MATHEMATICS

ALMOST PERIODIC FUNCTIONS AND DINI'S THEOREM

BY

GILBERT HELMBERG

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Let X be a topological group with unit e and let $\mathfrak{A}$ be the complex algebra (under pointwise operations) of all complex-valued continuous almost periodic (a.p.) functions on X. In the following we shall always assume the group X to be maximally a.p., that is, to possess sufficiently many continuous a.p. functions in order to distinguish points (if X is not maximally a.p., then it may be replaced by the maximally a.p. factor group of X with respect to the normal subgroup $Z=\{x \in X: f(x)=f(e)$ for all $f \in \mathfrak{A}\}$). It is well known that, in this case, a continuous and isomorphic image of X may be imbedded as a dense subgroup in a compact group X^* (its "Bohr compactification") in such a way that $\mathfrak{A}$ is the restriction to X of the set $\mathfrak{C}(X^*)$ of all complex-valued continuous functions on X ([13] 41C). Moreover, X^* is uniquely determined by X up to a topological isomorphism; in particular, $X^*=X$ if and only if X is compact. Furthermore, if μ^* is the normalized Haar measure on X^* and if $f^* \in \mathfrak{C}(X^*)$ is the continuous extension of a given function $f \in \mathfrak{A}$, then (M denoting the mean value on $\mathfrak{A}$)

$$Mf=\int_{X^*} f^*(x)d\mu^*(x) \qquad \text{for all } f \in \mathfrak{A}.$$

Of course, the properties of M of being linear, positive and normed on $\mathfrak{A}$ may be stated and proved without any reference to the Bohr compactification of X. By an application of the Daniell extension procedure as described e.g. in [13] 12, the mean value M could actually be written as a Lebesgue (i.e. countably additive) integral over X (in place of X^*), if it would only possess the following additional property of monotone continuity:

(1) *If $\{f_n\}$ is a monotone decreasing sequence of non-negative functions $f_n \in \mathfrak{A}$ that converges pointwise towards* 0, *then* $\lim_{n\to\infty} Mf_n=0$.

The validity of (1) is in fact a necessary and sufficient condition for the representability of M as an integral over X ([4], [6]) unless we want to use the concept of an integral with respect to a finitely (but not necessarily countably) additive measure on X as done in [1] 4 and [11] 2.

For a compact group X the validity of (1) is a consequence of the

boundedness of M (implied by the other mentioned properties of M) and of the validity (in this special case of a compact group X) of the so-called "Theorem of Dini" ([13] 16A) for $\mathfrak{A}$:

(2) *If $\{f_n\}$ is a monotone decreasing sequence of non-negative functions $f_n \in \mathfrak{A}$ that converges pointwise towards 0, then $\{f_n\}$ converges uniformly.*

In fact, even a sharper version of Dini's theorem is true in this case ([2] p. 105):

(2′) *If $\{f_\nu\}$ is a downwards directed set of non-negative functions $f_\nu \in \mathfrak{A}$ that converges pointwise towards 0, then $\{f_\nu\}$ converges uniformly.*

Here "downwards directed" means that for any two functions f_{ν_1}, f_{ν_2} there exists a function $f_{\nu_3} \leqq \min(f_{\nu_1}, f_{\nu_2})$ and convergence is meant with respect to the partial ordering $\leqq$ directing the set $\{f_\nu\}$.

It has already been pointed out by Bochner [1] that, if X is not compact, it will in general not be possible to write M as a Lebesgue integral (with respect to a suitable countably additive measure) over X. As a consequence, (1) and (2) will not be true in general. This combination of facts seems not to be too well known since statements to the countrary have recently been published and reviewed (erroneously) as results [7] [7a] [7b].

It is indeed possible to give topological criteria for the validity of (2) and (2′) (which conditions in turn imply the validity of (1)). Let us denote by X_1 the completely regular topological space obtained by endowing the set X with the weak topology induced by the functions in $\mathfrak{A}$. We note that $\mathfrak{A}$ is identical with the set of all bounded complex-valued continuous functions on X_1 ([14] theorem 2, criterion 2 and its corollary). Theorem 2 in [4] states (besides mentioning several other necessary and sufficient conditions) that (2) holds if and only if X_1 is pseudocompact in the sense as defined by Hewitt [10] (i.e. if and only if $\mathfrak{A}$ is actually the set of all complex-valued continuous functions on X_1). Theorem 4 in [9] states that (2′) holds if and only if X_1 is compact. Thus, as soon as $\mathfrak{A}$ is not simply the set of all complex-valued continuous functions on a (pseudo-)compact group X_1 (in which case M is actually the Haar integral), Dini's theorem (2′) (resp. 2)) cannot hold for $\mathfrak{A}$. It should be mentioned here that the investigations in [4] and [9] with respect to Dini's theorem have been extended in the meantime by Flachsmeyer [3].

The fact that the statements (1) and (2) (resp. (2′)) are not valid in general seems rather startling at first sight and aks for explicit counter-examples and for some intuitive explanation. In [1] 5 Bochner considers the sequence $f_n = (\sin x/n)^2$ consisting of non-negative continuous a.p. functions on the real line, converging pointwise towards 0 everywhere, and being uniformly bounded by 1, whereas $Mf_n = \frac{1}{2}$ for all $n \geqq 1$. This,

via Lebesgue's theorem on dominated convergence, disproves the validity of (1) and (2). However, the sequence $\{f_n\}$ itself is not monotone on the entire real line (even if it is eventually monotone on every bounded interval).

A simple direct counterexample for the statements (1) and (2) is obtained in the following way: Let X be the additive group of integers and let (for $m \geqq 1$, k an arbitrary integer) $E_{k,m}$ be the set of all integers congruent k modulo m. The characteristic function $\chi_{k,m}$ of $E_{k,m}$ is a.p. on X (even periodic with period m). Its mean value $M\chi_{k,m}$ is exactly equal to $1/m$. Let $\{m_i\}$ be a sequence of positive integers such that

$$s = \sum_{i=1}^{\infty} \frac{1}{m_i} < 1.$$

Let, furthermore, the whole group of integers be arranged into a sequence $\{k_i\}$ $(1 \leqq i)$. We set

$$f_n = 1 - \max\,(\chi_{k_i, m_i} : 1 \leqq i \leqq n).$$

Then f_n is a.p. on X (even periodic with period equal to least common multiple of m_i, $1 \leqq i \leqq n$) and

$$Mf_n \geqq 1 - M\Big(\sum_{i=1}^{n} \chi_{k_i, m_i}\Big) = 1 - \sum_{i=1}^{n} \frac{1}{m_i} \geqq 1 - s > 0 \qquad \text{for all } n \geqq 1.$$

The sequence $\{f_n\}$ is monotone decreasing and we have

$$\lim_{n\to\infty} f_n(x) = 0 \qquad \text{for all } x \in X$$

$$\lim_{n\to\infty} Mf_n \geqq 1 - s > 0.$$

Thus the sequence $\{f_n\}$ does not converge uniformly, even

$$\max_{x \in X} f_n(x) = 1 \qquad \text{for all } n \geqq 1.$$

The explanation for this seemingly pathological situation lies in the fact that, even if the sequence $\{f_n\}$ converges monotonically towards 0 everywhere on X, convergence of the corresponding continous functions f_n^* on the Bohr compactification X^* of X may take place only on a very small subset of X^*. This general background of the preceding special example is somewhat clarified in the proof of the following theorem:

Theorem: *Let X be a σ-compact non-compact maximally a.p. topological group and let $0 < \varepsilon < 1$ be given. Then there exists a monotone decreasing sequence of non-negative continuous a.p. functions $f_n \leqq 1$ on X such that*

$$\lim_{n\to\infty} f_n(x) = 0 \qquad \text{for all } x \in X$$

$$\lim_{n\to\infty} Mf_n \geqq 1 - \varepsilon$$

$$\max_{x \in X} f_n(x) = 1 \qquad \text{for all } n \geqq 1.$$

Proof: Let φ be the continuous isomorphism imbedding X into its Bohr compactification X^* and let $X = \bigcup_{i=1}^{\infty} C_i$ (C_i compact). Then $\varphi(C_i)$ is compact and closed in X^*. Therefore $\varphi(X) = \bigcup_{i=1}^{\infty} \varphi(C_i)$ is a Borel set in X^*. The crucial fact is that $\varphi(X)$ has Haar measure 0 in X^*. In order to show this we follow the reasoning of GLICKSBERG [5], some slight modifications being due to the fact that we do not assume X to be locally compact and abelian. Suppose $\mu^*(\varphi(X)) > 0$. Then there exists an index $i \geqq 1$ such that $\varphi(C_i)$ has positive Haar measure in X^* and the compact set $[\varphi(C_i)] \cdot [\varphi(C_i)]^{-1} \subset \varphi(X)$ contains an open neighbourhood of the identity in X^* ([8] 61 (3)). Thus $\varphi(X)$ is locally compact, open, and dense in X^*. As a consequence we get $X^* = \varphi(X)$. By Baire's category theorem ([12] p. 200), there must be an index $j \geqq 1$ such that $\varphi(C_j)$ contains an open set U in X^*. The continuous isomorphism φ is a homeomorphism on the open set $\varphi^{-1}(U)$ in X and, since U may be translated arbitrarily over $X^* = \varphi(X)$, on the whole of X. Thus X is both compact and non-compact, a contradiction.

In order to simplify the notation we shall identify from now on the group X and its image $\varphi(X)$. For every $i \geqq 1$, let V_i be an open set in X^* containing C_i and having the property that $\mu^*(\overline{V}_i) < \varepsilon/2^i$ ($\overline{V}_i$ denoting the closure of V_i in X^*; such a set V_i exists since μ^* is regular ([8] 64) and since X^* is normal). Let the function $h_i^* \in C(X^*)$ be chosen in such a way that

$$0 \leqq h_i^*(x) \leqq 1 \qquad \text{for all } x \in X^*$$

$$h_i^*(x) = \begin{cases} 1 & \text{for } x \in C_i \\ 0 & \text{for } x \notin V_i. \end{cases}$$

Let g_n be the restriction of $g_n^* = \max(h_i^* : 1 \leqq i \leqq n)$ to X. Then the sequence $\{g_n\} \subset \mathfrak{A}$ is monotone increasing and converges towards 1 everywhere on X. However,

$$Mg_n = \int_{X^*} g_n^*(x) d\mu^*(x) \leqq \sum_{i=1}^{n} \int_{X^*} h_i^*(x) d\mu^*(x) \leqq \sum_{i=1}^{n} \mu^*(V_i) \leqq \varepsilon \qquad \text{for all } n \geqq 1.$$

The open complement of $\bigcup_{i=1}^{n} \overline{V}_i$ in X^* is not empty since it has positive Haar measure. Since X is dense in X^* we have even

$$\min_{x \in X} g_n(x) = 0 \qquad \text{for all } n \geqq 1.$$

The sequence $f_n = 1 - g_n$ has therefore all the properties mentioned in the theorem.

The theorem applies, in particular, to every countable maximally a.p. group and to every compactly generated abelian group, thus also to every connected locally compact abelian group.

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