Commentationes Mathematicae Universitatis Carolinae

4, 1 (1963)

ON THE PRODUCT AND SUM OF A SYSTEM OF TRANSFORMATION SEMIGROUPS

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1. Introduction

If X is a set, then $(F;X)$ will denote a semigroup F of transformations of X into itself. Now if a system of transformation semigroups is given, $\{(F_\alpha;X_\alpha): \alpha \in A\}$, there are several ways to construct from these a transformation semigroup F operating on the set $X = \bigcup_{\alpha \in A} X_\alpha$. We will consider two methods; as they give us essentially the direct product and the direct sum in the case that the X_α are pairwise disjunct, we call the transformation semigroups $(F;X)$, constructed from the $(F_\alpha;X_\alpha)$ by the methods considered, the product and the sum of the transformation semigroups $(F_\alpha;X_\alpha)$.

We are mainly interested in the situation when the new transformation group $(F;X)$ turns out to be commutative. In the case of the product, it is sufficient to assume that all factors $(F_\alpha;X_\alpha)$ are commutative; in the case of the sum, another condition is needed.

In the last section, the theory is applied to obtain an embedding of a given commutative transformation semigroup $(F;X)$ into a commutative transformation semigroup $(G;X)$ that leaves the same subsets of X invariant as F does, and that is maximal in this respect. The semigroup $(G;X)$

turns out to be uniquely determined. Then the previous results are applied to generalise a theorem on the existence of a common fixed point of a commutative system of mappings. And finally we use them to prove that every commutative semigroup is contained in a product of algebraically generated transformation semigroups.

2. Notation

If X is a non-void set, the class of all mappings $f : X \to X$ will be denoted by X^X. This is a semigroup under functional composition o :

$$(f \circ g)(x) = f(g(x))$$

for all $f, g \in X^X$ and all $x \in X$.

If F is a subsemigroup of X^X, we will often write $(F;X)$, to indicate the set transformed by the elements of F.

A system $F \subset X^X$ is called <u>commutative</u> if $f \circ g = g \circ f$ for all $f, g \in F$.

A subset Y of X is said to be <u>invariant</u> under $F \subset X^X$ if $F(Y) \subset Y$. Here $F(Y) = \{f(y) : f \in F$ and $y \in Y\}$. If $f \in X^X$ and $A \subset X$, then $f \mid A$ denotes the restriction of f to A. If $F \subset X^X$ and $A \subset X$, then $F \mid A = \{f \mid A : f \in F\}$.

If $F \subset X^X$ and $x \in X$, then $F(x)$ is called the <u>orbit</u> of x under F; every orbit is an invariant set.

Let $\mathcal{J}$ be a family of subsets of a set X. A system $F \subset X^X$ is said to be $\mathcal{J}$ <u>-invariant</u> if every member of $\mathcal{J}$ is an invariant set under F. The system F is called a <u>maximal commutative</u> $\mathcal{J}$ <u>-invariant system</u> if it is commutative and $\mathcal{J}$ -invariant, and if there is no commutative $\mathcal{J}$ -invariant system $G \subset X^X$ such that $F \subset G$, $F \neq G$. The system F is called a <u>maximal commutative</u> system if it is a maximal commutative $\{\emptyset\}$ - invariant system. Here $\emptyset$ denotes the empty set.

A maximal commutative $\mathcal{J}$-invariant system is always a commutative semigroup containing the identity mapping $i : X \to X$.

The cartesian product of sets F_α, $\alpha \in A$, is denoted by $\prod_{\alpha \in A} F_\alpha$. If $f \in \prod_{\alpha \in A} F$, then f_α denotes the component of f in F_α, and we will also write $(f_\alpha)_{\alpha \in A}$ instead of f.

3. The product of a system of transformation semigroups

In this section and in the next one we consider a family $\{(F_\alpha;X_\alpha) : \alpha \in A\}$ of transformation semigroups: A is a non-void set of indices, and $F_\alpha \subset X_\alpha^{X_\alpha}$ for each $\alpha \in A$. The identity map of X_α onto itself will be denoted by i_α; it is assumed that $i_\alpha \in F_\alpha$ for each $\alpha \in A$. The union of all sets X_α will be denoted by X:

(3.1)
$$X = \bigcup_{\alpha \in A} X_\alpha ,$$

and the identity map of X onto itself will be denoted by i.

<u>Proposition 1</u>. Let S be the following subset of $\prod_{\alpha \in A} F_\alpha$:

(3.2)
$$S = \{(f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} F_\alpha : (\forall \alpha, \beta \in A)\ (f_\alpha | X_\alpha \cap X_\beta = f_\beta | X_\alpha \cap X_\beta)$$

Furthermore, let $F \subset X^X$ be defined in the following manner:

(3.3)
$$F = \{f \in X^X : (\exists s \in S)\ (\forall \alpha \in A)\ (f | X_\alpha = s_\alpha)\} .$$

Then F is a semigroup of transformations of X into itself, containing the identity map i. If F_α is commutative for every $\alpha \in A$, then F is also commutative.

Proof.

First we show the following: if $s = (s_\alpha)_{\alpha \in A} \in S$ and $t = (t_\alpha)_{\alpha \in A} \in S$, then also $(s_\alpha \circ t_\alpha)_{\alpha \in A}$ S.

As the F_α are semigroups, it is clear that $(s_\alpha \circ t_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} F_\alpha$. Now take $\alpha, \beta \in A$; we must show that

(3.4) $$s_\alpha \circ t_\alpha \mid X_\alpha \cap X_\beta = s_\beta \circ t_\beta \mid X_\alpha \cap X_\beta .$$

But we know that

(3.5) $$s_\alpha \mid X_\alpha \cap X_\beta = s_\beta \mid X_\alpha \cap X_\beta ,$$

(3.6) $$t_\alpha \mid X_\alpha \cap X_\beta = t_\beta \mid X_\alpha \cap X_\beta ,$$

as $s, t \in S$; this implies that $X_\alpha \cap X_\beta$ is invariant under s_α, s_β, t_α and t_β. The assertion (3.4) now follows from (3.5) and (3.6).

We now can prove that F is a semigroup. It is evident that F is non-void, as $(i_\alpha)_{\alpha \in A} \in S$, and hence $i \in F$. Take $f, g \in F$. There exist $s, t \in S$ such that for every $\alpha \in A$

(3.7) $$f \mid X_\alpha = s_\alpha , \qquad g \mid X_\alpha = t_\alpha .$$

It follows that $f(X_\alpha) \subset X_\alpha$ and $g(X_\alpha) \subset X_\alpha$; hence

(3.8) $$f \circ g \mid X_\alpha = s_\alpha \circ t_\alpha .$$

As $(s_\alpha \circ t_\alpha)_{\alpha \in A} \in S$, this shows that $f \circ g \in F$.

Finally, we assume that every F_α is commutative. Take again $f, g \in F$ and let $s, t \in S$ such that (3.7) holds. Then it follows from (3.8) that

$$f \circ g \mid X_\alpha = s_\alpha \circ t_\alpha = t_\alpha \circ s_\alpha = g \circ f \mid X_\alpha$$

for every $\alpha \in A$; hence $f \circ g = g \circ f$. Thus F is commutative.

Definition 1. The transformation semigroup $F \subset X^X$, defined in proposition 1 (by (3.2) and (3.3)), is called the product of the transformation semigroups $(F_\alpha;X_\alpha), \alpha \in A$, and is denoted by

$$\mathop{\mathbb{P}}_{\alpha \in A} F_\alpha \qquad \text{or} \qquad \mathbb{P}\{F_\alpha \ : \alpha \in A\} .$$

It follows from the construction of $F = \mathop{\mathbb{P}}_{\alpha \in A} F_\alpha$ that every set X_α is an invariant subset of X under F. Hence:

Proposition 2. The transformation semigroup $\mathop{\mathbb{P}}_{\alpha \in A} F_\alpha$ is $\{X_\alpha : \alpha \in A\}$ - invariant.

Proposition 3. If the sets X_α , $\alpha \in A$, are pairwise disjoint, then the abstract semigroup $(\mathop{\mathbb{P}}_{\alpha \in A} F, o)$ is isomorphic with the (unrestricted) direct product of the abstract semigroups (F, o) .

Proof. If S and F are as in (3.2) and (3.3), then, under the assumption that the X_α are pairwise disjoint, the set S is equal to the set $\prod_{\alpha \in A} F_\alpha$. If we define a multiplication $.$ in S by

$$s \, . \, t = (s_\alpha \circ t_\alpha)_{\alpha \in A} ,$$

then $(S, .)$ is even isomorphic with the direct product of the semigroups (F_α, o) . The proposition now follows from the fact that

$$f \longrightarrow (f|X_\alpha)_{\alpha \in A} \tag{3.9}$$

is an isomorphism of (F, o) onto $(S, .)$.

Proposition 4. If $X_\alpha = X$, for every $\alpha \in A$, then $\mathop{\mathbb{P}}_{\alpha \in A} F_\alpha = \bigcap_{\alpha \in A} F_\alpha$.

Proof. If again S and F are as defined in (3.2) and (3.3), then $(f_\alpha)_{\alpha \in A} \in S$ implies

$$f_\alpha = f_\alpha | X = f_\alpha | X_\alpha \cap X_\beta = f_\beta | X_\alpha \cap X_\beta = f_\beta | X = f_\beta$$

for all $\alpha, \beta \in A$. Conversely, if $(f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} F_\alpha$, and $f_\alpha = f_\beta$ for all $\alpha, \beta \in A$, then $(f_\alpha)_{\alpha \in A} \in S$. This proves the assertion, as $f_\alpha = f_\beta$ for all $\alpha, \beta \in A$ implies $f_\alpha \in \bigcap_{\alpha \in A} F_\alpha$.

4. <u>The sum of a system of transformation semigroups</u>

<u>Definition 2.</u> Let $\{(F_\alpha; X_\alpha) : \alpha \in A\}$ be a system of transformation semigroups, and let $X = \bigcup_{\alpha \in A} X_\alpha$. The transformation semigroup $F \subset X^X$, generated by the set

$$(4.1) \quad T = \{f \in X^X : (\exists \alpha \in A)(\exists f_\alpha \in F_\alpha)(f | X_\alpha = f_\alpha \text{ and } f | X \setminus X_\alpha = i | X \setminus X_\alpha)\}$$

is called the <u>sum</u> of the transformation semigroup $(F_\alpha; X_\alpha)$, and is denoted by

$$\mathfrak{S}_{\alpha \in A} F_\alpha \qquad \text{or} \quad \mathfrak{S}\{F_\alpha : \alpha \in A\}.$$

It follows from the definition that for every $\alpha \in A$ there is an isomorphism of F_α into $\mathfrak{S}_{\beta \in A} F_\beta$.

We are mainly interested in the case that $\mathfrak{S}_{\alpha \in A} F_\alpha$ is a commutative semigroup. By the above remark, every F_α then has to be commutative. But this is not sufficient; e.g. if $X_1 = X_2 = \{0, 1\}$, and if F_1 consists only of i and the map f_1 such that $f_1(0) = f_1(1) = 0$, while F_2 consists of i and the map f_2 such that $f_2(0) = f_2(1) = 1$, then $(F_1; X_1)$ and $(F_2; X_2)$ are commutative, but $\mathfrak{S}\{F_1, F_2\}$ is not commutative.

The following condition on the family $\{(F_\alpha; X_\alpha) : \alpha \in A\}$ will turn out to be sufficient, together with the commutativity of all F_α, in order to ensure that $\mathfrak{S}_{\alpha \in A} F_\alpha$ is commutative:

(C) for all $\alpha, \beta \in A$, the sets $X_\alpha \cap X_\beta$ and $X_\alpha \setminus X_\beta$ are invariant subsets of X_α under F_α, and if $f_\alpha \in F_\alpha$ and $f_\beta \in F_\beta$, then $f_\alpha | X_\alpha \cap X_\beta$ and $f_\beta | X_\alpha \cap X_\beta$ commute.

Proposition 5. Let $\{(F_\alpha;X_\alpha) : \alpha \in A\}$ be a family of commutative transformation semigroups, each contatining the identity mapping $i_\alpha : X \to X_\alpha$, and let condition (C) be satisfied. Then $\underset{\alpha\in A}{\$} F_\alpha$ is a commutative transformation semigroup containing the identity map.

Proof. Let T be as in (4.1), and let F be the subsemigroup of X^X generated by T. As it is evident that $i \in F$ we have only to show that T is commutative. Let $f, g \in T$. Then there are $\alpha, \beta \in A$ and $f_\alpha \in F_\alpha$, $f_\beta \in F_\beta$ such that

$$f|X_\alpha = f_\alpha \ ; \qquad g|X_\beta = f_\beta \ ;$$
$$f|X\setminus X_\alpha = i|X\setminus X_\alpha \ ;$$
$$g|X\setminus X_\beta = i|X\setminus X_\beta \ .$$

As condition (C) is assumed to be satisfied, $f|X_\alpha \cap X_\beta$ and $g|X_\alpha \cap X_\beta$ commute. Furthermore, $f|X\setminus(X_\alpha \cap X_\beta) =$ $= g|X\setminus(X_\alpha \cup X_\beta) = i|X\setminus(X_\alpha \cup X_\beta)$. Hence we need only check what happens with points in $X_\alpha\setminus X_\beta$ or in $X_\beta\setminus X_\alpha$. Because of the symmetry of the situation, we may restrict our attention to points in $X_\alpha\setminus X_\beta$.

Let $x \in X_\alpha\setminus X_\beta$. Then

$$(f \circ g)(x) = f(g(x)) = f(x) = f_\alpha(x) \ ;$$

as $X_\alpha\setminus X_\beta$ is supposed to be invariant under F_α,

$f_\alpha(x) \in X_\alpha\setminus X_\beta$; hence

$$f_\alpha(x) = g(f_\alpha(x)) = g(f(x)) = (g \circ f)(x) \ .$$

This finishes the proof.

Proposition 6. If the sets X_α, $\alpha \in A$, are pairwise disjoint, then the abstract semigroup $(\underset{\alpha\in A}{\$} F_\alpha, \circ)$ is isomorphic to the direct sum (restricted direct product) of the abstract semigroups $(F_\alpha, \circ)$, $\alpha \in A$.

Proof. Let T be defined by (4.1). Let φ be the mapping (3.9). Then maps T 1.1 onto the subset of $\prod_{\alpha \in A} F_\alpha$, consisting of all $(f_\alpha)_{\alpha \in A}$ such that $f_\alpha \neq i_\alpha$ for at most one $\alpha \in A$; and φ maps F 1.1 onto the subset of $\prod_{\alpha \in A} F_\alpha$ such that $f_\alpha \neq i_\alpha$ for only finitely many $\alpha \in A$. It is immediately seen that $\varphi | F$ is a homomorphism of $(F, \circ)$ into the direct product of the $(F_\alpha, \circ)$; hence $\varphi | F$ is an isomorphism, and $\varphi(F)$ is exactly the direct sum of the $(F_\alpha, \circ)$.

Proposition 7. Assume $X_\alpha = X$, for every $\alpha \in A$. Then condition (C) is satisfied if and only if $\bigcup_{\alpha \in A} F_\alpha$ is commutative, and $\mathfrak{S}_{\alpha \in A} F_\alpha$ is the subsemigroup of X^X generated by $\bigcup_{\alpha \in A} F_\alpha$.

Proof: evident.

5. Commutative semigroups that are maximal with respect to their system of invariant sets

In this section, $(F;X)$ is a commutative transformation semigroup, containing the identity transformation, and $\mathcal{J}$ will always denote a family of subsets of X that are invariant under F.

If $\mathcal{J}$ is such a family, then $\cup \mathcal{J}$ will denote the set $\cup\{A : A \in \mathcal{J}\}$, and $\mathbb{P}(\mathcal{J})$ will denote the semigroup

$$\mathbb{P}(\mathcal{J}) = \mathbb{P}\{F|A : A \in \mathcal{J}\}.$$

The following lemma is almost obvious:

Lemma 1. $f \in \mathbb{P}(\mathcal{J}) \Longleftrightarrow f|A \in F|A$ for all $A \in \mathcal{J}$.

From this lemma, the following propositions follow without difficulty:

Proposition 8. If $\cup \mathcal{J} = X$, then $F \subset \mathbb{P}(\mathcal{J}) \subset X^X$.

(If $\bigcup \mathcal{J} \neq X$, then certainly not $F \subset \mathbb{P}(\mathcal{J})$, as $\mathbb{P}(\mathcal{J})$ consists of mappings of $\bigcup \mathcal{J}$ into itself.)

Proposition 9. Let both $\mathcal{J}_1$ and $\mathcal{J}_2$ consist of subsets of X that are invariant under F. If $\bigcup \mathcal{J}_1 = \bigcup \mathcal{J}_2$, then $\mathcal{J}_1 \subset \mathcal{J}_2$ implies $\mathbb{P}(\mathcal{J}_1) \supset \mathbb{P}(\mathcal{J}_2)$.

If $\mathcal{J}_1$ and $\mathcal{J}_2$ are both families of subsets of a set X, we will say that $\mathcal{J}_1$ is a refinement of $\mathcal{J}_2$, and write

$$\mathcal{J}_1 \leqq \mathcal{J}_2 ,$$

if for every $A_1 \in \mathcal{J}_1$ there is an $A_2 \in \mathcal{J}_2$ such that $A_1 \subset A_2$.

Proposition 10. Let both $\mathcal{J}_1$ and $\mathcal{J}_2$ consist of subsets of X that are invariant under F. If $\bigcup \mathcal{J}_1 = \bigcup \mathcal{J}_2$ and $\mathcal{J}_1 \leqq \mathcal{J}_2$, then $\mathbb{P}(\mathcal{J}_1 \cup \mathcal{J}_2) = \mathbb{P}(\mathcal{J}_2)$.

Proof. By proposition 9, $\mathbb{P}(\mathcal{J}_1 \cup \mathcal{J}_2) \subset \mathbb{P}(\mathcal{J}_2)$ on the other hand,

$$f \in \mathbb{P}(\mathcal{J}_2) \Longleftrightarrow (\forall A \in \mathcal{J}_2)\ (f|A \in F(A)) \Rightarrow (\forall A \in \mathcal{J}_1 \cup \mathcal{J}_2)\ (f|A \in F|A)$$
$$f \in \mathbb{P}(\mathcal{J}_1 \cup \mathcal{J}_2) .$$

Example. If $X \in \mathcal{J}$, then $\mathbb{P}(\mathcal{J}) = F$.

Remark. If A is not an invariant subset of X, then $F|A$ is not a semigroup. However, if we define $F \| A = \{f|A : f \in F$ and $f(A) \subset A\}$ then $F \| A$ is a semigroup under composition. It is seen at once that

$$\mathbb{P}\,\{(F;X),\ (F\|A;A)\} = \{f \in F : fA \subset A\} ;$$

hence if A is not invariant, $F \not\subset \mathbb{P}(F, F\|A)$, although of course $X \cup A = X$.

Lemma 2. Let $\mathcal{J}_1$ be the class of all subsets of X that are invariant under F, and let $\mathcal{J}_2$ be the class of all orbits under F, and let $F \subset G \subset X^X$.

Then G is a commutative $\mathcal{J}_1$-invariant system if and only if G is a commutative $\mathcal{J}_2$-invariant system.

Proof. As $\mathcal{J}_2 \subset \mathcal{J}_1$, every $\mathcal{J}_1$-invariant system is $\mathcal{J}_2$ invariant. On the other hand, if $A \in \mathcal{J}_1$, then

$$A = F(A) = \bigcup \{F(x) : x \in A\} = \bigcup \{B \in \mathcal{J}_2 : B \subset A\} .$$

Hence every $\mathcal{J}_2$-invariant system is $\mathcal{J}_1$-invariant.

Lemma 3. Let $G \subset X^X$ be commutative. If there exists an $e \in X$ such that $G(e) = X$, then G is a maximal commutative semigroup.

Proof. Let $f \in X^X$ such that f commutes with every $g \in G$. We will show that $f \in G$. As $G(e) = X$, there exists a $g_0 \in G$ such that $f(e) = g_0(e)$. Let x be an arbitrary element of X; then there is a $g \in G$ such that $g(e) = x$, and it follows that

$$f(x) = f \circ g(e) = g \circ f(e) = g \circ g_0(e) = g_0 \circ g(e) =$$
$$= g_0(x) .$$

Hence $f = g_0 \in G$.

In particular, we have the following:

Lemma 4. If $F \subset X^X$ is a commutative semigroup, containing the identity map, then for every orbit $F(x)$ under F, $F | F(x)$ is a maximal commutative semigroup of mappings $F(x) \to F(x)$.

Theorem 1. Let $F \subset X^X$ be a commutative semigroup, containing the identity map. Let $\mathcal{J}$ be the class of all subsets of X that are invariant under F. Then there exists one and only one maximal commutative $\mathcal{J}$-invariant semigroup $G \subset X^X$ containing F; and

$$G = \mathbb{P} \{F | F(x) : x \in X\} .$$

Proof. Let g be any mapping $X \to X$ that commutes with every $f \in F$ and that maps every $A \in \mathcal{J}$ into itself. We will show that $g \in G$.

Take any $x \in X$. Then $g|F(x)$ maps $F(x)$ into itself, as $F(x) \in \mathcal{J}$, and $g|F(x)$ commutes with every mapping in $F|F(x)$. But by lemma 4, $F|F(x)$ is a maximal commutative semigroup; hence $g|F(x) \in F|F(x)$. It now follows from lemma 1 that $g \in G$.

An immediate consequence is that $F \subset G$ (this also follows from proposition 8). So it remains only to be proved that G is $\mathcal{J}$-invariant. But by proposition 2, G is $\mathcal{J}_2$-invariant, where $\mathcal{J}_2 = \{F(x) : x \in X\}$; now apply lemma 2.

<u>Corollary</u>: If $F \subset X^X$ is a maximal commutative transformation semigroup, then

$$F = \mathbf{P}\{F|F(x) : x \in X^X\} .$$

A family of orbits $\{F(x) : x \in Y\}$, where Y is a subset of X, is called an <u>F-orbit cover</u>, or shortly an <u>F-cover</u> of X, if $F(Y) = X$.

From proposition 10 and theorem 1 we deduce at once:

<u>Theorem 2</u>. If $\{F(x) : x \in Y\}$ is an F-cover of X, then $\mathbf{P}\{F|F(x) : x \in Y\}$ is the maximal commutative $\mathcal{J}$-invariant semigroup containing F (where $\mathcal{J}$ is the family of all subsets of X that are invariant under F).

In [1] the following theorem was proved ([1], Theorem 1):

"Let F be a maximal commutative semigroup of mappings of a set X into itself, and let $r(F) \neq \emptyset$. If each $f \in F$ has a fixed point, then all mappings in F have precisely one common fixed point."

Here $r(F) = \{f \in F : (\forall f_1 \in F)\ (\exists f_2 \in F)\ (f = f_1 \circ f_2)\}$ is the set of all mappings $f \in F$ that are common multiples of all mappings in F .

Using the concepts developed in this paper, we may generalise this theorem as follows:

<u>Theorem 3</u>. Let $F \subset X^X$ be a maximal commutative $\mathcal{J}$-invariant transformation semigroup (where $\mathcal{J}$ again is the family of all subsets of X that are invariant under F). If $r(F) \neq \emptyset$ and if each $f \in F$ has a fixed point, then all mappings in F have a common fixed point.

The proof is exactly the same as the first part of the proof of [1], Theorem 1 . It is easily seen that the mapping g , constructed in [1], leaves all sets of $\mathcal{J}$ invariant; hence the weaker assumption that F is maximally $\mathcal{J}$-invariant suffices in order to conclude that $g \in F$.

Finally we will give one more application of the above product construction. In order to do so, however, we need the concept of an <u>algebraically generated</u> transformation semigroup.

Take an abstract semigroup $(X; .)$ and consider all left multiplications in X , i.e. all mappings f_a, $a \in X$, defined by

(5.1) $$f_a(x) = a.x \quad .$$

These mappings constitute a semigroup $F \subset X^X$. If X has an identity element, it is even true that the abstract semigroup $(F; \circ)$ is isomorphic with $(X; .)$. (In fact, in that case the correspondence $a \rightarrow f_a$ is an isomorphism of $(X; .)$ onto $(F; \circ)$.) Now transformation semigroups of this kind will be called algebraically generated. More explicitly:

<u>Definition 3</u>. A transformation semigroup $F \subset X^X$ is called <u>algebraically generated</u> if there exists a binary operation . om X such that

(i) $(X; .)$ is a semigroup with unit;

(ii) $F = \{f_a : a \in X\}$, where f_a is as defined in (5.1).

Using lemma 3, it is easy to give a complete characterisation of all commutative transformation semigroups that are algebraically generated.

Lemma 5. A commutative transformation semigroup $F \subset X^X$ is algebraically generated if and only if there exists an $e \in X$ such that $F(e) = X$.

Proof.

First assume F to be algebraically generated, say by the semigroup structure $(X; .)$. Then if e is the unit element of $(X; .)$, it is immediate that $F(e) = X$.

Conversely, assume $F(e) = X$, for some $e \in X$. From the proof of lemma 3 it follows at once that the mapping $\varphi : F \to X$, defined by

$$\varphi(f) = f(e)$$

is a 1-1-mapping of F onto X. Define a binary operation $.$ in X by

$$x . y = \varphi(\varphi^{-1}(x) \circ \varphi^{-1}(y)) .$$

Then $(X; .)$ is a commutative semigroup, with e as unit element and $F = \{f_a : a \in X\}$, as

$$f_a(x) = a.x = \varphi(\varphi^{-1}(a) \circ \varphi^{-1}(x)) = (\varphi^{-1}(a) \circ \varphi^{-1}(x))(e) =$$
$$= \varphi^{-1}(a)(x) .$$

We now prove the following theorem, which states in effect that every commutative transformation semigroup can be built up, using the product construction of section 3, from algebraically generated semigroups:

Theorem 4. Let $F \subset X^X$ be a commutative transformation semigroup, and let m be the cardinal number of an F-cover of X. Then F is a subsemigroup of a product of m algebraically generated commutative semigroups.

Proof.

Theorem 2 assers that F is a subsemigroup of a product of m semigroups $F/F(x)$, and lemma 5 shows that all these semigroups are algebraically generated (as $(F/F(x))(x) = F(x)$).

References:

[1]. Z. HEDRLÍN, Two theorems concerning common fixed points of commutative mappings, CMUC, 3,2 (1962).