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On Square-Compact Cardinals

by

I. Juhász

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Introduction. It is well-known that in general compactness in "higher cardinalities" does not enjoy the productive property (i.e. fails to satisfy the corresponding "Tychonov theorem"). For instance, weakly compact cardinals must be very "rare" even among the (strongly) inaccessible cardinals, where α is weakly compact iff the product of α α-compact spaces, each of weight $\leq \alpha$, is α-compact. (A space is α-compact if from every open covering one can select a subcovering of power $< \alpha$.)

S. Mrówka raised the following question [1]: What can be said about cardinals α for which already the product of two α-compact spaces need not be α-compact? The two main results of this paper give the following answers to the above question: Assuming G C H every accessible α, and assuming the axiom of constructibility (V = L) every non-weakly compact α has the above mentioned property.

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1) Oral communication

Our notation and terminology follow that of [1]

Definition 1. A cardinal α is called square-compact if for any α-compact space X, $X^2 = X \times X$ is also α-compact.

Proposition 1. If X and Y are α-compact spaces and α is square-compact then $X \times Y$ is also α-compact.

Proof. Let $R = X \oplus Y$ be the topological sum of X and Y. Then R is α-compact, hence $R \times R$ is also. However, $X \times Y$ is a closed subset of $R \times R$, and thus is α-compact as well.

Proposition 2. If λ is a singular cardinal then λ is not square-compact.

Proof. Let X be the space whose points are the ordinals $\xi \leqq \lambda$ and in which every $\xi < \lambda$ is isolated, furthermore a basis of neighbourhoods of λ consist of the final segments $[\xi, \lambda]$ with $\xi < \lambda$. Clearly, X is λ-compact.

Let us now consider the open covering $\mathfrak{U}$ of X consisting of the following sets:

1) $X \times [cf(\lambda), \lambda]$;

2) $[\alpha_\xi, \lambda] \times \{\xi\}$ for $\xi < cf(\lambda)$, where the α_ξ's are chosen so that $\alpha_\xi < \lambda$ and $\sum_{\xi < cf(\lambda)} \alpha_\xi = \lambda$;

3) the singletons $\{(\eta, \xi)\}$ with $\xi < cf(\lambda)$ and $\eta < \alpha_\xi$.

Obviously, $\mathfrak{U}$ is a disjoint open covering of X^2 with $|\mathfrak{U}| = \lambda$, hence X^2 is not α-compact, consequently α is not square-compact.

Proposition 3. For any fixed α, let

$$\sum_{\gamma < \alpha} 2^\gamma = 2^{\underset{\smile}{\alpha}} \quad \text{and} \quad \beta = (2^{\underset{\smile}{\alpha}})^+.$$

Then there are two β-compact spaces X_1 and X_2 such that $X_1 \times X_2$ is not 2^α-compact.

Proof. Let C_α be the set of all 0-1 sequences of type α, which are not identically 0 and contain cofinally many 0's, that is, for $(a_\xi)_{\xi<\alpha} \in C_\alpha$ and $\xi < \alpha$ there always is a ξ' with $\xi < \xi' < \alpha$ such that $a_{\xi'} = 0$.

C_α, provided with the lexicographic order, is a densely ordered space, with no first or last element.

Now let X_1 be the space on C_α, for which the half open intervals $[x,y)$ (with $x,y \in C_\alpha$, $x < y$) constitute a basis.

Similarly, X_2 is the space on C_α with the basis consisting of the half open intervals $(x,y]$.

We claim that both X_1 and X_2 are α-compact (in fact, even hereditarily α-compact). By [2], Lemma 1, this follows from the fact that every right-separated subset of X_1 or X_2 is of cardinality $\leq 2^{\alpha} < \beta$. This assertion, however, is proved in [3], Theorem, in a slightly different (but equivalent) formulation.

Now we show that $X_1 \times X_2$ is not 2^{α}-compact. This follows from the fact that the diagonal Δ of this product is a closed discrete subset of power $|C_\alpha| = 2^{\alpha}$. Indeed, Δ is obviously closed in $X_1 \times X_2$ and for $(p,p) \in \Delta$, $[p,-) \times (-,p]$ (product of "half lines") is a neighbourhood of (p,p) in Δ such that

$$\Delta \cap \big([p,-) \times (-,p]\big) = \{(p,p)\} ,$$

hence Δ is discrete.

Theorem 1. Suppose G C H. Then every square-compact α is (strongly) inaccessible.

Proof. If α is accessible, then either α is singular or $\alpha = \beta^+$ for some β. In the former case Prop. 2, implies that α is not square-compact and in the latter case $\beta = 2^{\beta}$ holds by G.C.H., hence $2^{\beta} = \beta^+ = \alpha$ is not square-compact by Prop. 3.

Theorem 2. Suppose V = L. Then every square-compact α is weakly compact.

Proof. Since V = L implies GCH, it suffices to show that every non-weakly compact inaccessible α is not square-compact. For this we employ a result of R.B. Jensen [4], which says that for such an α there is a "Suslin-ordering", i.e. an ordered set L of power α in which every disjoint collection of open intervals is of cardinality less than α. We can assume that L is densely ordered.

Now let X_1 and X_2 be the spaces on L, with a basis consisting of the intervals half open to the right and left respectively.

Next we show that every discrete subset of X_i $(i = 1,2)$, say D, is of power $< \alpha$. Indeed, for each $x \in D$ there is a half open interval I_x, $[x,y_x)$ if $i = 1$ (or $(z_x,x]$ if $i = 2$), such that $I_x \cap D = \{x\}$. Now it is obvious that $\{I_x : x \in D\}$ is disjoint, hence $\{I_x \setminus \{x\} : x \in D\}$ is a disjoint collection of open intervals, which implies $|D| < \alpha$. (Here we used that $I_x \setminus \{x\} \neq \emptyset$, because L is densely ordered.)

Applying Theorem 1 of [5] we obtain that X_1 and X_2 are α-compact (in fact even hereditarily α-compact).

However if we take the product $X_1 \times X_2$, it is not α-compact, since its diagonal Δ can be shown to be a closed discrete subset, similarly as in the proof of Prop. 3.

References

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[4] R.B. Jensen, Suslin's hypothesis = weak compactness in L, Notices of the Ann. Math. Soc., 16 (1969), p. 842.

[5] A. Hajnal and I. Juhász, Discrete subspaces of topological spaces, II, Indag. Math. 31 (1969), 18-30.