{f}(\mathcal{M})$. Clearly $\bar{f}$ is well defined. If $x \in X$, then $\bar{f}[\{S \in \mathcal{S} \mid x \in S\}]$ is the unique m.l.s. of $\mathcal{T}$ containing $\{T \in \mathcal{T} \mid x \in f^{-1}T\} = \{T \in \mathcal{T} \mid f(x) \in T\} = f(x)$. It follows that $\bar{f}$ is an extension of f. To see the continuity of $\bar{f}$, let $T_1 \in \mathcal{T}$ and suppose that $\mathcal{M} \notin \bar{f}^{-1}[T_1^+]$. Then there exists $T_2 \in \mathcal{T}$ with $f^{-1}[T_2] \in \mathcal{M}$ and $T_1 \cap T_2 = \emptyset$. Using the normality of $\mathcal{T}$, there exists $T_3, T_4 \in \mathcal{T}$ with $T_3 \cup T_4 = Y$, $T_3 \cap T_2 = \emptyset$ and $T_1 \cap T_4 = \emptyset$, i.e. $\lambda_{\mathcal{S}}X \setminus f^{-1}[T_3]^+$ is open, contains $\mathcal{M}$ and is disjoint from $\bar{f}^{-1}[T_1]$. Hence $\bar{f}^{-1}[T_1]$ is closed and so $\bar{f}$ is continuous. If f is onto and $\mathcal{N} \in \lambda_{\mathcal{T}}Y$, then $\{f^{-1}[T] \mid T \in \mathcal{N}\}$ is a linked system of $\mathcal{S}$ and hence any m.l.s. of $\mathcal{S}$ containing $\{f^{-1}[T] \mid T \in \mathcal{N}\}$ is mapped onto $\mathcal{N}$ by $\bar{f}$.

We remark that the restriction of the above defined $\bar{f}$ to $\beta_{\mathcal{S}}X = X^-$ yields a mapping into $Y^- = \beta_{\mathcal{T}}Y$.

If Y is T_2 and compact and $\mathcal{T}$ is all of the zerosets of Y (or more generally, $\mathcal{T}$ is weakly normal), then $\beta_{\mathcal{T}}Y = Y$. Thus we obtain a kind of Stone extension theorem for these cases. It should be noted however, we have more, in the sense that we can extend such functions over all of $\lambda_{\mathcal{S}}X$ into $\lambda_{\mathcal{T}}Y$.

Applying this result to the case when Y is the unit interval, $\mathcal{T}$ is the collection of all closed intervals of Y, and $\mathcal{S}$ is a T_1-subbase for X which contains all zerosets of X, we can conclude that all bounded real-valued continuous functions on X can be extended over $\lambda_{\mathcal{S}}X$, and hence

COROLLARY 1. If $\mathcal{Z}$ is the collection of zerosets of a $T_{3\frac{1}{2}}$ space X, then $\beta_{\mathcal{Z}}(X) = \beta(X)$, the Čech-Stone-compactification.

As a corollary to theorem 2.1, we also have:

COROLLARY 2. If $\mathcal{S}$ is a normal T_1-subbase for X, then the identity function on X has a closed continuous extension from λX onto $\lambda_{\mathcal{S}}X$.

It should be noted, however, that $\lambda_{\mathcal{S}}X$ need not always be a quotient space or even a continuous image of λX (c.f. examples 2 and 3 in §4).

Related to the extension problem, we also have the following proposition, the proof of which is straightforward.

PROPOSITION 2.1. Let $\mathcal{S}$ be a T_1-subbase for X and let f be a continuous mapping of X into the unit interval I. Then

(1) $\overline{f}$: $\lambda_{\mathcal{S}}X \to I$ defined by $\overline{f}(\mathcal{M}) = \inf \{\sup f(S) | S \in \mathcal{M}\}$ is an upper semi-continuous extension of f.

(2) $\underline{f}$: $\lambda_{\mathcal{S}}X \to I$ defined by $\underline{f}(\mathcal{M}) = \sup \{\inf f(S) | S \in \mathcal{M}\}$ is a lower semi-continuous extension of f.

(3) $\underline{f} \leq \overline{f}$.

(4) If $\mathcal{S}$ contains all zerosets of X, then $\underline{f} = \overline{f}$ is continuous.

(5) If $\mathcal{S}$ contains all zerosets of X, and one uses as a subbase for I all of the closed intervals of I, then $\underline{f} = \overline{f}$ is the $\overline{f}$ of theorem 2.1.

NOTATION. If X is an infinite T_1-space, we let w(X) denote the weight of X (i.e. the minimal cardinality of a subbase for X).

LEMMA 1. If X is an infinite compact Hausdorff space, then $w(\lambda X) = w(X)$.

Proof. Let $\mathcal{B}$ be a base for the topology of X such that $|\mathcal{B}| = w(X)$ and $\mathcal{B}$ is closed under the taking of finite unions and finite intersections. Since X is compact Hausdorff, it is easy to see that if F_1 and F_2 are disjoint closed subsets of X, then there exist $B_1, B_2 \in \mathcal{B}$ screening F_1 and F_2. It follows that $\{B^+ | B \in \mathcal{B}\}$ is a subbase for the topology of λX.

THEOREM 2.2. If X is an infinite compact Hausdorff space and if $\mathcal{S}$ is a normal T_1-subbase for X, then $w(\lambda_{\mathcal{S}} X) = w(X)$.

Proof. This follows from lemma 1 and the corollary 2 to theorem 2.1, since the extension of the identity function on X induces an upper semicontinuous decomposition of λX whose numbers are compact.

PROPOSITION 2.2. If X is compact Hausdorff and zero-dimensional, and if $\mathcal{S}$ is a subbase for X containing all of the clopen sets of X, then $\lambda_{\mathcal{S}} X$ is zero-dimensional.

Proof. If $\mathcal{B}$ is the base of all clopen sets of X (or even if $\mathcal{B}$ is a base of clopen sets which is closed under the taking of complements, finite unions, and finite intersections), then once can argue as in lemma 1 that $\{B^+ | B \in \mathcal{B}\}$ is a subbase for $\lambda_{\mathcal{S}} X$ and by proposition 1.2(2) and 1.2(8), each B^+, for $B \in \mathcal{B}$ is clopen in $\lambda_{\mathcal{S}} X$.

THEOREM 2.3. If X is the Cantor space, then λX is homeomorphic to X.

Proof. Since a Cantor space is completely characterized by the properties; second axiom of countability, zerodimensional, compact Hausdorff and dense-it-itself, theorem 2.2 and proposition 2.2 imply that it is sufficient to prove that λX has no isolated points. We do this by showing that every nonempty basic open set of X contains at least two points. We may assume a basic open set is of the form $\lambda X \setminus \bigcup_{i=1}^{n} S_i^+$, where $S_1, \ldots, S_n$ are clopen in X. Assume $\mathcal{M} \notin \bigcup_{i=1}^{n} S_i^+$ and let $T_i = X \setminus S_i$, $i = 1, \ldots, n$. Then $T_i \in \mathcal{M}$, $i = 1, \ldots, n$. Since $T_i \cap T_j \neq \emptyset$, X contains no isolated points, and $T_i \cap T_j$ is clopen in X, we can conclude that $T_i \cap T_j$ contains infinitely many points of X. Pick $p_{ij} \neq p_{ij}^1$ in $T_i \cap T_j$ $(i,j = 1,\ldots,n)$ so that $\{p_{ij} | i,j = 1,\ldots,n\} \cap \{p_{ij}^1 | i,j = 1,\ldots,n\} = \emptyset$. For each $j = 1,\ldots,n$, let $H_j = \{p_{ij} | i = 1,\ldots,n\}$. Then H_j is a closed subset of X and $H_j \subset T_j$, $j = 1,\ldots,n$. The collection $\{H_j | j = 1,\ldots,n\}$ is linked and so is contained in some maximal linked system $\mathcal{N}_1$ in the base of all closed sets of X. It is clear that $\mathcal{N}_1 \notin \bigcup_{i=1}^{n} S_i^+$. Similarly, using the $\{p_{ij}^1 | i,j = 1,\ldots,n\}$, we obtain a maximal linked system $\mathcal{N}_2$ such that $\mathcal{N}_2 \notin \bigcup_{i=1}^{n} S_i^+$.

Since $\{p_{ij} | i,j = 1,\ldots,n\} \in \mathcal{N}_1$ and $\{p^1_{ij} | i,j = 1,\ldots,n\} \in \mathcal{N}_2$, then $\mathcal{N}_1 \neq \mathcal{N}_2$.

THEOREM 2.4. Let M be a compact metric space with metric ρ and let $\overline{U}_a(S) = \{x \in M | \rho(x,s) \leq a\}$ for $S \subset M$ and a a real number. Then $\tilde{\rho}: \lambda M x \lambda M \to \mathbb{R}$, defined by

$$\tilde{\rho}(\mathcal{M},\mathcal{N}) = \inf\{a \in \mathbb{R} | \forall S \in \mathcal{M},\ T \in \mathcal{N},\ \overline{U}_a(T) \in \mathcal{M}, \overline{U}_a(s) \in \mathcal{N}\},$$

is a metric for λM.

Proof. It is clear that $\tilde{\rho}(\mathcal{M},\mathcal{N}) = \tilde{\rho}(\mathcal{N},\mathcal{M})$ for every $\mathcal{M},\mathcal{N} \in \lambda M$. We prove next that

$$\tilde{\rho}(\mathcal{M},\mathcal{N}) = \min\{a \in \mathbb{R} | \forall S \in \mathcal{M}, T \in \mathcal{N},\ \overline{U}_a(T) \in \mathcal{M}, \overline{U}_a(S) \in \mathcal{N}\}.$$

Let $c = \tilde{\rho}(\mathcal{M},\mathcal{N})$. Suppose that there is an $S \in \mathcal{M}$ with $\overline{U}_c(S) \notin \mathcal{N}$. Then there exists $T \in \mathcal{N}$ such that $T \cap \overline{U}_c(S) = \emptyset$ and hence an $\varepsilon > 0$ such that $T \cap \overline{U}_{c+\varepsilon}(S) = \emptyset$, a contradiction. Thus $\overline{U}_c(S) \in \mathcal{N}$ for every $S \in \mathcal{M}$. Similarly, $\overline{U}_c(T) \in \mathcal{M}$ for every $T \in \mathcal{N}$. Hence the desired assertion. Since S closed in M implies that $\overline{U}_0(S) = S$, it follows immediately from the definition that $\tilde{\rho}(\mathcal{M},\mathcal{N}) = 0$ iff $\mathcal{M} = \mathcal{N}$. Thus we only need to prove the triangle inequality. Let $\mathcal{M},\mathcal{N},\mathcal{P} \in \lambda M$ and suppose that $\tilde{\rho}(\mathcal{M},\mathcal{N}) = a$ and $\tilde{\rho}(\mathcal{N},\mathcal{P}) = b$. Then for each $S \in \mathcal{M}$, $T \in \mathcal{N}$ and $P \in \mathcal{P}$, $\overline{U}_a(S) \in \mathcal{N}$, $\overline{U}_a(T) \in \mathcal{M}$, $\overline{U}_b(T) \in \mathcal{P}$ and $\overline{U}_b(P) \in \mathcal{N}$. Thus, for every $S \in \mathcal{M}$ and $P \in \mathcal{P}$, $\overline{U}_b(\overline{U}_a(S)) \in \mathcal{P}$ and $\overline{U}_a(\overline{U}_b(P)) \in \mathcal{M}$. Now $\overline{U}_b(\overline{U}_a(S)) \subset \overline{U}_{a+b}(S)$ and $\overline{U}_a(\overline{U}_b(P)) \subset \overline{U}_{a+b}(P)$ imply that $a + b \geq \tilde{\rho}(\mathcal{M},\mathcal{P})$ and the triangle inequality follows. Before showing that the metric topology on λM is compatible with the superextension topology, it is useful to remark that

$$\tilde{\rho}(\mathcal{M},\mathcal{N}) = \min\{a | \forall S \in \mathcal{M},\ \overline{U}_a(S) \in \mathcal{N}\}.$$

We need only prove $\leq$, since the reverse inequality is obvious. Suppose we have an a such that for every $S \in \mathcal{M}$, $\overline{U}_a(S) \in \mathcal{N}$. Then $\forall S \in \mathcal{M}$, $T \in \mathcal{N}$, it follows that $\overline{U}_a(S) \cap T \neq \emptyset$, and hence $\overline{U}_a(T) \cap S \neq \emptyset$; i.e. $\overline{U}_a(T) \in \mathcal{M}$, $\forall T \in \mathcal{N}$. It follows that $a \geq \tilde{\rho}(\mathcal{M},\mathcal{N})$.

To see that $\tilde{\rho}$ is compatible with the topology on λM, we show that the topology induced by $\tilde{\rho}$ on λM is weaker than the superextension topology (which is compact) and hence these two topologies must coincide.

Let $\varepsilon > 0$ and $\mathcal{M} \in \lambda M$. Let $p_1, \ldots, p_n$ be an $\varepsilon/3$-net of M and let $U_a(p) = \{x \in M | \rho(x,p) < a\}$, where $a \in \mathbb{R}$ and $p \in M$. We let $\mathcal{O}$ be the finite collection of open sets which are unions of sets of the form $U_{\varepsilon/3}(p_i)$ $(i = 1, \ldots, n)$. Let $0 = \cap\{U^+ | U \in \mathcal{O}$ and $\mathcal{M} \in U^+\}$. It is clear that 0 is an open set in M containing $\mathcal{M}$. Moreover, if $S \in \mathcal{M}$, then there exists $U \in \mathcal{O}$ such that $S \subset U \subset U_{2\varepsilon/3}(S)$, namely, $U = \cup \{U_{\varepsilon/3}(p_i) | \rho(p_i, S) < \varepsilon/3\}$. Therefore, $0 \subset U^+ \subset (U_{2\varepsilon/3}(S))^+$ and hence $0 \subset \bigcap_{S \in \mathcal{M}} (U_{2\varepsilon/3}(S))^+$. It follows that if $\mathcal{N} \in 0$, then for each $T \in \mathcal{N}$, $T \cap U_{2\varepsilon/3}(S) \neq \emptyset$ for every $S \in \mathcal{M}$. Thus $U_{2\varepsilon/3}(T) \cap S \neq \emptyset$ for every $S \in \mathcal{M}$ and so $\bar{U}_{2\varepsilon/3}(T) \in \mathcal{M}$ for every $T \in \mathcal{N}$. Hence $\tilde{\rho}(\mathcal{M}, \mathcal{N}) \leq \frac{2}{3}\varepsilon < \varepsilon$ and hence $\{\mathcal{N} \in \lambda M | \tilde{\rho}(\mathcal{N}, \mathcal{M}) < \varepsilon\}$ is open.

We can derive more information about $\tilde{\rho}$. We list a few of the results but omit the proofs.

(1) $\tilde{\rho}(\mathcal{M}, \mathcal{N}) = \sup_{S \in \mathcal{M}} \inf_{T \in \mathcal{N}} \sup_{x \in T} \rho(x, S)$.

(2) $\tilde{\rho}(\mathcal{M}, \mathcal{N}) = \max_{S \in \mathcal{M}} \min_{T \in \mathcal{N}} \sup_{x \in T} \rho(x, S)$.

(3) $\tilde{\rho}(\mathcal{M}, \mathcal{N}) = \sup_{S \in \mathcal{M}} \inf_{T \in \mathcal{N}} d(T, S)$, where d is the Hausdorff metric on the collection of all closed sets of M.

(4) $\tilde{\rho} | M \times M = \rho$.

(5) $U_\varepsilon(\mathcal{M}) = \bigcup_{0<a<\varepsilon} \bigcap_{S \in \mathcal{M}} (U_a(S))^+$ and $\bar{U}_\varepsilon(\mathcal{M}) = \bigcap_{S \in \mathcal{M}} (\bar{U}_\varepsilon(S))^+$.

THEOREM 2.5. <u>Let $\{X_\alpha; \phi_{\alpha\beta}\}$ be an onto inverse spectrum of spaces such that $\phi_{\alpha\alpha}$ is the identity on X_α with $\mathcal{S}_\alpha$ a normal T_1-subbase for X_α and such that $\phi_{\alpha\beta}^{-1}[\mathcal{S}_\alpha] \subset \mathcal{S}_\alpha$ for each $\alpha > \beta$. Then $\varprojlim \lambda_{\mathcal{S}_\alpha} X_\alpha = \lambda_{\mathcal{S}} \varprojlim X_\alpha$ where $\mathcal{S}$ is the relativized natural subbase for the product topology of $\prod_\alpha X_\alpha$.</u>

<u>Proof</u>. Let π_β be the β<u>th</u> projection of $\prod_\alpha X_\alpha$. Then $\varprojlim X_\alpha = \{x \in \prod_\alpha X_\alpha | \pi_\beta(x) = \phi_{\alpha\beta} \circ \pi_\alpha(x)$ for all $\alpha > \beta\}$, and $\mathcal{S}$ is the restriction to $\varprojlim X_\alpha$ of the collection $\{\pi_\alpha^{-1}[\mathcal{S}_\alpha]\ S \in \mathcal{S}_\alpha;$ all $\alpha\}$ which is a subbase for the product topology. It is not difficult to see that $\mathcal{S}$ is a normal T_1-subbase for $\varprojlim X_\alpha$. By theorem 2.1, each $\phi_{\alpha\beta}$ has an extension $\bar{\phi}_{\alpha\beta}$ from $\lambda_{\mathcal{S}\alpha} X$ onto $\lambda_{\mathcal{S}\beta} X_\beta$.

Making use of the definition of $\bar{\phi}_{\alpha\beta}$, one can show that if $\alpha > \beta > \gamma$, then $\bar{\phi}_{\alpha\gamma} = \bar{\phi}_{\beta\gamma} \circ \bar{\phi}_{\alpha\beta}$ so that $\{\lambda_{\mathcal{S}_\alpha} X_\alpha; \bar{\phi}_{\alpha\beta}\}$ is an onto inverse spectrum. Considering each π_α as a mapping of $\varprojlim X_\alpha$ onto X_α, then $\pi_\alpha^{-1}[S] \in \mathcal{S}$ for each $S \in \mathcal{S}_\alpha$. Thus π_α has an extension $\bar{\pi}_\alpha$ from $\lambda_{\mathcal{S}} \varprojlim X_\alpha$ onto $\lambda_{\mathcal{S}_\alpha} X_\alpha$ with the property that $\bar{\pi}_\beta(\mathcal{M}) = \bar{\phi}_{\alpha\beta} \circ \bar{\pi}_\alpha(\mathcal{M})$ for all $\alpha > \beta$. It is now easy to exhibit a map of $\lambda_{\mathcal{S}} \varprojlim X_\alpha$ onto $\varprojlim \lambda_{\mathcal{S}_\alpha} X_\alpha$ and to show that the map is a homeomorphism.

We conlcude this section with a result on super-connected spaces [3] which are of interest only for non-T_2-spaces. We omit the easy proof.

PROPOSITION 2.3. Let X be a nonempty T_1-space. Then (i) through (vi) below are equivalent, (vi) implies (vii) and (viii), and (viii) through (x) are equivalent.

	(i)	X is super-connected.
<=>def	(ii)	Each open set of X is connected.
<=>	(iii)	Each nonempty open set of X is dense.
<=>	(iv)	Each pair of nonempty open sets of X has nonempty intersection.
<=>	(v)	Each n nonempty open sets of X have nonempty intersection.
<=>	(vi)	The open topology of X is centered.
(v)=>	(vii)	X is dense in λX.
(v)=>	(viii)	The open subbase of λX is centered.
<=>	(ix)	The open topology of λX is centered.
<=>	(x)	λX is superconnected.

3. Finitely determined maximal linked systems.

In this section X is a fixed T_1-space and $\mathcal{S}$ is a fixed subbase for X which contains all of the finite subsets of X.

We define a special kind of m.l.s. of $\mathcal{S}$ and with the aid of this special m.l.s., we obtain some results on the connectivity of the super-extension $\lambda_{\mathcal{S}}X$.

DEFINITION 3.1. Let F be a finite subset of X. An m.l.s. with respect to F is a maximal linked system of the collection of all subsets of F.

An "m.l.s. with respect to a finite set F" is usually not an m.l.s. of $\mathcal{S}$, but we have:

PROPOSITION 3.1. If $\mathcal{M}$ is an m.l.s. with respect to a finite subset F of X, then $\mathcal{M}$ is contained in a unique m.l.s. of $\mathcal{S}$; we denote this unique m.l.s. by $\underline{\mathcal{M}}$.

Proof. Let $\mathcal{N} = \{S \in \mathcal{S} | S$ contains a member of $\mathcal{M}\}$. Since $\mathcal{M}$ is linked, also $\mathcal{N}$ is linked. If $T \in \mathcal{S}$ and T meets every member of $\mathcal{N}$, then T meets every member of $\mathcal{M}$. Thus $T \cap F$ meets every member of $\mathcal{M}$ so that $T \cap F \in \mathcal{M}$ (proposition 1.2 (5)), since $\mathcal{S}$ contains all finite subsets of X. Hence T contains a member of $\mathcal{M}$, and so $T \in \mathcal{N}$, i.e. $\mathcal{N}$ is an m.l.s. in $\mathcal{S}$. Clearly any m.l.s. in $\mathcal{S}$ that contains $\mathcal{M}$, must contain $\mathcal{N}$, and thus equals $\mathcal{N}$, proving that $\mathcal{N}$ is the unique m.l.s. containing $\mathcal{M}$.

DEFINITION 3.2. If $\mathcal{M}$ is an m.l.s. with respect to a finite subset F of X and if $\mathcal{N} \in \lambda_{\mathcal{S}}X$ and $\mathcal{N} \supset \mathcal{M}$, then we say that $\mathcal{N}$ is defined on F and that $\mathcal{N}$ is generated by $\mathcal{M}$.

PROPOSITION 3.2.(a). If F is a finite subset of X and $\mathcal{N} \in \lambda_{\mathcal{S}}X$, then $\mathcal{N}$ is defined on F iff $\{S \in \mathcal{N} | S \subset F\}$ is an m.l.s. with respect to F.

(b). If F_1 and F_2 are finite subsets of X with $F_1 \subset F_2$ and if $\mathcal{N} \in \lambda_{\mathcal{S}}X$ with $\mathcal{N}$ defined on F_1, then $\mathcal{N}$ is defined on F_2.

(c). If an m.l.s. $\mathcal{N} \in \lambda_{\mathcal{S}}X$ is defined on a finite subset of X, then there is a smallest among the subsets of X on which $\mathcal{N}$ is defined.

Proof. The (a) is obvious. Note that $\{S\in\mathcal{N}\mid S\subset F\} = \{S\cap F\mid S\in\mathcal{N}\}\subset\mathcal{N}$.

(b): Let $\mathcal{M}_1 = \{S\in\mathcal{N}\mid S\subset F_1\}$. $\mathcal{N}$ is defined on F_1 means: $\mathcal{M}_1$ generates $\mathcal{N}$. If $F_2\supset F_1$ then $\{T\subset F_2\mid T$ contains a member of $M_1\}$ is a mls with respect to F_2 (c.f. the proof of proposition 3.1.). It is readily verified that this collection is contained in $\mathcal{N}$, i.e. $\mathcal{N}$ is defined on F_2.

(c): Let $\mathcal{N}^*$ be the collection of minimal sets in $\mathcal{N}$. Since $\mathcal{N}$ is defined on a finite set, say F, any element $S\in\mathcal{N}$ contains a finite set, which belongs to $\mathcal{N}$ and hence S contains a set $T\in\mathcal{N}^*$. It is easily chequed that $\cup\mathcal{N}^*$ is the smallest set on which $\mathcal{N}$ is defined.

DEFINITION 3.3. If $\mathcal{N}\in\lambda_{\mathcal{S}}X$ and $\mathcal{N}$ is defined on some finite subset of X, then $\mathcal{N}$ is called a <u>finitely determined maximal linked system</u> (<u>f.m.l.s.</u>) of $\mathcal{S}$. If $\mathcal{N}$ is defined on a finite set of at most n elements, then $\mathcal{N}$ is called an <u>n maximal linked system</u> (n-m.l.s.) of $\mathcal{S}$. We let

$$\lambda_{f\mathcal{S}}X = \{\mathcal{N}\in\lambda_{\mathcal{S}}X\mid\mathcal{N} \text{ is an f.m.l.s. of } \mathcal{S}\},$$

$$\lambda_f X = \{\mathcal{N}\in\lambda X\mid\mathcal{N} \text{ is an f.m.l.s. of the base of all closed sets of } X\},$$

and

$$\lambda_{n\mathcal{S}}X = \{\mathcal{N}\in\lambda_{\mathcal{S}}X\mid\mathcal{N} \text{ is an n-m.l.s. of } \mathcal{S}\}.$$

It is easy to see from the definitions that

PROPOSITION 3.3. $X\approx\lambda_{1\mathcal{S}}X = \lambda_{2\mathcal{S}}X\subset\lambda_{3\mathcal{S}}X\subset\ldots\subset\bigcup_{n\in\mathbb{N}}\lambda_{n\mathcal{S}}X = \lambda_{f\mathcal{S}}X$.

REMARK. Using the technique of the following proof, one can show that if $\{A_1,\ldots,A_n\}$ is a linked system of subsets of X, then $(\bigcap_{i=1}^{n} A_i^+)\cap\lambda_{f\mathcal{S}}X \neq \emptyset$. We use this remark without further comment.

PROPOSITION 3.4. Let $A_1,\ldots,A_n\subset X$ with $\{A_1,\ldots,A_n\}$ linked. Then $\bigcap_{i=1}^{n} (cl_X A_i)^+\subset cl_{\lambda_{\mathcal{S}}X}((\bigcap_{i=1}^{n} A_i^+)\cap\lambda_{f\mathcal{S}}X)\subset cl_{\lambda_{\mathcal{S}}X}(\bigcap_{i=1}^{n} A_i^+)$.

Proof. Let $\mathcal{M} \in \bigcap_{i=1}^{n} (cl_X A_i)^+$. Then there exists $S_1,\ldots,S_n \in \mathcal{M}$ such that $S_i \subset cl_X A_i$, $i = 1, \ldots, n$. If we let $\lambda_f X \setminus \bigcup_{i=1}^{m} T_i^+ = \bigcap_{i=1}^{m} (X \setminus T_i)^+$ be any basic open set containing $\mathcal{M}$, then there exist $U_1,\ldots,U_m \in \mathcal{M}$ with $U_i \subset X \setminus T_i$, $i = 1, \ldots, m$. Since $\{S_1,\ldots,S_n,U_1,\ldots,U_m\}$ is a linked system, then $\{cl_X A_1,\ldots,cl_X A_n, X \setminus T_1,\ldots,X \setminus T_m\}$ is linked. It follows that $\{A_1,\ldots,A_n,X \setminus T_1,\ldots,X \setminus T_m\}$ is linked. For each $i,j = 1,\ldots,n + m$ we choose $p_{ij} = p_{ji} \in X$; for $i,j = 1,\ldots,n$, pick $p_{ij} \in A_i \cap A_j$; for each $i = 1,\ldots,n$ and $j = n + 1,\ldots,n + m$, pick $p_{ij} \in A_i \cap (X \setminus T_{j-n})$; for each $i,j = n + 1,\ldots, n + m$, pick $p_{ij} \in (X \setminus T_{i-n}) \cap (X \setminus T_{j-n})$. Let $F = \{p_{ij} \mid i,j = 1,\ldots,n + m\}$ and let $H_i = \{p_{ij} \mid j = 1,\ldots,n + m\}$ for $i = 1,\ldots,n + m$. Then $\{H_i \mid i=1,\ldots,n+m\}$ is a linked system of F and is contained in an m.l.s. $\mathcal{N}$ with respect to F. It follows that $H_1,\ldots,H_{n+m} \in \mathcal{N}$ and since $H_i \subset A_i$ for $i = 1,\ldots,n$, then $\mathcal{N} \in \bigcap_{i=1}^{n} A_i^+$. Moreover, $H_{i+n} \subset X \setminus T_i$ for $i = 1,\ldots,m$ implies that $\mathcal{N} \in \bigcap_{i=1}^{m} (X \setminus T_i)^+$. Hence every basic open set containing $\mathcal{M}$ contains an element of $((\bigcap_{i=1}^{n} A_i^+) \cap \lambda_{fg} X)$; i.e. $\mathcal{M} \in cl_{\lambda_f X}((\bigcap_{i=1}^{n} A_i^+) \cap \lambda_{fg} X)$. The last inclusion of the proposition is trivial.

COROLLARY. If $A_1,\ldots,A_n \subset X$, then $\bigcap_{i=1}^{n} A_i^+ \subset cl_{\lambda_f X}((\bigcap_{i=1}^{n} A_i^+) \cap \lambda_{fg} X)$.

<u>Proof.</u> If $\bigcap_{i=1}^{n} A_i^+ = \emptyset$, then the inclusion is trivial. If $\bigcap_{i=1}^{n} A_i^+ \neq \emptyset$, then $\{A_1,\ldots,A_n\}$ is linked and the result follows from proposition 4.

PROPOSITION 3.5. A subset A of a T_1-space X is nowhere dense in X iff A^+ is nowhere dense in λX.

<u>Proof.</u> Suppose first that A is nowhere dense in X. Let $O_1^+ \cap \ldots \cap O_n^+$ be a nonempty basic open set in λX, where $O_1,\ldots,O_n$ are open in X. For each $i,j = 1,\ldots,n$, let $O_{ij} = O_i \cap O_j \cap (X \setminus cl_X A)$. Then O_{ij} is a nonempty open set of X since $O_i \cap O_j$ is a nonempty open set and A is nowhere dense. For $i = 1,\ldots,n$, let $U_i = O_{i1} \cup \ldots \cup O_{in}$. It follows easily that $\bigcap_{i=1}^{n} U_i^+$ is a nonempty open set in λX which is disjoint from $cl_{\lambda X} A^+$ and which is contained in $\bigcap_{i=1}^{n} O_i^+$. Thus A^+ is nowhere dense in λX.

Conversely, if A^+ is nowhere dense in λX, then $\lambda X \setminus cl_{\lambda X} A^+$ is dense in λX. By proposition 3.4, $(cl_X A)^+ \subset cl_{\lambda X} A^+$ so that $\lambda X \setminus cl_{\lambda X} A^+ \subset \lambda X \setminus (cl_X A)^+ =$ $= (X \setminus cl_X A)^+$ and $(X \setminus cl_X A)^+$ is dense in λX.

It follows that $cl_X A$ can contain no nonempty open set of X and hence A is nowhere dense in X.

We remark that in general it is not the case that if A is nowhere dense in X, then A^+ is nowhere dense in $\lambda_{\mathcal{S}}X$ (c.f. example 2 in §4).

THEOREM 3.1. If $\mathcal{S}$ is a T_1-subbase containing all finite sets then the set of all fmls-s is dense in the superextension with respect to $\mathcal{S}$, i.e.: $(\lambda_{f\mathcal{S}}X)^- = \lambda_{\mathcal{S}}X$.

<u>Proof</u>. $\lambda_{\mathcal{S}}X = X^+ \subset cl_{\lambda_{\mathcal{S}}X}(\lambda_{f\mathcal{S}}X \cap X^+)$.

PROPOSITION 3.6. Define a function $\phi: \lambda_{f\mathcal{S}}X \times \lambda_{f\mathcal{S}}X \times \lambda_{f\mathcal{S}}X \to \lambda_{f\mathcal{S}}X$

by

$$\phi(\mathcal{M},\mathcal{N},\mathcal{P}) = (\mathcal{M}\cap\mathcal{N}) \cup (\mathcal{M}\cap\mathcal{P}) \cup (\mathcal{N}\cap\mathcal{P}).$$

Then

(i) If $\mathcal{M} = \mathcal{N}$, then $\phi(\mathcal{M},\mathcal{N},\mathcal{P}) = \mathcal{M}$.

(ii) If $\mathcal{M}, \mathcal{N} \in \bigcap_{i=1}^{n} A_i^+$, $A_1,\ldots,A_n \subset X$, then $\phi(\mathcal{M},\mathcal{N},\mathcal{P}) \in (\bigcap_{i=1}^{n} A_i^+)$

(iii) ϕ is continuous.

<u>Proof</u>. We first show that the range of ϕ is contained in $\lambda_{f\mathcal{S}}X$. Let $\mathcal{M},\mathcal{N},\mathcal{P} \in \lambda_{f\mathcal{S}}X$. By proposition 3.2 (b), we may assume $\mathcal{M},\mathcal{N},\mathcal{P}$ are all defined on the same finite subset F of X. Let $\mathcal{M}_1 = \{S \in \mathcal{M} | S \subset F\}$, let $\mathcal{N}_1 = \{S \in \mathcal{N} | S \subset F\}$, and let $\mathcal{P}_1 = \{S \in \mathcal{P} | S \subset F\}$. By proposition 3.2 (a), $\mathcal{M}_1$, $\mathcal{N}_1$, $\mathcal{P}_1$, and $\mathcal{P}_1$ are m.l.s.'s with respect to F. Let $\mathcal{K} = (\mathcal{M}_1 \cap \mathcal{N}_1) \cup (\mathcal{M}_1 \cap \mathcal{P}_1) \cup (\mathcal{N}_1 \cap \mathcal{P}_1)$. We will show that $\mathcal{K}$ is an m.l.s. with respect to F. If $S,T \in \mathcal{K}$, then S,T belong to at least one of $\mathcal{M}_1,\mathcal{N}_1,\mathcal{P}_1$ both and hence $S \cap T \neq \emptyset$. Suppose now that $T \subset F$ and $\mathcal{K} \cup \{T\}$ is linked. If T is not in at least two of $\mathcal{M}_1, \mathcal{N}_1, \mathcal{P}_1$, say $T \notin \mathcal{M}_1$ and $T \notin \mathcal{N}_1$, then there exist $S_1 \in \mathcal{M}_1$ and $S_2 \in \mathcal{N}_1$ such that $S_1 \cap T = \emptyset$ and $S_2 \cap T = \emptyset$. It follows that $S_1 \cup S_2 \subset F$ so that $S_1 \cup S_2 \in \mathcal{M}_1 \cap \mathcal{N}_1 \subset \mathcal{K}$ and $T \cap (S_1 \cup S_2) = \emptyset$, contrary to $\mathcal{K} \cup \{T\}$ being linked. Thus T is in at least two of $\mathcal{M}_1, \mathcal{N}_1$, and $\mathcal{P}_1$ and hence $T \in \mathcal{K}$. Therefore $\mathcal{K}$ is an m.l.s. with respect to F and hence $\mathcal{K}$ is contained in a unique m.l.s. $\underline{\mathcal{K}}$ of $\mathcal{S}$ by proposition 3.1.

It is easy to show now that $\phi(\mathcal{M},\mathcal{N},\mathcal{P}) = \underline{\mathcal{K}}$ and since $\underline{\mathcal{K}}$ is defined in F, the range of ϕ is contained in $\lambda_{f\mathcal{S}}X$.

(i) If $\mathcal{M} = \mathcal{N}$, then $\mathcal{M}\cap\mathcal{N} = \mathcal{M} \subset \phi(\mathcal{M},\mathcal{N},\mathcal{P})$ and hence $\phi(\mathcal{M},\mathcal{N},\mathcal{P}) = \mathcal{M}$.

(ii) Let $A_1,\ldots,A_n \subset X$ and suppose that $\mathcal{M},\mathcal{N} \in \bigcap_{i=1}^n A_i^+$. Then there exist $S_i,\ldots,S_n \in \mathcal{M}_1$ and $T_1,\ldots,T_n \in \mathcal{N}_1$ such that $S_i, T_i \subset A_i$, for i = 1,...,n. Thus $S_i \cup T_i \subset A_i$, i=1,...,n and $S_i \cup T_i \in \mathcal{M}_1 \cap \mathcal{N}_1 \subset \phi(\mathcal{M},\mathcal{N},\mathcal{P})$. Therefore $\phi(\mathcal{M},\mathcal{N},\mathcal{P}) \in (\bigcap_{i=1}^n A_i^+)$.

(iii) Let $S \in \mathcal{S}$ and $V = S^+ \cap \lambda_{f\mathcal{S}}X$, then
$U = (V \times V \times \lambda_{f\mathcal{S}}X) \cup (V \times \lambda_{f\mathcal{S}}X \times V) \cup (\lambda_{f\mathcal{S}}X \times V \times V)$ is closed in $\lambda_{f\mathcal{S}}X \times \lambda_{f\mathcal{S}}X \times \lambda_{f\mathcal{S}}X$.
Let $(\mathcal{M},\mathcal{N},\mathcal{P}) \in U$, say $\mathcal{M} \in V \subset S^+$ and $\mathcal{N} \in V \subset S^+$. By (ii), $\phi(\mathcal{M},\mathcal{N},\mathcal{P}) \in T^+ \cap \lambda_{f\mathcal{S}}X = V$; i.e. $U \subset \phi^{-1}(V)$.
Suppose now that $(\mathcal{M},\mathcal{N},\mathcal{P}) \in \phi^{-1}(V)$. Then $\phi(\mathcal{M},\mathcal{N},\mathcal{P}) \in V$ so $S \in \phi(\mathcal{M},\mathcal{N},\mathcal{P})$. Since S must belong to at least two of $\mathcal{M},\mathcal{N},\mathcal{P}$, then $(\mathcal{M},\mathcal{N},\mathcal{P}) \in U$. It follows that $\phi^{-1}(V) = U$ and hence ϕ is continuous.

PROPOSITION 3.7. Let $\mathcal{P}$ be an f.m.l.s. that is defined on $P = \{p_1,\ldots,p_n\} \subset X$ and let $\tau: P \to X$ be a function. If we let $\tau(p_i) = x_i$ and let $F = \{x_1,\ldots,x_n\}$, then the system $\mathcal{P}_\tau$ defined by $\mathcal{P}_\tau = \{S \in \mathcal{S} \mid \exists T \in \mathcal{P}\ \ \tau(T) \subset S\}$ is an f.m.l.s. of $\mathcal{S}$ and is the unique m.l.s. of $\mathcal{S}$ containing $\{\tau(S) \mid S \subset P$ and $S \in \mathcal{P}\}$.

<u>Proof</u>. It is easy to verify that $\mathcal{P}_\tau$ is a linked system of $\mathcal{S}$ which containes $\{\tau(S) \mid S \subset P$ and $S \in P\}$. If $T \in \mathcal{S}$ and $T \cap S \neq \emptyset$ for every $S \in \mathcal{P}_\tau$, then $T \cap \tau(S) \neq \emptyset$ for every $S \subset P$ with $S \in \mathcal{P}$. Then $S \cap \tau^{-1}T \neq \emptyset$ for every $S \subset P$ with $S \in \mathcal{P}$. Since $\mathcal{P}$ is defined on P, then this implies that $\tau^{-1}T \in \mathcal{P}$ and since $T \supset \tau(\tau^{-1}T)$, this means: $T \in \mathcal{P}_\tau$. Uniqueness follows immediately.

THEOREM 3.2. <u>Let $\mathcal{P}$ be an f.m.l.s. of $\mathcal{S}$ defined on</u> $P = \{p_1,\ldots,p_n\}$. <u>For each</u> $X = (x_1,\ldots,x_n) \in X^n$, <u>let</u> $\tau_x: P \to X$ <u>be defined by</u> $\tau_x(p_i) = x_i$ $(i = 1,\ldots,n)$. <u>Let</u> $\psi_n : X^n \to \lambda_{f\mathcal{S}}X$ <u>be defined by</u> $\psi_n((x_1,\ldots,x_n)) = \mathcal{P}_{\tau_x}$, <u>where</u> $x = (x_1,\ldots,x_n)$.
<u>Then ψ_n is a continuous function</u>.

<u>Proof</u>. It is clear from proposition 7 that ψ_n is well defined. Let $S \in \mathcal{S}$, i.e. S^+ is a subbasic closed set of $\lambda_{\mathcal{S}}X$. For each $T \in \mathcal{P}$ with $T \subset P$,

let $S_T = \prod_{\{j|p_j\in T\}} S \times \prod_{\{j|p_j\in T\}} X$. We show that $\psi_n^{-1}(S^+\cap \lambda_{f\mathcal{S}}X) = \{S_T | T\in\mathcal{P}$ and $T\subset P\}$, which is closed in X^n.

Let $(x_1,\ldots,x_n) = x\in\psi_n^{-1}(S^+\cap\lambda_{f\mathcal{S}}X)$. Then $\mathcal{P}_{\mathcal{T}_x}\in S^+$, and so $S\in\mathcal{P}_{\mathcal{T}_x}$. Thus there exists $T\in\mathcal{P}$ with $T\subset P$ and such that $\mathcal{T}_x(T)\subset S$. This implies that $(x_1,\ldots,x_n)\in S_T$, for this particular T.

If on the other hand $(x_1,\ldots,x_n)\in S_T$ for some $T\in\mathcal{P}$, then $\mathcal{T}_x(T)\subset S$ and hence $\mathcal{P}_{\mathcal{T}_x}\in S^+\cap\lambda_{f\mathcal{S}}X$. Thus $\psi_n^{-1}(S^+\cap\lambda_{f\mathcal{S}}X)$ is closed and so ψ_n is continuous.

PROPOSITION 3.8. If X is a connected space, then for $A_1,\ldots,A_k\subset X$, $A_1^+\cap\ldots\cap A_k^+$ ($=\emptyset$ or) is connected.

<u>Proof</u>. In view of the corollary to proposition 4, it is sufficient to prove that $(\bigcap_{i=1}^k A_i^+)\cap\lambda_{f\mathcal{S}}X$ is connected. We may also assume that $|X|\geq\aleph_0$ since the case when $|X| = 0$ or 1 is trivial. Let $\mathcal{M},\mathcal{N}\in(\bigcap_{i=1}^k A_i^+)\cap\lambda_{f\mathcal{S}}X$ and suppose that $\mathcal{M}$ is defined on $M = \{p_1,\ldots,p_m\}$ and $\mathcal{N}$ is defined on $N = \{q_1,\ldots,q_n\}$. By the cardinality assumption on X, we may choose mn distinct points of X disjoint from $M\cup N$; denote this set by $R = \{r_{ij} | i=1,\ldots,m; j=1,\ldots,n\}$; Let $\mathcal{T}_1: R\to M$ be defined by $\mathcal{T}_1(r_{ij}) = p_i$. Similarly, we can define a function $\mathcal{T}_2 : R\to N$ by $\mathcal{T}_2(r_{ij}) = q_j$. Let

$$\mathcal{B} = \{\mathcal{T}_1^{-1}(S)\cap\mathcal{T}_2^{-1}(T) | S\in\mathcal{M} \text{ with } S\subset M,\ T\in\mathcal{N} \text{ with } T\subset N\}.$$

It is clear that $\mathcal{B}$ is a linked system of subsets of R. Let $\mathcal{P}$ be any m.l.s. of $\mathcal{S}$ which is defined on R and which contains $\mathcal{B}$. Suppose that $S\in\mathcal{P}$. If $\mathcal{T}_1(S\cap R)\notin\mathcal{M}$, then $M\setminus\mathcal{T}_1(S\cap R)\in\mathcal{M}$ since $\mathcal{M}$ is defined on M. Thus $\mathcal{T}_1^{-1}(M\setminus\mathcal{T}_1(S\cap R))\in\mathcal{B}\subset\mathcal{P}$ and $S\cap\mathcal{T}_1^{-1}(M\setminus\mathcal{T}_1(S\cap R)) = \emptyset$, contradiction. Therefore, for every $S\in\mathcal{P}$, $\mathcal{T}_1(S\cap R)\in\mathcal{M}$, and similarly $\mathcal{T}_2(S\cap R)\in\mathcal{M}$. By the uniqueness in proposition 3.7, $\mathcal{P}_{\mathcal{T}_1} = \mathcal{M}$ and $\mathcal{P}_{\mathcal{T}_2} = \mathcal{N}$. Let ψ_{mn}: $X^{m+n}\to\lambda_{f\mathcal{S}}X$ be the continuous function defined in Theorem 3.2, where the domain of the $\mathcal{T}_x$'s is now R and the m.l.s. defined on R is the $\mathcal{P}$ from above. Since $\mathcal{P}_{\mathcal{T}_1} = \mathcal{M}$ and $\mathcal{P}_{\mathcal{T}_2} = \mathcal{N}$, then $\mathcal{M},\mathcal{N}$ are in the image of ψ_{mn}. For the $\mathcal{M}$ and $\mathcal{N}$ as fixed above, we can define a function Θ : $\lambda_{f\mathcal{S}}X\to\lambda_{f\mathcal{S}}X$ by

$$\Theta(\mathcal{K}) = \phi(\mathcal{M},\mathcal{N},\mathcal{K}), \text{ where } \phi \text{ defined in proposition 3.6.}$$

It follows from proposition 3.6 that Θ is continuous, $\Theta(\mathcal{M}) = \mathcal{M}$ and $\Theta(\mathcal{N}) = \mathcal{N}$. Also, since $\mathcal{M}, \mathcal{N} \in \bigcap_{i=1}^{k} (A_i^+)$, then $\Theta(\mathcal{K}) \in \bigcap_{i=1}^{k} A_i^+$ for all $\mathcal{K} \in \lambda_{f\mathcal{S}}X$. Therefore, $\Theta o \psi_{mn}: X^{m+n} \to \lambda_{f\mathcal{S}}X$ is a continuous function such that $\mathcal{M}$ and $\mathcal{N}$ are in $\mathcal{I}m(\Theta o \psi_{mn})$ and $\mathcal{I}m(\Theta o \psi_{mn}) \subset \mathcal{I}m\,\Theta \subset (\bigcap_{i=1}^{k} A_i^+) \cap \lambda_{f\mathcal{S}}X$. Since X^{m+n} is connected, then $\mathcal{I}m(\Theta o \psi_{mn})$ is connected. It follows immediately that $(\bigcap_{i=1}^{k} A_i^+) \cap \lambda_{f\mathcal{S}}X$ is connected.

With a somewhat different technique one can prove that if A is a connected subset of the space X, then A^+ is a connected subset of $\lambda_{\mathcal{S}}X$. It was conjectured that if A is connected subset of a space X and $A_1, \ldots, A_n \subset X$, then $A^+ \cap A_1^+ \cap \ldots \cap A_n^+$ is connected. This is false (c.f. example 10 in §4).

THEOREM 3.3. _If_ X _is_ _a_ _connected_ _space_ _and_ $\mathcal{S}$ _is_ _a_ T_1-_subbase_ _containing_ _all_ _finite_ _subsets_ _of_ X _or_ _if_ $\mathcal{S}$ _is_ _a_ _normal_ T_1-_subbase_ _for_ X, _then_ $\lambda_{\mathcal{S}}X$ _is_ _connected_ _and_ _locally_ _connected_. (c.f. example 4.7).

Proof. Suppose first that $\mathcal{S}$ is a T_1-subbase for X which contains all finite subsets of X. By proposition 3.8, $\lambda_{\mathcal{S}}X$ is connected and locally connected. In case $\mathcal{S}$ is a normal T_1-subbase for X, then by Th 2.1, coro 2 $\lambda_{\mathcal{S}}X$ is a quotient of λX which is connected and locally connected by the previous remark. Thus $\lambda_{\mathcal{S}}X$ is connected and locally connected.

We conclude with some remarks concerning the existance of example and/or counterexamples.

1. If $\mathbb{N}$ is the natural numbers with the usual topology, then $\lambda\mathbb{N}$ is not locally connected although $\mathbb{N}$ is locally connected (example 4.6).

2. If $\lambda_{\mathcal{S}}X$ is connected, then X need not be connected (c.f. [4] or example 4.8).

3. If I is the unit interval and $\mathcal{S}$ is merely a T_1-subbase for I, then $\lambda_{\mathcal{S}}I$ need not be locally connected or connected (c.f. examples 4.2 and 4.3).

4. Examples.

This section is primarily devoted to the construction of various counterexamples that were mentioned in the previous sections. We also include some problems that we have been unable to solve as of now.

We begin this section with a lemma which is useful in the computations of the examples.

LEMMA 1. Let X be a compact space and $\mathcal{S}$ a T_1-subbase for X. If $\mathcal{M}\in\lambda_{\mathcal{S}}X$ and $\mathcal{F}\subset\mathcal{M}$ with $\mathcal{F}$ a filterbase and $\cap\mathcal{F}\in\mathcal{S}$, then $\cap\mathcal{F}\in\mathcal{M}$.

The proof is a straightforward application of compactness.

EXAMPLE 4.1. There exists a weakly normal T_1-subbase $\mathcal{S}$ for the unit interval I such that $\lambda_{\mathcal{S}}I$ is not Hausdorff.

Proof. Let $\mathcal{S}$ be the set of all intervals and all doublets which are contained in I. If $0 \leq x \leq y \leq z \leq 1$, we define

$$\mathcal{L}_{xyz} = \{S\in\mathcal{S} \mid S \text{ contains at lest one of } \{x,y\},\{y,z\},\{x,z\}\},$$

$$x\mathcal{L}_{yz} = \{S\in\mathcal{S} \mid S \text{ contains } [y,z] \text{ or S contains a set of the form } \{x,y\} \text{ for } u\in[y,z]\},$$

$$xy\mathcal{L}_{z} = \{S\in\mathcal{S} \mid S \text{ contains } [x,y] \text{ or S contains a set of the form } \{u,z\} \text{ for } u\in[x,y]\}.$$

It is easy to show that for each $0 \leq x \leq y \leq z \leq 1$, $\mathcal{L}_{xyz}$, $x\mathcal{L}_{yz}$, and $xy\mathcal{L}_z$ are maximal linked systems of $\mathcal{S}$. We will show later that all m.l.s.'s of $\mathcal{S}$ are one of these types. Note that if x =y or y =z, then $\mathcal{L}_{xyz} = x\mathcal{L}_{yz} = xy\mathcal{L}_z = \{S\in\mathcal{S}|y\in S\}$ and hence corresponds to the point of y of I. We first hos that $\mathcal{L}_{0\frac{1}{4}\frac{1}{2}}$ and $\mathcal{L}_{0\frac{1}{4}\frac{3}{4}}$ do not have disjoint basic open neighborhoods in $\lambda_{\mathcal{S}}I$. On the contrary, suppose they do have such neighborhoods. Then there exist $S_1,\ldots,S_n\in\mathcal{S}$ such that $\bigcup_{i=1}^{n} S_i^+ = \lambda_{\mathcal{S}}I$ and no S_i^+ contains both $\mathcal{L}_{0\frac{1}{4}\frac{1}{4}}$ and $\mathcal{L}_{0\frac{1}{4}\frac{3}{4}}$; i.e. there exist $[a_1,b_1],\ldots[a_r,b_r]$, $\{a_{r+1},b_{r+1}\},\ldots\{a_n,b_n\}$, in $\mathcal{S}$ such that none of these sets belong to both $_{0\frac{1}{4}\frac{1}{2}}$ and $_{0\frac{1}{4}\frac{3}{4}}$ and such that $[a_1,b_1]^+ \ldots [a_r,b_r]^+\cup\ldots\cup\{a_n,b_n\}^+ = \lambda_{\mathcal{S}}I$. If $x \geq \frac{3}{4}$, then $\mathcal{L}_{0\frac{1}{4}x}\in[a_i,b_i]^+$ implies that $\mathcal{L}_{0\frac{1}{4}\frac{1}{2}}$ and $\mathcal{L}_{0\frac{1}{4}\frac{3}{4}}\in[a_i,b_i]^+$. Thus it must be the case that $\{\mathcal{L}_{0\frac{1}{4}x}|\frac{3}{4} \leq x \leq 1\}\subset\{a_{r+1},b_{r+1}\}^+\cup\ldots\cup\{a_n,b_n\}^+$. It follows that there exist i such that $r < i \leq n$ and infinitely many x's such that

$\{a_i,b_i\} \in \mathcal{L}_{0\frac{1}{4}x}$. This implies that $\{a_i,b_i\} = \{0,\frac{1}{4}\}$ and $\{0,\frac{1}{4}\}$ belongs to both $\mathcal{L}_{0\frac{1}{4}\frac{1}{2}}$ and $\mathcal{L}_{0\frac{1}{4}\frac{3}{4}}$, contrary to the assumption on the S_i. Thus $\lambda_{\mathcal{S}}X$ is not Hausdorff.

We next prove that every m.l.s of $\mathcal{S}$ is of one of the types $\mathcal{L}_{xyz}$, $x\mathcal{L}_{yz}$ or $xy\mathcal{L}_z$ for some $0 \leq x \leq y \leq z \leq 1$. Let $\mathcal{M}$ be a m.l.s. of and suppose that $\cap\mathcal{M} = \emptyset$. It is easy to see that $\mathcal{M}$ must contain some doublet {a,b} since any linked system consisting of only intervals is centered. Since $a \notin \cap\mathcal{M}$ one of the following cases hold:

i) $\exists$ c,d $a \notin \{c,d\} \in \mathcal{M}$, hence $b \in \{c,d\}$, say b = c
ii) $\exists$ c,d $a \notin [c,d] \in \mathcal{M}$
Similarly for b
iii) $\exists$ e,f $b \notin \{e,f\} \in \mathcal{M}$, hence $a \in \{e,f\}$, say a = e.
or iv) $\exists$ e,f $b \notin [e,f] \in \mathcal{M}$

If i) and iii) then $\{c,d\} \cap \{e,f\} = \emptyset$ implies d = f. It is easily chequed that the only doublets in $\mathcal{M}$ are {a,b}, {b,d} and {a,d}. So $\mathcal{M}$ is of the form $\mathcal{F}_{xyz}$. If i) and iv), but $\mathcal{M}$ is not a 3m.l.s., then {a,d} is not in $\mathcal{M}$. So any doublet contains b = c, and the other point of the doublet must be in $\cap\{[u,v] \mid b \notin [u,v] \in \mathcal{M}\}$. It is readily verified that this intersection itself belongs to $\mathcal{M}$, and that $\mathcal{M}$ is of the type $x\mathcal{L}_{yz}$ or $xy\mathcal{L}_z$. The case ii) and iii) is similar to i) and iv). If not i) and not iii) then {a,b} is the only doublet in $\mathcal{M}$, let a < b, and let $c = \inf\{c \mid a \in [0,c] \in \mathcal{M}\}$. It is easy to show that $c = \sup\{c \mid b \in [c,1] \in \mathcal{M}\}$ and that hence {a,c} and $\{b,c\} \in \mathcal{M}$, a contradiction.

EXAMPLE 4.2. There exists a nicely screening (non-normal) T_1-subbase for the unit interval I such that

(i) $\lambda_{\mathcal{S}}I$ not connected.
(ii) I has a nowhere dense subset A with A^+ not nowhere dense in $\lambda_{\mathcal{S}}I$.

<u>Proof</u>. Let $\mathcal{S}_0 = \{[(n-1)2^{-k}, n2^{-k}] \mid k=1,2,\ldots ; n=1,2,\ldots,2^k\}$ and let $\mathcal{S} = \mathcal{S}_0 \cup \{0,1\}$. It is obvious that $\mathcal{S}_0$ is a nicely screening (non-normal)

T_1-subbase for I such that $\lambda_{\mathcal{S}_0} I = I$. Since $[(n-1)^{2k}, n2^{-k}]$ and $\{0,1\}$ can be nicely screened by $\{[(i-1)2^{-k-1}, i2^{-k-1}] \mid i=1,\ldots,2^{k+1}\}$ then $\mathcal{S}$ is also a nicely screening T_1-subbase for I. First, we note that $\{\{0,1\}, [0,\frac{1}{2}], [\frac{1}{2},1]\}$ is an m.l.s. of $\mathcal{S}$. This follows easily from the fact that any other element of $\mathcal{S}$ must be contained properly in $[0,\frac{1}{2}]$ or $[\frac{1}{2},1]$. Next, we note that this is the only m.l.s. $\mathcal{M}$ of $\mathcal{S}$ for which $\cap\mathcal{M} = \emptyset$. For if $\cap\mathcal{M} = \emptyset$, $\{0,1\} \in \mathcal{M}$ and there exists k, k', n, n' such that $[(n-1)2^{-k}, n2^{-k}]$ and $[(n'-1)2^{-k'}] \in \mathcal{M}$ with $0 \notin [(n-1)2^{-k}, n2^{-k}]$ and $1 \notin [(n'-1)2^{-k'}, n'2^{-k'}]$. Since $\mathcal{M}$ is linked, it must be the case that k=1, n=2, k'=1, n'=1. Since $\lambda_{\mathcal{S}} I$ is Hausdorff and I is compact, it follows that $\lambda_{\mathcal{S}} I$ is homeomorphic to the union of an interval and an isolated point. Thus $\lambda_{\mathcal{S}} I$ is not connected. Moreover, $\{0,1\}$ is nowhere dense in I but $\{0,1\}^+$ is not nowhere dense in $\lambda_{\mathcal{S}} I$.

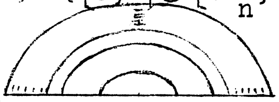
Note that (i) and theorem 3.3 imply that $\lambda_{\mathcal{S}} I$ is not a quotient of λI or even a continuous image of λI.

REMARK. It is worth noting that example 4.2 can be modified so as to obtain a superextension of I which is the disjoint union of two intervals. If we let $\mathcal{S}_1 = \mathcal{S} \cup \{[0,a]\} \mid a \in [\frac{1}{3},\frac{2}{3}]\} \cup \{[a,1] \mid a \in [\frac{1}{3},\frac{2}{3}]\}$ where $\mathcal{S}$ is as defined in example 4.2, then one can show that for each m.l.s. $\mathcal{M}$ of $\mathcal{S}_1$, there exists an $a \in [\frac{1}{3},\frac{2}{3}]$ such that

$$\mathcal{M} = \{\{0,1\}, [0,d], [c,1] \mid \tfrac{1}{3} \leq c \leq a \leq d \leq \tfrac{2}{3}\}.$$

EXAMPLE 4.3. There exists a weakly normal T_1-subbase $\mathcal{S}$ for the unit interval I such that $\lambda_{\mathcal{S}} I$ is not locally connected.

<u>Proof</u>. Let $\mathcal{S} = \{[a,b] \mid [a,b] \subset I\} \cup \{[0,\frac{1}{n}] \cup [1-\frac{1}{n},1] \mid n \in \mathbb{N}\} \cup \{\{0,1\}\}$.

We show that $\lambda_{\mathcal{S}} I$ is homeomorphic to 

For any $\mathcal{M} \in \lambda_{\mathcal{S}} I$, $\cap\{[a,b] \mid [a,b] \in \mathcal{M}\}$ is a singleton because if it were an interval $[c,d]$, then $[0,\frac{c+d}{2}]$ and $[\frac{c+d}{2},1]$ must belong to $\mathcal{M}$. If $\mathcal{M} \in \lambda_{\mathcal{S}} I$ with $\cap\mathcal{M} = \emptyset$, we define m and k by

(i) $\quad m \in \cap\{[a,b] \mid [a,b] \in \mathcal{M}\}$

(ii) $\quad k = \sup\{n-1 \mid [0,\frac{1}{n}] \cup [1-\frac{1}{n},1] \in \mathcal{M}\}$.

We note that $\{n|[0,\frac{1}{n}]\cup[1-\frac{1}{n},1]\in\mathcal{M}\}\neq\emptyset$ (however k may be infinite) and that $m\in(\frac{1}{k},1-\frac{1}{k})$ since $\cap\mathcal{M}=\emptyset$. Suppose that $\mathcal{M}'\in\lambda_{\mathcal{J}}I$ with $\mathcal{M}\neq\mathcal{M}'$ and $\cap\mathcal{M}'=\emptyset$. If k' and m' are defined for $\mathcal{M}'$ in the same way as k and m were defined, then $\mathcal{M}\neq\mathcal{M}'$ yields that either there exists $n\in\mathbb{N}$ with $[0,\frac{1}{n}]\cup[1-\frac{1}{n},1]\in\mathcal{M}$ and $[0,\frac{1}{n}]\cup[1-\frac{1}{n},1]\notin\mathcal{M}'$ or there exists $[a,b]\in\mathcal{M}$ with $[a,b]\notin\mathcal{M}'$. In the first case we obtain $k\neq k'$ and in the second case we obtain $m\neq m'$. On the other hand, if $k'\in\mathbb{N}$ or k' is infinite and $m'\in(\frac{1}{k'},1-\frac{1}{k'})$ with either $k\neq k'$ or $m\neq m'$, then

$$\mathcal{M}'=\{S\in\mathcal{J}|S \text{ contains } [0,\tfrac{1}{k'}]\cup[1-\tfrac{1}{k'},1] \text{ or } S \text{ contains } [\tfrac{1}{k},m] \text{ or } S \text{ contains } [m,\tfrac{1}{k}]\}$$

defines an m.l.s. $\mathcal{M}'\neq\mathcal{M}$ and the m and k defined by (i) and (ii) for ' are m' and k'. It follows that if we let $\mathcal{P}(m,k)=\{S\in\mathcal{J}|S$ contains $[0,\frac{1}{k}]\cup[1-\frac{1}{k},1]$ or S contains $[\frac{1}{k},m]$ or S contains $[m,\frac{1}{k}]\}$, then $\lambda_{\mathcal{J}}I=I\cup\{\mathcal{P}(m,k)|k\in\mathbb{N}$ or $k=\infty$ and $m\in(\frac{1}{k},1-\frac{1}{k})\}$. To complete the proof, we describe the topology of $\lambda_{\mathcal{J}}I$ by describing a neighborhood basis of each of its points.

If $m\in I$ and $\frac{1}{k+1}<m<\frac{1}{k}$ or $\frac{1}{k+1}<1-m<\frac{1}{k}$, then for sufficiently small ε, $(m-\varepsilon,m+\varepsilon)^+=(m-\varepsilon,m+\varepsilon)$ so that a neighborhood basis for m consists of $\{(m-\varepsilon,m+\varepsilon)|\varepsilon\in\mathbb{R}^+$ and ε sufficiently small $\}$.

If m is of the form $\frac{1}{k}$ $(k\neq 1)$, then a neighborhood basis for m is the collection of sets of the form $(\frac{1}{k+1},1-\frac{1}{k+1})^+\cap[0,\frac{1}{k}+\varepsilon)^+\cap(\frac{1}{k}-\varepsilon,1]^+=(\frac{1}{k}-\varepsilon,\frac{1}{k}+\varepsilon)\cup\{\mathcal{P}(\mu,k)|\mu\in(\frac{1}{k},\frac{1}{k}+\varepsilon)\}$ for ε sufficiently small.

A neighborhood basis for $1-\frac{1}{k}$ $(k\neq 1)$ is the collection of sets of the form $(1-\frac{1}{k}-\varepsilon,1-\frac{1}{k}+\varepsilon)\cup\{\mathcal{P}(\mu,k)|\mu\in(1-\frac{1}{k}-\varepsilon,1-\frac{1}{k})\}$ for ε sufficiently small.

For m=0 or 1, merely put $k=\infty$ in the above two descriptions and restrict the intervals to I.

A neighborhood basis of $\mathcal{P}(m,\infty)$ is the collection of sets of the form $([0,\varepsilon)\cup(1-\varepsilon,1])^+\cap[0,m+\varepsilon)^+\cap(m-\varepsilon,1]^+=\{\mathcal{P}(\mu,k)|\mu\in(m-\varepsilon,m+\varepsilon)$ and $k>\frac{1}{\varepsilon}$ or $k=\infty\}$ for ε sufficiently small.

Finally, a neighborhood basis of $\mathcal{P}(m,k)$ is the collection of sets of the form $([0,\frac{1}{k}+\varepsilon)\cup(1-\frac{1}{k}-\varepsilon,1])^+\cap[0,m+\varepsilon)^+\cap(m-\varepsilon,1]^+\cap(\frac{1}{k+1},1-\frac{1}{k+1})^+=\{\mathcal{P}(\mu,k)|\mu\in(m-\varepsilon,m+\varepsilon)\}$ for ε sufficiently small.

EXAMPLE 4.4. Let $D_4 = \{1,2,3,4\}$ be a discrete space with four points. There exists normal T_1-subbases $\mathcal{S}(n)$ for D_4, $n=4,\ldots,12$, such that $\lambda_{\mathcal{S}(n)}D_4$ is discrete with n points.

Proof. Let $\mathcal{S} = \{\{1\},\{2\},\{3\},\{4\}\}$ and for $n=4,\ldots,10$, let $\mathcal{S}(n)$ be as follows:

$$\begin{aligned}
\mathcal{S}_4 &= \mathcal{S}\cup\{\{2,3,4\},\{3,4\},\{1,2\},\{1,2,3\}\},\\
\mathcal{S}_5 &= \mathcal{S}\cup\{\{1,2\},\{1,3,4\},\{2,3,4\},\{3,4\},\{1,2,3\}\},\\
\mathcal{S}_6 &= \mathcal{S}\cup\{\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{2,4\}\},\\
\mathcal{S}_7 &= \mathcal{S}_6\cup\{\{1,3,4\}\},\\
\mathcal{S}_8 &= \mathcal{S}\cup\{\text{doublets of } D_4\},\\
\mathcal{S}_9 &= \mathcal{S}_8\cup\{\{2,3,4\}\},\\
\mathcal{S}_{10} &= \mathcal{S}_9\cup\{\{1,3,4\}\},\\
\mathcal{S}_{11} &= \mathcal{S}_{10}\cup\{\{1,2,4\}\},\\
\mathcal{S}_{12} &= 2^{D_4}.
\end{aligned}$$

It might have been conjectured that if $\mathcal{M}$ is an m.l.s. of a T_1-subbase for a space X such that every element of $\mathcal{M}$ is finite, then $\mathcal{M}$ must be an f.m.l.s. The following example disproves this.

EXAMPLE 4.5. Let X be $\mathbb{N}$ with the cofinite topology. There exists an m.l.s. $\mathcal{M}$ of the base of all closed sets of X such that every element of $\mathcal{M}$ is finite but $\mathcal{M}$ is not an f.m.l.s.

Proof. Let $\mathcal{B}_2 = \{1,2\}$ and for each $n \geq 2$, let $\mathcal{B}_{2n-1} = \{1,2n-1\}\cup \{j\in X \mid j \text{ is even and } 3 \leq j \leq 2n-1\}$ and $\mathcal{B}_{2n} = \{2,2n\}\cup\{j\in X \mid j \text{ is odd and } 2 \leq j \leq 2n-1\}$. These for each $n \geq 2$, we have defined $\mathcal{B}_n$. It is easy to verify that $\{\mathcal{B}_n \mid n \geq 2\}$ is an m.l.s. of the base of all closed sets but is not an f.m.l.s.

EXAMPLE 4.6. Consider $\mathbb{N}$ with the discrete topology. Then $\lambda\mathbb{N}$ has the following properties:

(i) $\lambda_f\mathbb{N}$ is a countable, discrete, dense subset of $\lambda\mathbb{N}$.

(ii) $\lambda\mathbb{N}$ contains converging sequences and hence is not homeomorphic to a Čech-Stone compactification $\beta\mathbb{N}$.

(iii) $\lambda\mathbb{N}$ is not locally connected even though $\mathbb{N}$ is.

Proof. (i) Since $\mathbb{N}$ is countable, $\lambda_f\mathbb{N}$ is countable so that we only need to show that is discrete. To see this, let $\mathcal{M}\in\lambda_f\mathbb{N}$. Then there is a finite subset $M\subset\mathbb{N}$ such that $\mathcal{M}$ is defined on M. It follows that $\mathcal{M}= \bigcap\{S^+|S\subset M, S\in\mathcal{M}\}$ and the latter set is open in $\lambda\mathbb{N}$.

(ii) For each $n\in\mathbb{N}$, $n\neq 1$, let $\mathcal{M}_n$ be the unique m.l.s. containing $\{\{1,i\}|\ i=2,\ldots,n\}\cup\{\{2,\ldots,n\}\}$ and let $\mathcal{M}$ be the unique m.l.s. containing $\{\{1,i\}|\ i=2,3,\ldots\}\cup\{\mathbb{N}\setminus\{1\}\}$. We show that $\mathcal{M}_n$ converges to $\mathcal{M}$. If $\mathcal{M}\notin S_0^+\cup\ldots\cup S_m^+$, then we may assume that $S_0=\{1\}$ and $1\notin S_i$ for $i=1,\ldots,m$. For $i=1,\ldots,m$, it must be the case that $S_i\neq\mathbb{N}\setminus\{1\}$ and so there exists a natural number k_i such that $k_i\notin S_i$. For $n>k_i$, $\{1,k_i\}\in\mathcal{M}_n$ and hence $\mathcal{M}_n\notin S_0^+\cup\ldots\cup S_i^+$. It follows that if we let $M=\max\{k_1,\ldots,k_m\}$, then for $n>M$ we have $\mathcal{M}_n\notin S_0^+\cup\ldots\cup S_n^+$.

(iii) Since any open neighborhood of a point in $\lambda\mathbb{N}\setminus\lambda_f\mathbb{N}$ must contain (infinitely many) points of $\lambda_f\mathbb{N}$, then because of (ii), $\lambda\mathbb{N}$ is not locally connected.

We next consider an example to illustrate a very special part of Theorem 3.3.

EXAMPLE 7. Let $I=\{(x,y)\in\mathbb{R}\times\mathbb{R}\mid x=0,\ -1\leq y\leq 1\}$, let $X=I\cup\{(x,y)\in\mathbb{R}\times\mathbb{R}\mid y=\sin\frac{1}{x},\ 0\leq x\leq\frac{1}{\pi}\}$, and let ρ be the usual metric of $\mathbb{R}\times\mathbb{R}$ restricted to X. We illustrate a special consequence of Theorem 3 in §3 by showing the following: for each $p\in I$ and U a neighborhood in λX, there exists $\epsilon>0$ such that if $q\in U_\epsilon(p)=\{x\in X|\rho(x,p)<\epsilon\}$, then q and p cannot be separated in U.

Proof. Since U is a neighborhood of p in λX, it must contain a finite intersection of pluses of open set containing p. Take ϵ so small that $U_\epsilon(p)$ lies in the intersection of these open sets. It follows that $U_\epsilon(p)^+\subset U$. Let $q\in U_\epsilon(p)$ and for each $x\in X$, let $\mathcal{M}_x=\{S|S$ is a closed subset of X containing at least 2 points of $\{p,q,x\}\}$. It is easy to show that $\{\mathcal{M}_x|x\in X\}$ is homeomorphic to X and hence is connected. Moreover, $\{p,q\}\in\mathcal{M}_x$ for each $x\in X$ and so $\mathcal{M}_x\in\{p,q\}^+\subset U_\epsilon(p)^+\subset U$. Since $\mathcal{M}_p=p$ and $\mathcal{M}_q=q$, the assertion is proved.

EXAMPLE 4.8. Let X be a linearly ordered set with ordering $\leq$ and let $\mathcal{S}$ be the ussual closed subbase for the interval topology of X ; i.e. $\mathcal{S} = \{\{x\in X | x \leq a\} | a\in X\} \cup \{\{x\in X | a \leq x\} | a\in X\}$. Then $\lambda_{\mathcal{S}}X$ is (homeomorphic to) the completion of X by cuts, where the completion is equipped with its interval topology. It follows, in this case, that $\lambda_{\mathcal{S}}X = \beta_{\mathcal{S}}X$. Moreover, if X is the rational numbers, then X is not connected but $\lambda_{\mathcal{S}}X = \mathbb{R} \cup \{\pm\infty\}$ is connected.

<u>Proof</u>. c.f. [4] for the details.

EXAMPLE 4.9. Let $X = \mathbb{N} \cup \{a,b,c\}$ be the T_1-space consisting of a sequence ($\mathbb{N}$) converging to three different points ($\{a,b,c\}$). Then X is compact (compare example 1 in §1) but $\beta_{\mathcal{S}}X \neq X$, where $\mathcal{S}$ is the set of all closed subsets of X. It can be verified that $\beta_{\mathcal{S}}X = X \cup \{\{S\in\mathcal{S}\} \mid |S\cap\{a,b,c\}| \geq 2\}$.

This space is also one of the few compact, non-supercompact spaces we know. Whether there exists compact non-supercompact T_2-spaces is unknown. C.f. Introduction, [1] and [5] .

EXAMPLE 4.10. There exists a space X with a connected subset A and subsets A_1, A_2 of X such that $A^+ \cap A_1^+ \cap A_2^+$ is not connected in λX.

<u>Proof</u>. Let $X = [0,1] \cup \{2,3\}$ with the relative topology from $\mathbb{R}$. Let $A = [0,1]$, $A_1 = \{0,2,3\}$, and $A_2 = \{1,2,3\}$. It is easy to show that $A^+ \cap A_1^+ \cap A_2^+ = \{\mathcal{M}_1, \mathcal{M}_2\}$, where $\mathcal{M}_1$ is the unique m.l.s. generated by $\{\{0,1\}, \{0,2\}, \{1,2\}\}$ and $\mathcal{M}_2$ is the unique m.l.s. generated by $\{\{0,1\}, \{0,3\}, \{1,3\}\}$. Since λX is Hausdorff, $\mathcal{M}_1$ and $\mathcal{M}_2$ are separated in $A^+ \cap A_1^+ \cap A_2^+$.

Let $\mathcal{S}$ be a T_1-subbase for a space X. In [1], de Groot defined the following operation $*$ on subsets of X: if $A \subset X$, define $A^* = \{\mathcal{M} \in \lambda_{\mathcal{S}}X \mid \forall S \in \mathcal{M}, S\cap A \neq \emptyset\}$.

It is not difficult to prove that $*$ satisfies the following properties:

(i) If $A \in \mathcal{S}$, then $A^+ = A^* = \{\mathcal{M} \in \lambda_{\mathcal{S}}X \mid A \in \mathcal{M}\}$.

(ii) For every $A \subset X$, $A^+ \subset A^*$.

(iii) If $A,B \subset X$ with $A \subset B = X$, then $A^+ \cup B^* = X$.

(iv) If $A,B \subset X$ with $A \cap B = \emptyset$, then $A^{+} \cap B^{*} = \emptyset$.

(v) For every $A \subset X$, $\lambda_{\mathcal{S}} X \setminus A^{*} = (X \setminus A)^{+}$.

EXAMPLE 4.11. The operations + and * are different.

<u>Proof</u>. In example 3, let $A = \{\frac{1}{k}, 1 - \frac{1}{k} \mid k \in \mathbb{N}\} \cup \{0,1\}$. Then

$$A^{+} = \underset{A}{\widehat{\ldots\ \cdot\ \ldots}} = A \cup \{\mathcal{P}(m,\infty] \mid m \in (0,1)\} \quad \text{and}$$

$$A^{*} = A \cup (\lambda_{\mathcal{S}} X \setminus X).$$

We conclude this section with a list of unsolved problems.

1. Let $\mathcal{S}$ be a T_1-subbase for a space X. Are the following true?

(i) For every closed subset A of X, A^{+} is closed in $\lambda_{\mathcal{S}} X$.

(ii) For every closed subset A of X, A^{*} is closed in $\lambda_{\mathcal{S}} X$.

(iii) For every open subset A of X, A^{+} is open in $\lambda_{\mathcal{S}} X$.

(iv) For every open subset A of X, A^{*} is open in $\lambda_{\mathcal{S}} X$.

Note that by the properties of * and +, (i) and (iv) are equivalent and (ii) and (iii) are equivalent.

2. If $\mathcal{S}$ is T_1-subbase for X such that $\lambda_{\mathcal{S}} X$ is Hausdorff, then it is not the case that $\mathcal{S}$ must be nicely screening.

3. If X is a compact, connected metric space, then λX is homeomorphic to the Hilbert cube. In particular, if I is the unit interval, then λI is the Hilbert cube.

4. If $A \subset X$, it appears that the formula $(\mathrm{cl}_X A)^{+} \subset \mathrm{cl}_{\lambda X} A^{+}$ plays a role in trying to generalize the theorem on connectedness and locally connectedness. What conditions must be imposed on $\mathcal{S}$ in order to make the formula valid for $\lambda_{\mathcal{S}} X$?

5. Example 2 shows that the normality condition on $\mathcal{S}$ in the extension theorem cannot be weakened to nicely screening and hence not to weak normality. Can it be weakened in another way?

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LIST OF SYMBOLS

$+$	For $A\subset X$ we have $A^{+} = \{\mathcal{M}\in\lambda_{\mathcal{S}}X \mid \exists\, S\in\mathcal{M}\ \ S\subset A\}$	5.
$*$	For $A\subset X$ we have $A^{*} = \{\mathcal{M}\in\lambda_{\mathcal{S}}X \mid \forall\, S\in\mathcal{M}\ \ S\cap A\neq\emptyset\}$	29,30.
$\bar{f}$	The extension of f, as defined in theorem 2.1.	16.
$\underline{\mathcal{A}}$	If $\mathcal{A}\subset\mathcal{S}$, then often $\underline{\mathcal{A}} = \{S\in\mathcal{S} \mid \exists\, A\in\mathcal{A}\ \ A\subset S\}$	16.