stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZW 17/73 DECEMBER

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SCHNEIDER'S METHOD IN FIELDS OF CHARACTERISTIC $p \neq 0$

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

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AMS (MOS) subject classification scheme (1970): 10F35

Schneider's method in fields of characteristic $p \neq 0$

by

J.M. Geijsel

Abstract

In this report two theorems on transcendental elements of fields of characteristic $p > 0$, namely L.I. Wade's result on the analogue of the Gelfond-Schneider theorem for fields of characteristic p (see Duke Math. J. 13 (1946), 79-85), and my result on the trancendency of certain values of the Carlitz-Bessel functions (see Math. Centre Report ZW 2/71, Amsterdam) are generalized for a wider class of so called E-functions.

Let $\mathbb{F}_q$ be a finite field of characteristic $p \neq 0$ with $p = q^{n_0}$ elements. We denote by $\mathbb{F}_q[x]$ the ring of polynomials with coefficients in $\mathbb{F}_q$ and by $\mathbb{F}_q\{x\}$ its quotientfield.

For $0 \neq E \in \mathbb{F}_q[x]$ we define the (logarithmic) valuation

$$\mathrm{dg}\, E = \text{degree of } E \text{ and } \mathrm{dg}\, 0 = -\infty.$$

For $Q \in \mathbb{F}_q\{x\}$ where $Q = \frac{E}{F}$ with $E,F \in \mathbb{F}_q[x]$ and $F \neq 0$ we define

$$\mathrm{dg}\, Q = \mathrm{dg}\, E - \mathrm{dg}\, F.$$

The completion of $\mathbb{F}_q\{x\}$ with respect to the valuation is denoted by $\mathcal{F}$ and the completion of the algebraic closure of $\mathcal{F}$ by Φ. The valuation dg on $\mathbb{F}_q\{x\}$ can be extended to Φ in a unique way and also will be denoted by dg.

A function $f : \Phi \to \Phi$ given by a power series

$$f(t) = \sum_{i=0}^{\infty} a_i\, t^i \quad \text{with } a_i \in \Phi,$$

which converges for all t with $\mathrm{dg}\, t < R$ is called *linear* if

$$\begin{cases} f(t+u) = f(t) + f(u) & \forall t,u \in \Phi \text{ with } \mathrm{dg}\, t < R,\ \mathrm{dg}\, u < R, \\ f(ct) = cf(t) & \forall t \in \Phi \text{ with } \mathrm{dg}\, t < R \text{ and } c \in \mathbb{F}_q. \end{cases}$$

For linear functions we define for all t for which the involving series converge the operators Δ^r $(r=1,2,\ldots)$ by

$$\Delta f(t) = f(xt) - x\, f(t),$$
$$\Delta^r f(t) = \Delta^{r-1} f(xt) - x^{q^{r-1}}\, \Delta^{r-1} f(t), \qquad r \geq 2.$$

For purpose of notation we define $\Delta^0 f(t) = f(t)$.

A function $f : \Phi \to \Phi$ is said to be *entire* if f can be written as a power series with coefficients in Φ, which converges for all $t \in \Phi$.

For entire linear functions f we have an "expansion formula" (see [1], or [2] lemma 2.1), namely :

for every $M \in \mathbb{F}_q[x]$ we have

$$f(Mt) = \sum_{\nu=0}^{\mathrm{dg}\, M} \frac{\psi_\nu(M)}{F_\nu} \Delta^\nu f(t),$$

where

$$F_\nu := (x^{q^\nu}-x)(x^{q^\nu}-x^q) \ldots (x^{q^\nu}-x^{q^{\nu-1}}), \quad \nu \geq 1$$

$$F_0 := 1$$

$$\psi_\nu(t) := \prod_{\substack{\mathrm{dg}\, E<\nu \\ E\in\mathbb{F}_q[x]}} (t-E).$$

Now we introduce a special class of linear functions.

Definition. A linear function $f : \Phi \to \Phi$ given by

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^{q^k}}{F_k}$$

is called an *E-function* if there exists a finite separable algebraic extension K of F of degree h such that:

(1) $a_k \in K, \quad k = 0,1,\ldots$

(2) $\exists c \in \mathbb{R},\ c > 0$ such that $\mathrm{dg}\, a_k < cq^k$

(3) $\forall k \in \mathbb{N} \cup \{0\}$ $\exists Q_k \in \mathbb{F}_q[x]$ of minimal degree such that $Q_k a_0, Q_k a_1, \ldots, Q_k a_k$ are integers in K and $\mathrm{dg}\, Q_k = O(kq^k),\ k \to \infty$.

Remarks

(i) From (1) and (2) we have that every E-function is entire.

(ii) The functions $\psi(t)$ and $J_n(t)$ (see [3]) are E-functions.

(iii) Linear polynomials with separable algebraic coefficients in Φ are E-functions.

(iv) If f and g are E-functions then $\Delta^r f$ $(r\geq 1)$, f^{q^r}, $f + g$ are E-functions.

(v) If P is a linear polynomial with separable algebraic coefficients in Φ and f is an E-function, then $P(f(t))$ is an E-function.

Lemma 1. Let K be a separable finite algebraic extension of $\mathbb{F}\{x\}$ of degree h. Let r , s $\in \mathbb{N}$ with $0 < r < s$. Then the system of linear equations

$$\sum_{i=1}^{s} \alpha_{ki} X_i = 0 \qquad (k=1,\ldots,r),$$

where α_{ki} are algebraic integers in K and

$$a = \max_{k,i} (\mathrm{dg}\ \alpha_{ki}, 0)$$

has a non-trivial solution $\{X_i\}_{i=1}^{s}$ with

$$X_i \in \mathbb{F}_q[x]$$

such that

$$\mathrm{dg}\ X_i < \frac{cs + ar}{s - r} \qquad (i=1,\ldots,s),$$

where c is a positive constant only depending on the field K.

Proof. We use the following lemma which will be proved in an appendix.

Lemma: Let K be a separable finite algebraic extension of $\mathbb{F}_q\{x\}$ of degree h. Then there exists a basis $\beta_1 , \ldots , \beta_h$ of algebraic integers of K such that every algebraic integer $\xi \in K$ can be written uniquely as

$$\xi = \sum_{i=1}^{h} A_i \beta_i \quad \text{with} \quad A_i \in \mathbb{F}_q[x].$$

Further we use the methods of lemma 4.2 in [3]. □

Now we can formulate the main result of this paper.

Theorem 1. Let $f_1(t) , \ldots , f_n(t)$ be $\mathcal{E}$-functions, not all polynomials. Suppose

$$\Delta^r f_\nu(t) = R_{\nu r}(f_1(t),\ldots,f_n(t)) \quad , \quad r = 0,1,\ldots; \ \nu = 1,\ldots,n,$$

where $R_{\nu r}$ are n-linear polynomials of n variables $f_1 , \ldots , f_n$ of total degree $<q^r$ with coefficients in $\mathbb{F}_q[x]$ of degree $<q^r$.

Let $\alpha \neq 0$, $\beta \notin \mathbb{F}_q\{x\}$ and $f_\nu(t) \neq 0$. Then at least one of the elements

$$\{\beta , f_1(\alpha) , \dots , f_n(\alpha) , f_1(\alpha\beta) , \dots , f_n(\alpha\beta)\}$$

is transcendental over $\mathbb{F}_q\{x\}$.

Corollary 1.

a) With the choice $f_1(t) = \psi(t)$, $\alpha = \lambda(\alpha^*)$, where α^* is not a zero of $\lambda(t)$, and $\beta \notin \mathbb{F}_q\{x\}$ we get the analogue of the theorem of Gelfond-Schneider: at least one of the elements $\{\beta , \alpha^* = \psi(\lambda(\alpha^*)) , \psi(\beta\lambda(\alpha^*))\}$ is transcendental over $\mathbb{F}_q\{x\}$. This result was proved by Wade in [5].

b) With $f_1(t) = J_n(t)$, $f_2(t) = \Delta J_n(t)$ and $\alpha \neq 0$, $\beta \notin \mathbb{F}_q\{x\}$ we get: at least one of the elements

$$\{\beta , J_n(\alpha) , \Delta J_n(\alpha) , J_n(\alpha\beta) , \Delta J_n(\alpha\beta)\}$$

is transcendental over $\mathbb{F}_q\{x\}$. This result was essentially proved in [3], where the theorem said under the same conditions for α and β: at least one element of the set $V = \{\alpha , \beta , J_n(\alpha) , \Delta J_n(\alpha) , J_n(\alpha\beta) , \Delta J_n(\alpha\beta)\}$ is transcendental over $\mathbb{F}_q\{x\}$. At the begin of the proof in [3] we supposed α to be algebraic over $\mathbb{F}_q\{x\}$ but we didn't use this fact, hence α can be omitted in V.

Corollary 2. If we choose $f_1(t) = \psi(\alpha_1^* t),\dots,f_n(t) = \psi(\alpha_n^* t)$, where $\alpha_\nu^* \neq 0$, $\nu=1,\dots,n$, and if $\alpha = 1$, $\beta \notin \mathbb{F}_q\{x\}$ then at least one of the elements $\{\beta, \psi(\alpha_1^*),\dots,\psi(\alpha_n^*) , \psi(\alpha_1^*\beta),\dots,\psi(\alpha_n^*\beta)\}$ is transcendental over $\mathbb{F}_q\{x\}$.

If now α_i is not a zero of $\lambda(t)$, $i=1,\dots,n$, and $\alpha_i^* := \lambda(\alpha_i)$, then at least one of the elements of the set $\{\beta,\alpha_1,\dots,\alpha_n,\psi(\beta\lambda(\alpha_1)),\dots \dots,\psi(\beta\lambda(\alpha_n))\}$ is transcendental over $\mathbb{F}_q\{x\}$. For $n = 1$ we have the result of corollary 1a.

For $\beta \notin \mathbb{F}_q\{x\}$, α algebraic and $\lambda(\alpha) \neq 0$ we have: at least one of the elements $\{\beta, \psi(\beta\lambda(\alpha)),\psi(\beta\lambda(\alpha^q),\dots , \psi(\beta\lambda(\alpha^{q^n}))\}$, $n \geq 1$, is transcendental over $F_q\{x\}$. Equivalent with this last result is: at least one of the elements $\{\beta, \psi(\lambda(\alpha)), \psi(\beta\lambda(x\alpha)), \ldots, \psi(\beta\lambda(x^n\alpha))\}$, $n \geq 1$, is transcendental over $\mathbb{F}_q\{x\}$, since $\Delta\lambda(t) = \lambda(xt) - x\lambda(t) = \lambda(t^q)$.

Proof of theorem 1. Suppose $\beta, f_1(\alpha),\ldots,f_n(\alpha),f_1(\alpha\beta),\ldots,f_n(\alpha\beta)$ are algebraic over $\mathbb{F}_q\{x\}$, then, for some $e \in \mathbb{N}$, β^{q^e}, $f_1^{q^e}(\alpha),\ldots, f_n^{q^e}(\alpha)$, $f_1^{q^e}(\alpha\beta),\ldots, f_n^{q^e}(\alpha\beta)$ are separable over $\mathbb{F}_q\{x\}$ and they generate a separable extension K of $\mathbb{F}_q\{x\}$ of degree h.

Let $\Gamma \in \mathbb{F}_q[x]$ be such that $\Gamma\beta^{q^e}$, $\Gamma f_\nu^{q^e}(\alpha)$, $\Gamma f_\nu^{q^e}(\alpha\beta)$, $\nu=1,\ldots,n$, are algebraic integers of K.

The natural numbers k, ℓ with $k < \frac{1}{3}\ell$ will be chosen later.

Define

$$L(t) := \sum_{\nu=1}^{n} \sum_{j=0}^{q^{2\ell}-1} \sum_{i=0}^{q^{2k}-1} X_{ij\nu}\, t^{jq^e} f_\nu^{iq^e}(\alpha t),$$

where the polynomials $X_{ij\nu}$ will be determined by the following:

$$L(A+\beta B) = 0 \quad \text{for all } A,B \in \mathbb{F}_q[x] \text{ with } \operatorname{dg} A < m,\ \operatorname{dg} B < m,$$

$$\text{where } m := k + \ell - 1.$$

Since $\beta \notin \mathbb{F}_q\{x\}$ we get a linear system of at most q^{2m} equations in $nq^{2k+2\ell}$ variables $X_{ij\nu}$ with algebraic coefficients:

$$(1)\quad L(A+\beta B) = \sum_{\nu=1}^{n} \sum_{j=0}^{q^{2\ell}-1} \sum_{i=0}^{q^{2k}-1} X_{ij\nu}\, (A+\beta B)^{jq^e} f_\nu^{iq^e}(\alpha(A+\beta B)) = 0,$$

$$\operatorname{dg} A < m,\ \operatorname{dg} B < m.$$

$f_i(\alpha A+\alpha\beta B) = f_i(\alpha A) + f_i(\alpha\beta B)$ since f_i is linear. Since

$$\Delta^\mu f_i(t) = R_{i\mu}(f_1(t),\ldots,f_n(t)) =$$

$$=: \sum_{0\leq j_1+\ldots+j_n\leq\mu} A_{i\mu j_1\ldots j_n} f_1^{q^{j_1}}(t)\ldots f_n^{q^{j_n}}(t)$$

with $A_{i\mu j_1 \dots j_n} \in \mathbb{F}_q[x]$ and dg $A_{i\mu j_1 \dots j_n} < q^\mu$, the expansion formula gives:

$$f_i(\alpha A) = \sum_{\mu=0}^{\mathrm{dg}\,A} \frac{\psi_\mu(A)}{F_\mu} \sum_{j_1+\dots+j_n \le \mu} A_{i\mu j_1 \dots j_n} f_1^{q^{j_1}}(\alpha) \dots f_n^{q^{j_n}}(\alpha).$$

Since $\mathrm{dg}\frac{\psi_\mu(A)}{F_\mu} \le \max_{0\le\mu\le \mathrm{dg}\,A} \mathrm{dg}\,\frac{\psi_\mu(A)}{F_\mu} = \max_{0\le\mu\le \mathrm{dg}\,A} (q^\mu \mathrm{dg}\,A - \mu q^\mu) \le \mathrm{dg}\,A \,.\, q^{\mathrm{dg}\,A}$,

$$\mathrm{dg}\,f_i(\alpha A) \le mq^m + q^m + q^m \max\{\mathrm{dg}\,f_1(\alpha),\dots,\mathrm{dg}\,f_n(\alpha),0\}.$$

Since $f_i^{q^e}(\alpha A)$ resp. $f_i^{q^e}(\alpha\beta A)$ is a polynomial in $f_\nu^{q^e}(\alpha)$ resp. $f_\nu^{q^e}(\alpha\beta)$, $i,\nu \in (1,\dots,n)$, of degree $\le q^m$ with coefficients in $\mathbb{F}_q\{x\}$, the coefficients of $X_{ij\nu}$ in (1) are polynomials in

$$\beta^{q^e} \text{ of degree} \le q^{2\ell} - 1$$

$$f_\nu^{q^e}(\alpha)\ ,\ f_\nu^{q^e}(\alpha\beta) \quad \text{of degree} \le (q^{2k}-1)q^m$$

with coefficients in $\mathbb{F}_q\{x\}$.

Since $q^{2\ell} - 1 + 2n(q^{2k}-1)q^m < q^{2\ell+2n}$ we can get a system of equations with integral algebraic coefficients in K by multiplying each equation with the factor

$$\Gamma^{q^{2\ell+2n}} (F_m^{q^e})^{q^{2k}-1}.$$

This gives the system of equations:

$$\Gamma^{q^{2\ell+2n}} (F_m^{q^e})^{q^{2k}-1} L(A+\beta B) = 0 \qquad \text{for dg}\,A, \mathrm{dg}\,B < m;\ A,B \in \mathbb{F}_q[x]$$

which we denote by

$$(2) \quad \sum_{\nu=1}^{n} \sum_{j=0}^{q^{2\ell}-1} \sum_{i=0}^{q^{2k}-1} X_{ij\nu} D_{ij\nu} = 0 \qquad \text{for } A,B \in \mathbb{F}_q[x];\ \mathrm{dg}\,A, \mathrm{dg}\,B < m.$$

Since $m = k + \ell - 1$ the number of equations, q^{2m}, is less than the number of variables $n\,q^{2k+2\ell}$. Furthermore

$$\mathrm{dg}\ D_{ij} \leq q^{2\ell+2n}\ \mathrm{dg}\ \Gamma + mq^{m+2k+e} + q^{2\ell+e}(m+c_1) + q^{2k+e}(mq^m+q^m+q^m c_0)$$

where $c_1 = \max(\mathrm{dg}\beta, 0)$; $c_0 = \max(\mathrm{dg}\ f_\nu(\alpha);\ \mathrm{dg}\ f_\nu(\alpha\beta),\ (\nu=1,\ldots,n);\ 0)$, which gives (since $k < \frac{1}{3}\ell$):

$$\mathrm{dg}\ D_{ij\nu} \leq (3m+c_2)q^{2\ell+e} \qquad \text{where } c_2 \geq 0.$$

According to lemma 1 with $r = q^{2m}$, $s = nq^{2k+2\ell}$ and $a = \max\limits_{i,j,\nu}(\mathrm{dg}\ D_{ij\nu}, 0)$ we have that there exist polynomials $X_{ij\nu} \in \mathbb{F}_q[x]$, not all zero, such that (1) is satisfied and

(3) $$\mathrm{dg}\ X_{ij\nu} \leq (3m+c_3)q^{2\ell+e} \qquad \text{where } c_3 \geq 0.$$

Now we shall prove that, for all $A, B \in \mathbb{F}_q[x]$, $L(A+\beta B) = 0$. Let $\mu \geq m$ and $\eta = \mu - k + 1$, then $\eta \geq \ell$. Furthermore let

$$\mathcal{B}(\mu) = \{A + \beta B \mid \mathrm{dg}\ A < \mu,\ \mathrm{dg}\ B < \mu,\ A \text{ and } B \text{ not both } 0\}.$$

Suppose $L(t) = 0$ for all $t \in \mathcal{B}(\mu)$. Let $\xi \in \mathcal{B}(\mu+1) \setminus \mathcal{B}(\mu)$, then $\mathrm{dg}\ \xi = \mu + g$ with $g \geq 0$. We choose ℓ such that $m + g < 2m$. By assumption

$$L(t) \Big/ \prod_{\mathcal{B}(\mu)} (t-A-\beta B)$$

is an entire function since $L(t)$ is entire. According to the maximum-modulus-principle

$$\mathrm{dg}\left(\frac{L(\xi)}{\prod_{\mathcal{B}(\mu)}(\xi-A-\beta B)}\right) \leq \max_{\mathrm{dg}\, t = 2\mu} \mathrm{dg}\left(\frac{L(t)}{\prod_{\mathcal{B}(\mu)}(t-A-\beta B)}\right).$$

Hence

$$\mathrm{dg}\ L(\xi) \leq \max_{\mathrm{dg}\, t = 2\mu} \mathrm{dg}\ L(t) - 2\mu(q^{2\mu}-1) + (\mu+g)(q^{2\mu}-1).$$

From the definition of $L(t)$ we get

$$\max_{\mathrm{dg}\, t = 2\mu} \mathrm{dg}\ L(t) \leq \max_{i,j,\nu} \mathrm{dg}\ X_{ij\nu} + 2\mu q^{2\ell+e} + q^{2k+e} \max_{\mathrm{dg}\, t = 2\mu} \mathrm{dg}\ f_\nu(\alpha t).$$

Since f_ν is an E-function we have

$$f_\nu(t) = \sum_{k=0}^{\infty} a_{\nu k} \frac{t^{q^k}}{F_k}, \qquad \nu = 1,\dots,n,$$

where $\exists c > 0$ such that $\deg a_{\nu k} < cq^k$ for $k > k_0$ and $\nu = 1,\dots,n$. Hence

$$\max_{\deg t = 2\mu} \deg f_\nu(\alpha t) \le \max_{k\ge 0} (\deg a_{\nu k} + 2\mu q^k - kq^k + q^k \deg \alpha) <$$

$$< \max_{k\ge 0} (c+2\mu-k+\deg \alpha)q^k \le c_4\, q^{2\mu}$$

where $c_4 > 0$ and $\mu \ge m$. This gives

$$\deg L(\xi) \le (3m+c_3)q^{2\ell+e} + 2\mu q^{2\ell+e} + c_4 q^{2\mu+2k+e} - (\mu-g)q^{2\mu}.$$

Since $\mu \ge m$, $\eta \ge \ell$ and $\mu = \eta + k - 1$ we get

(4) $\quad \deg L(\xi) \le q^{2\eta+e} (5\mu + c_5 q^{4k} - (\mu-g)q^{2k-e}).$

$L(\xi)$ is a polynomial in β^{q^e} of degree $q^{2\ell}-1$ and in each of the $f_\nu^{q^e}(\alpha)$, $f_\nu^{q^e}(\alpha\beta)$ of degree $(q^{2k}-1)q^\mu$, hence $L(\xi)$ is algebraic; since

$$q^{2\ell} + 2nq^{2k+\mu} < q^{2\eta+2n},$$

$F_\mu^{q^{2k+e}} \; \Gamma^{q^{2\eta+2n}} \; L(\xi)$ is an algebraic integer of K. Therefore

$$N(F_\mu^{q^{2k+e}} \; \Gamma^{q^{2\eta+2n}} \; L(\xi)) \in \mathbb{F}_q[x]$$

and

$$\deg N(F_\mu^{q^{2k+e}} \; \Gamma^{q^{2\eta+2e}} \; L(\xi)) \le$$

$$\le h[\mu q^{2k+e+\mu} + q^{2\eta+2n} \deg \Gamma + q^{2\eta+e} (5\mu + c_5 q^{4k} - (\mu-g)q^{2k-e})] \le$$

$$\le h\, q^{2\eta+e} [(6\mu + c_6 q^{4k}) - (\mu-g)q^{2k-e}],$$

which is negative for sufficiently large k and ℓ. Now choose k and ℓ such that $\mathrm{dg}\, N(F_\mu^{q^{2k+e}}\, \Gamma^{q^{2\eta+2e}}\, L(\xi))$ is negative. Hence $L(\xi) = 0$.

In the following k and ℓ are fixed. Since $\beta \notin \mathbb{F}_q\{x\}$ all $A + \beta B$ are different and L(t) has an infinite number of zero's. Since L(t) is entire and not a polynomial L(t) is a transcendental function (see [4] or [2]).

Let N be the set of zero's $\neq 0$ of L(t), then N is countable and for any $\nu \in \mathbb{N}$

$$L(t) = \gamma_0\, t^\rho \prod_{\xi\in\mathcal{B}(\nu)} \left(1 - \frac{t}{\xi}\right) \prod_{\substack{\xi\notin\mathcal{B}(\nu)\\ \xi\in N}} \left(1 - \frac{t}{\xi}\right) \quad \text{with } \rho \geq 0 \text{ and } \gamma_0 \in \Phi.$$

Let ν_0 be the minimum of the degrees of the zero's $\neq 0$ of L(t), then

$$\max_{\mathrm{dg}\,t = 2\nu} \mathrm{dg} \prod_{\xi\in N\setminus\mathcal{B}(\nu)} \left(1 - \frac{t}{\xi}\right) \geq \max_{\mathrm{dg}\,t = \frac{\nu_0}{2}} \mathrm{dg} \prod_{\xi\in N\setminus\mathcal{B}(\nu)} \left(1 - \frac{t}{\xi}\right) = 0.$$

Furthermore

$$\prod_{\xi\in\mathcal{B}(\nu)} \left(1 - \frac{t}{\xi}\right) = \frac{\prod_{\mathcal{B}(\nu)} (A+\beta B-t)}{\prod_{\mathcal{B}(\nu)} (A+\beta B)},$$

hence

$$\max_{\mathrm{dg}\,t = 2\nu} \mathrm{dg}\, L(t) \geq c_7 + 2\nu\rho + 2\nu(q^{2\nu}-1) - (\nu+g)(q^{2\nu}-1),$$

where c_7 is a constant only depending on L(t). This gives

$$(5) \qquad \max_{\mathrm{dg}\,t = 2\nu} \mathrm{dg}\, L(t) \geq (c_8\nu + c_9)q^{2\nu} \quad \text{with } c_8 > 0.$$

On the other hand we have proved

$$(6) \qquad \max_{\mathrm{dg}\,t = 2\nu} \mathrm{dg}\, L(t) \leq (3m+c_3)q^{2\ell+e} + 2\nu q^{2\ell+e} + c_4 q^{2\nu+2k+e}.$$

For ν large enough (5) and (6) are contradictory, which completes the proof of the theorem. □

Remark. The theorem is also true for systems $\{f_1,\dots,f_n\}$ for which the following relation is true:

$$\Delta^2 f_\nu(t) = R_{\nu r}(f_1(t),\dots,f_n(t)), \quad r=0,1,\dots;\ \nu=1,2,\dots,n,$$

where

$$R_{\nu r}(f_1(t),\dots,f_n(t)) = \sum_{j_1+\dots+j_n\le r} Q_{\nu r j_1\dots j_r} f_1^{q^{j_1}}(t)\dots f_n^{q^{j_n}}(t)$$

with $Q_{\nu r j_1\dots j_r} \in \mathbb{F}_q\{x\}$, such that for all $r \ge 0$ there exists a polynomial A_r such that $Q_{\nu\rho j_1\dots j_r} A_r \in \mathbb{F}_q[x]$ for $\rho=0,1,\dots,r$; $1 \le \nu \le n$; $j_1+\dots+j_n \le r$ and $\mathrm{dg}\, A_r < q^r$.

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APPENDIX

Lemma. Let K be a separable finite algebraic extension of $\mathbb{F}_q\{x\}$ of degree h. Then there exists a basis $\beta_1, \ldots, \beta_h$ of algebraic integers of K such that every algebraic integer $\xi \in K$ can be written uniquely as

$$\xi = \sum_{i=1}^{h} A_i \beta_i \quad \text{with } A_i \in \mathbb{F}_q[x].$$

Proof. According to the theorem of the primitive element [*], since K is a separable finite extension of $\mathbb{F}_q\{x\}$ of degree h there exists an element $\theta \in K$ such that $K = \mathbb{F}_q\{x\}(\theta)$. θ is a separable algebraic element of K, hence there is a polynomial P in $\mathbb{F}_q[x]$ such that $P\theta$ is an algebraic integer of K. Denote $P\theta$ again by θ. Let $\theta_1 = \theta, \theta_2, \ldots, \theta_h$ be the conjugate elements of the algebraic integer θ. The discriminant $\Delta(1,\theta,\ldots,\theta^{h-1})$ of the basis $1,\theta,\ldots,\theta^{h-1}$ of $K / \mathbb{F}_q\{x\}$ is a Van der Monde determinant and since θ is separable, $\theta_i \neq \theta_j (i \neq j)$; hence $\Delta(1,\theta,\ldots,\theta^{h-1}) \neq 0$. Furthermore is

$$\Delta(1,\theta,\ldots,\theta^{h-1}) = \prod_{1 \le i < j \le h} (\theta_i - \theta_j)^2$$

a symmetric polynomial in the conjugate elements of θ and can be expressed as a polynomial in the coefficients of the minimal polynomial of θ; hence $\Delta(1,\theta,\ldots,\theta^{h-1}) \in \mathbb{F}_q[x]$.

For every base $\{w_1, \ldots w_h\}$ of $K / \mathbb{F}_q\{x\}$ with w_i algebraic integer in K we have

$$\Delta(w_1,\ldots,w_h) = (\det(a_{ij}))^2 \cdot \Delta(1,\theta,\ldots,\theta^{h-1})$$

where $w_i = a_{i1} + a_{i2}\theta + \ldots + a_{ih}\theta^{h-1}$ $(i=1,\ldots,h)$ with $a_{ij} \in \mathbb{F}_q\{x\}$, and $\det a_{ij} \neq 0$. On the other hand as a symmetric polynomial in the algebraic integers $w_1, \ldots, w_h$ and its conjugates $\Delta(w_1,\ldots,w_h) \in \mathbb{F}_q[x]$. Consider all bases $\{w_1,\ldots,w_h\}$ for $K / \mathbb{F}_q\{x\}$ with algebraic integers $w_1, \ldots, w_h$. Then $\mathrm{dg}\, \Delta(w_1,\ldots,w_h) \in \mathbb{N} + \{0\}$, hence there exists a basis $\{\beta_1,\ldots,\beta_h\}$ with $\mathrm{dg}\, \Delta(\beta_1,\ldots,\beta_h)$ minimal and $\beta_1, \ldots, \beta_h$ algebraic integers. We shall prove that this basis $\{\beta_1,\ldots,\beta_h\}$ is a basis for the ring of algebraic integers in K over $\mathbb{F}_q[x]$.

Suppose $\{\beta_1,\dots,\beta_h\}$ is not a basis for the integers in K over $\mathbb{F}_q[x]$, then there exists an algebraic integer $\xi \in K$ such that $\xi = a_1\beta_1+\dots+a_h\beta_h$ with $a_i \in \mathbb{F}_q\{x\}$ and not all $a_i \in \mathbb{F}_q[x]$. Suppose $a_1 \notin \mathbb{F}_q[x]$. $a_1 = A + r$ with $A \in \mathbb{F}_q[x]$ and $r \in \mathbb{F}_q\{x\} \setminus \mathbb{F}_q[x]$, dg $r < 0$ and $r \neq 0$. Now define

$$\beta_1^* = \xi - A\beta_1 = (a_1-a)\beta_1 + a_2\beta_2 + \dots + a_h\beta_h$$
$$\beta_i^* = \beta_i , \qquad i=2,\dots,h.$$

The system $\{\beta_1^*,\dots,\beta_h^*\}$ is a basis for $K / \mathbb{F}_q\{x\}$ and β_i^* $(i=1,\dots,h)$ are algebraic integers.

$$\Delta(\beta_1^*,\dots,\beta_h^*) = \det \begin{pmatrix} a_1-a & a_2 & a_3 & \dots & a_h \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & & 1 \end{pmatrix}^2 \Delta(\beta_1,\dots,\beta_h)$$
$$= r^2 \,\Delta(\beta_1,\dots,\beta_h).$$

$\mathrm{dg}\, \Delta(\beta_1^*,\dots,\beta_h^*) = 2\, \mathrm{dg}\, r + \mathrm{dg}\, \Delta(\beta_1,\dots,\beta_h) < \mathrm{dg}\, (\beta_1,\dots,\beta_h)$.
This contradicts the minimality of dg $\Delta(\beta_1,\dots,\beta_h)$ and proves the lemma. □

[*] B.L. van der Waerden, *Algebra* I, §43.