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On the orbits of the hat-function, and on countable maximal commutative semigroups of continuous mappings of the unit interval into itself

by

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1. Introduction

In this report we study some properties of the so-called hat-function. This is the mapping of the unit interval $I=[0,1]$ into itself that takes the value 0 in 0 and 1, the value 1 in $\frac{1}{2}$, while it is linear both in $[0,\frac{1}{2}]$ and in $[\frac{1}{2},1]$.

In section 2 the orbits of points $x \in I$ under f are studied. The results are the following: if x is rational, its orbit is finite. If x is irrational, its orbit is either nowhere dense or everywhere dense. Both the set of all irrational x with a nowhere dense orbit and the set of all x with an everywhere dense orbit are themselves everywhere dense.

In section 3 these results are generalized to "multi-hats". These functions were brought to our attention by Z. Hedrlín, who also showed that every two multi-hats commute (proposition 6 in section 3). In section 4 we communicate some results about continuous mappings $I \to I$ that commute with the hat.

The last section contains a proof of the fact that the multi-hats f_n, $n=0,1,2,\ldots$, form a maximal commutative semigroup of continuous mappings $I \to I$. This semigroup was first studied by Z. Hedrlín, as mentioned above. As far as we know this is the first non-trivial example of a <u>countable maximal</u> commutative semigroup of continuous transformations. Of course it leads at once to a whole class of (topologically equivalent) maximal commutative semigroups of continuous mappings, as is also shown in section 4.

Finally it is shown that the maximal commutative semigroup constructed (which is countable) is contained in an uncountable commutative semigroup of (no longer necessarely continuous) transformations of the unit segment into itself.

2. The orbits of the hat

We will denote by f_2 the mapping $I \longrightarrow I$ such that

$$(2.1)\qquad \begin{aligned} f_2(x) &= 2x && \text{if } 0 \leq x \leq \tfrac{1}{2}; \\ f_2(x) &= 2-2x && \text{if } \tfrac{1}{2} \leq x \leq 1. \end{aligned}$$

The reason for this notation will become clear in section 3.

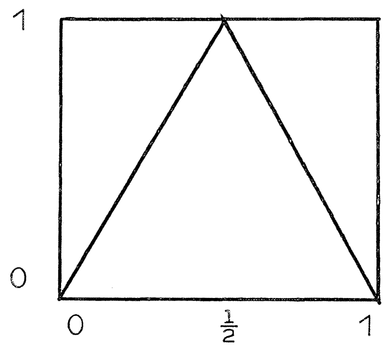

Fig. 1: graph of f_2

If $x \in I$, then $\theta(x) = \{f_2^k(x) : k=0,1,2,\ldots\}$ is called the orbit of x (under f_2).

In order to study these orbits it is useful to write the numbers $a \in I$ in dyadic expansion: $a = 0.a_1a_2a_3a_4\ldots$, where each a_k is either 0 or 1, and where $a = \sum_{k=1}^{\infty} \frac{a_k}{2^k}$.

Then the following fact is almost immediate:

Proposition 1 Let $a \in I$, and let $0.a_1a_2a_3\ldots$ be a dyadic expansion of a. Then, for $k=1,2,3,\ldots$,

$$(2.2)\qquad \begin{aligned} f_2^k(a) &= 0.a_{k+1}\,a_{k+2}\,a_{k+3}\ldots && \text{if } a_k=0; \\ f_2^k(a) &= 0.\bar{a}_{k+1}\,\bar{a}_{k+2}\,\bar{a}_{k+3}\ldots && \text{if } a_k=1. \end{aligned}$$

Here $\bar{a}_i = 1-a_i$.

<u>Theorem 1</u> $\theta(a)$ is finite if and only if a is rational.

<u>Proof</u>

Let a be rational. Then a has a dyadic expansion of the form

(2.3) $$a = 0.a_1a_2 \ldots a_m\overline{b_1b_2 \ldots b_n} \quad ,$$

the right hand side being an abbreviation for a the infinite dyadic expansion $0.a_1a_2a_3\ldots$ with $a_{m+kn+r}=a_{m+r}$ $(k=1,2,\ldots;$ $r=1,2,\ldots,n)$.

Then it follows from proposition 1 that

$$f_2^{m+2n}(a)=f_2^{m+n}(a)=0.\overline{b_1b_2\ldots b_n} \quad \text{if } b_n=0;$$

$$f_2^{m+2n}(a)=f_2^{m+n}(a)=0.\overline{\bar{b}_1\bar{b}_2\ldots\bar{b}_n} \quad \text{if } b_n=1.$$

Hence the orbit of a is finite. (In fact, this orbit consists of a loop of at most n points, and a tail of at most m points, as sketched in Fig.2).

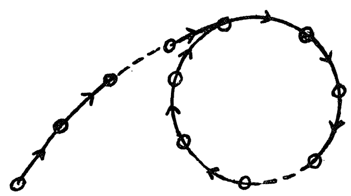

Fig.2: $\theta(a)$ if a is rational

On the other hand, if the orbit $\theta(a)$ is finite, it must look like the one sketched in fig.2; this implies that a has a dyadic expansion of the form (2.3), hence that a is rational.

<u>Proposition 2</u> There are irrational numbers $a \in I$, such that $\theta(a)$ is everywhere dense.

Proof

Choose any enumeration of the (countable) set of all finite dyadic fractions:

$$a^{(1)} = 0.a_1^{(1)}\, a_2^{(1)} \dots a_{k_1}^{(1)} \;;$$

$$(2.4) \qquad a^{(2)} = 0.a_2^{(2)}\, a_2^{(2)} \dots a_{k_2}^{(2)} \;;$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Let

$$(2.5) \quad a=0.a_1^{(1)}\, a_2^{(1)} \dots a_{k_1}^{(1)}\; 0\; a_1^{(2)}\, a_2^{(2)} \dots a_{k_2}^{(2)}\; 0\; a_1^{(3)} a_2^{(3)} \dots a_{k_3}^{(3)} 0 \dots$$

Assertion: $\theta(a)$ is dense. For let U be any open set in I. Then U contains a finite dyadic fraction $b=0.b_1b_2\dots b_k$; and there exists a natural number $n > k$ such that $x \in U$ for all x such that $|b-x| < \frac{1}{2^n}$. Define $c \in I$ by

$$c = 0.b_1b_2 \dots b_k \underbrace{0\,0\dots0}_{n-k \text{ zeros}}\,1 \quad .$$

Then $c = a^{(m)}$ for some m, and if $s = \sum_{i=1}^{m-1} (k_i+1)$, then

$$f_2^s(a) = 0.b_1b_2\dots b_k\; 0\; 0 \dots 0\; 1\; 0\; a_1^{(m+1)} \dots \;;$$

hence $|b-f_2^s(a)| < \frac{1}{2^n}$; which implies that $f_2^s(a) \in U$.

Corollary Let E be the set of all $a \in I$ such that $\theta(a)$ is everywhere dense. The set E is itself everywhere dense.

Proof

This follows at once from the fact that $\theta(a) \subset E$, where a is the point defined by (2.5).

Theorem 2 For every $a \in I$, either $\theta(a)$ is everywhere dense or $\theta(a)$ is nowhere dense.

Proof

Assume $\theta(a)$ is not nowhere dense. Then there is an open interval $U \subset \overline{\theta(a)}$. By the corollary to proposition 2 there is an $x \in U$ such that $\overline{\theta(x)}=I$.

Let V be any open subset of I. Then $f_2^n(x) \in I$, for some natural number n. Now f_2^n is continuous; hence $f_2^n(y) \in \underline{I}$ for all y in some open neighbourhood W of x. As $\emptyset \neq W \cap U \subset \overline{\theta(a)}$, $W \cap U$ must contain a point $z \in \theta(a)$; say $z=f^m(a)$. Then $f^{m+n}(a) \in V \cap \theta(a)$.

As V was an arbitrary open subset of I, it follows that $\theta(a)$ is dense in I.

Proposition 3 There is an irrational number $a \in I$ such that $\theta(a)$ is nowhere dense. In fact, $a=\sum_{n=1}^{\infty} 2^{-\frac{1}{2}n(n+1)}$ is such a number.

Proof

Let $a=\sum_{n=1}^{\infty} 2^{-\frac{1}{2}n(n+1)}$. Then a admits the following dyadic expansion:

(2.6) $$a=0.1010010001000010\ldots .$$

It is easily checked that e.g. the interval $0{,}101001 < x < 0{,}1010001$ contains no point of $\theta(a)$. Hence $\theta(a)$ is not everywhere dense; it now follows from theorem 2 that $\theta(a)$ is nowhere dense.

Corollary Let N be the set of all irrational $x \in I$ such that $\theta(x)$ is nowhere dense. The set N is everywhere dense.

Proof

N contains all numbers with a dyadic expansion of the form

$$0.p_1p_2p_3\ldots p_n\ 1010010001000010\ldots ;$$

hence N is dense in the set of all numbers with a finite dyadic expansion. But the latter set is dense in I.

3. Generalisation to multi-hats

For n=1,2,3,..., let f_n be the mapping $I \longrightarrow I$ defined as follows:

$$f_n\left(\frac{2k}{n}\right) = 0 \qquad (k=0,1,2,.., [\tfrac{n}{2}]) ;$$

$$f_n\left(\frac{2k+1}{n}\right) = 1 \qquad (k=0,1,2,.., [\tfrac{n}{2}]) ;$$

$$f_n \Big| \left[\frac{k}{n}, \frac{k+1}{n}\right] \text{ is linear } \quad (k=0,1,2,...,n-1) .$$

In particular, f_1 is the identity map, and f_2 is the hat-function defined in section 2. We may also define f_0 to be the function that is identically zero on I. The functions f_n with $n \geq 3$ may be called multiple hats or multi-hats; if n is odd, the multi-hat f_n has a flap at the right.

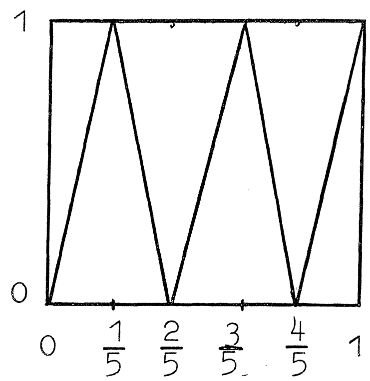

Fig.3: graph of f_5

The results of section 2 can also be obtained for the functions f_n, n=3,4,... . This depends on the following easy generalisation of proposition 1:

Proposition 4 Let $n \geq 2$ be an integer. Let

$$a = 0.a_1 \, a_2 \, a_3 \, a_4 \, \ldots\ldots$$

be an n-adic expansion of a I. Then

$$f_n^k(a) = 0,a_{k+1} \, a_{k+2} \ldots \quad \text{if } a_k \text{ is even ;}$$

$$f_n^{(k)}(a) = 0,\bar{a}_{k+1} \, \bar{a}_{k+2} \ldots \quad \text{if } a_k \text{ is odd .}$$

Here $\bar{a}_i$ denotes the integer $n-1-a_i$.

Using this proposition, one can establish in a manner exactly analogous to the methods of section 2:

Theorem 3 Let $n \geq 2$ be a natural number. For $a \in I$, let $\theta(a)$ be the orbit of a under f_n.

1. $\theta(a)$ is finite if and only if a is rational.
2. $\theta(a)$ is either everywhere dense or nowhere dense.
3. Let E be the set of all $x \in I$ such that $\theta(x)$ is everywhere dense, and let N be the set of all irrational $x \in I$ such that $\theta(x)$ is nowhere dense. Both N and E are everywhere dense.

4. Continuous mappings commuting with the hat

We say that two mappings f,g of a set A into itself commute if $f \circ g = g \circ f$, i.e. if

$$f(g(x)) = g(f(x))$$

for all $x \in A$.

Proposition 5 If n and m are non-regative integers, then f_n and f_m commute.

This is an immediate consequence of

<u>Proposition 6</u> If n and m are non-negative integers, then $f_n \circ f_m = f_{n.m}$.

<u>Proof</u>

From the definition of the multi-hats it follows that $f_n \circ f_m \left(\frac{k}{n.m}\right)$ is either 0 or 1 $(k=0,1,2,\ldots,nm)$, and that $f_n \circ f_m$ is linear on each of the intervals $\frac{k}{n.m} \leq x \leq \frac{k+1}{n.m}$ $(k=0,1,\ldots,n.m-1)$. As $f_n \circ f_m$ is continuous, and as $f_n \circ f_m(0)=0$, it follows that $f_n \circ f_m$ must coincide with $f_{n.m}$.

Another example of a continuous map $g:I\longrightarrow I$ that commutes with f_2 is the constant map $g(x)=\frac{2}{3}$, for all $x \in I$.

As 0 and $\frac{2}{3}$ are the only fixed points of f_2, we must either have $g(0)=0$ on $g(0)=\frac{2}{3}$, for any mapping $g:I\longrightarrow I$ (continuous or not) that commutes with f_2. The following proposition gives some information about the case $g(0)=0$. It plays an important role in the proof of theorem 4 in the next section.

<u>Proposition 7</u> Let g be a continuous map $I\longrightarrow I$ such that $g(0)=0$. If g commutes with f_2, then $x^{-1}g(x)$ is bounded on I.

<u>Proof</u>

We may assume that g is not identically zero on I. In that case $g(x)=1$ for at least one x. For take any z such that $g(z)\neq 0$, and any natural number k such that $\frac{1}{2^k} < g(z)$. Then $\frac{1}{2^k} \in g\,I$; hence

$$1=(f_2)^k \left(\frac{1}{2^k}\right) \in f_2^k\, g\, I = g\, f_2^k\, I = g\, I.$$

Let $a = \inf g^{-1}(1)$. As g is continuous, $g(a)=1$; as $g(0)=0$, $a\neq 0$.

<u>Assertion</u> $$g\left[0, \frac{a}{2^k}\right) \subset \left[0, \frac{1}{2^k}\right), \quad \text{for } k=0,1,2,\ldots .$$

Indeed, assume this were false for a certain k; then there would exist an x, $0 < x < \frac{a}{2^k}$, such that $g(x) = \frac{1}{2^k}$. It would follow that

$$g \circ f_2^k(x) = f_2^k\, g(x) = 1;$$

as $f_2^k(x) < a$, this contradicts the definition of a.

We now are able to show that $g(x) < \frac{2x}{a}$ on $[0,a)$. For this is true on each of the intervals $\left[\frac{a}{2^{k+1}}, \frac{a}{2^k}\right)$, $k=0,1,2,\ldots$, and it holds trivially if $x=0$.

5. A maximal commutative semigroup of continuous mappings $I \to I$

Let M be the semigroup of all f_n, $n=0,1,2,\ldots$. We have seen already that M is commutative. We will now show that M is a maximal commutative semigroup of continuous mappings.

Theorem 4 The semigroup M is a maximal commutative semigroup of continuous mappings $I \longrightarrow I$. I.e. if $g: I \longrightarrow I$ is continuous, and $g \circ f_n = f_n \circ g$ for $n=0,1,2,\ldots$, then $g \in M$.

Proof

As $g \circ f_0 = f_0 \circ g$, we have $g(0)=0$. Then by proposition 7 the function $x^{-1}g(x)$ is bounded on I. Let k be the smallest non-negative integer such that $g(x) \leq kx$ for all $x \in I$; we will show: $g(x) = f_k$. This is trivial if $k=0$; hence we may assume $k > 0$.

Assertion 1 $g(x)$ is linear on $[0, \frac{1}{k}]$.
Let n be any natural number. Then $x \leq \frac{1}{kn}$ implies $g(x) \leq \frac{1}{n}$; hence

$$n.g(x) = f_n \circ g(x) = g \circ f_n(x) = g(nx)$$

for all $x \in [0, \frac{1}{nk}]$. Putting $nx=y$ we get

$$g\left(\frac{y}{n}\right) = \frac{1}{n}\, g(y)$$

for all $y \in [0, \frac{1}{k}]$. This holds for every natural number n.

Now consider any rational number $r=\frac{n}{m}\leq 1$. Then we see that

$$\frac{1}{m}\, g(y) = g(\frac{y}{m}) = \frac{1}{n}\, g(\frac{n}{m}\, y) \;;$$

hence

$$g(ry) = rg(y) \;.$$

As g is continuous, it follows that g is linear on $[0, \frac{1}{k}]$.

Assertion 2 $\quad g(x)= nx$ on $[0, \frac{1}{k}]$, where n is a suitable non-zero integer.

Let $g(x)=\alpha x$ on $[0, \frac{1}{k}]$. Then $g(\frac{1}{k})= \frac{\alpha}{k}$; hence $g(1)=g \circ f_k(\frac{1}{k})=f_k(\frac{\alpha}{k})$. Now $g(1)=0$ or $g(1)=1$, as $f_2 \circ g(1)= =g \circ f_2(1)=g(0)=0$. It follows that $f_k(\frac{\alpha}{k})$ is either 0 or 1, and this again implies that α is an integer n. As we assumed $k\neq 0$, it is not possible that $\alpha=0$. For in that case it would follow, for any $x \in I$:

$$g(x) = g \circ f_k\, (\frac{x}{k}) = f_k \circ g\, (\frac{x}{k}) = f_k(0)=0.$$

This proves the assertion.

Assertion 3 $\quad g(x)=f_k$.

We already know: $g(x)=nx$ on $[0, \frac{1}{k}]$, where n is an integer $\neq 0$. As $g(\frac{1}{k})= \frac{n}{k} \leq 1$, we have $n \leq k$. Take any x such that $\frac{1}{k}\leq x \leq \frac{1}{n}$; we will show that again $g(x)=nx$. Let s be a natural number such that $\frac{x}{n^s}<\frac{1}{k}$. The

$$g(x)=g \circ f_{n^s}\, (\frac{x}{n^s})=f_{n^s} \circ g(\frac{x}{n^s})=f_{n^s}(\frac{x}{n^{s-1}}) = nx.$$

Next we show that $g=f_n$. Take any $x \in [\frac{1}{n}, 1]$. Then $\frac{x}{n} \in [0, \frac{1}{n}]$; hence

$$g(x)=g \circ f_n\, (\frac{x}{n})=f_n \circ g\, (\frac{x}{n}) = f_n(x).$$

Finally from the fact that $g(x) \leq nx$ on I and the definition of k it follows that n=k.

Theorem 5 Let τ be any topoligical map of I onto itself. Then $\tau M \tau^{-1} := \{\tau f_n \tau^{-1} : n=0,1,2,\ldots\}$ is a maximal commutative semigroup of continuous mappings.

Proof

It is clear that $\tau M \tau^{-1}$ is a commutative semigroup. If F is a commutative semigroup of continuous mappings such that $F \supset \tau M \tau^{-1}$, then $\tau^{-1} F \tau$ is a commutative semigroup containing M; hence $\tau^{-1} F \tau = M$, or $F = \tau M \tau^{-1}$. Thus $\tau M \tau^{-1}$ is maximal.

If we take $\tau(x)=1-x$, we get the semigroup $\{\bar{f}_o, \bar{f}_1, \bar{f}_2, \ldots\}$, the first four members of which have graphs as indicated below.

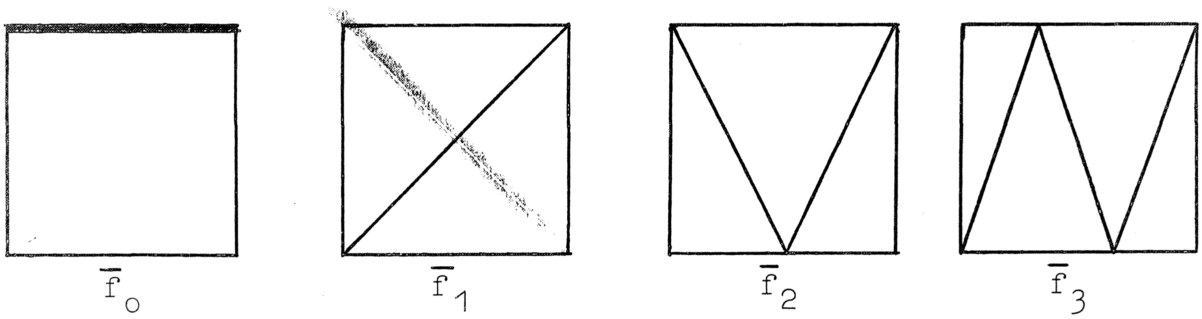

Fig. 4

It is also possible to obtain countable maximal semigroups of continuous mappings $I \longrightarrow I$, all elements of which are differentiable on (0,1). For instance, if $\tau(x) = \frac{1+\cos x}{2}$, then $\tau f_n(x) = \frac{1+\cos nx}{2}$, and hence $\tau f_n \tau^{-1}$ is differentiable on (0,1), for n=0,1,2,... .

Definition A mapping $f: I \longrightarrow I$ is called a topological multi-hat if it is of the form $f = \tau f_n \tau^{-1}$, for some $f_n \in M$ and some homeomorphism τ of I onto itself.

Every topological multi-hat is open. On the other hand, Z. Hedrlín pointed out to us that every continuous open map f of I onto itself is of the following form: there is a partition $0=a_0 < a_1 < a_2 < \dots < a_n = 1$ of I such that the values $f(a_i)$ are alternatingly 0 or 1, while f is strictly monotone on each interval $[a_i, a_{i+1}]$. Hence the graph of such an open continuous mapping f of I onto itself looks like a "topological" deformation of the graph of an f_n on an $\overline{f}_n$. However, not every open continuous f: $I \longrightarrow I$ (onto), say with $f(0)=0$, is a topological multi-hat; e.g. the mappings outlined in fig.5 are not.

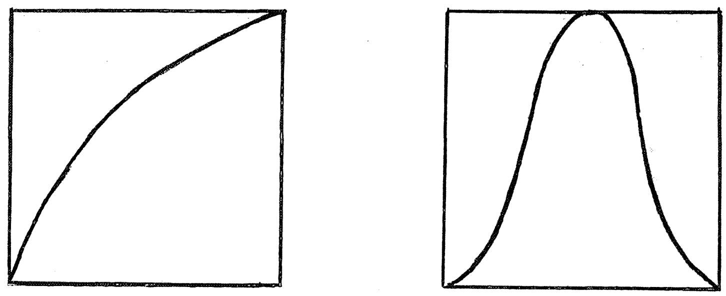

Fig. 5

For the first cannot be of the form $\tau f_1 \tau^{-1}$, as $\tau f_1 \tau^{-1} = f_1$; the second cannot be of the form $\tau f_2 \tau^{-1}$, as it has three fixed points where f_2 has only two of them.

It should be emphasized that all maximal commutative semigroups $\tau M \tau^{-1}$ are countable. If we drop the condition that the functions are commutative, then M is no longer maximal, and there are uncountable commutative semigroups of transformations: $I \longrightarrow I$, containing M.

The fact that M is no longer maximal if we drop the continuity condition follows at once from the fact that every f_n maps the rationals into the rationals, and the irrationals into the irrationals. Hence if we define $f: I \longrightarrow I$ as follows: $f(x)=f_2(x)$ for all rational x, and $f(x)=f_3(x)$ for all irrational x, then $f \notin M$, but f commutes with all functions in M.

If we want to show that M is contained in an uncountable commutative semigroup, we must be more precise. Let $x \in I$; we will write $M(x)$ for the set $\{f_n(x): n=0,1,2,\dots\}$. Suppose $M(x) \cap M(y)$ contain a point $z \neq 0$. Then $z=f^n(x)=f^m(y)$, for some natural numbers n and m. It follows that there is a rational algebraic relation between x and y.

Now there exists an uncountable subset S of I that is algebraically independent over the field of rationals. It follows that $M(x) \cap M(y)=\{0\}$ for $x,y \in S$, $x \neq y$. Because S is uncountable, there are uncountably many mappings $f: I \longrightarrow I$ with the following properties:

1) $f \mid I \setminus \bigcup_{x \in S} M(x) = f_n \mid I \setminus \bigcup_{x \in S} M(x)$, for some $f_n \subset M$;

2) for every $x \in S$ there exists an $f_n \in M$ such that

$$f \mid M(x) = f_n \mid M(x) .$$

It follows at once from the fact that $M(x) \cap M(y)=\{0\}$ if $x,y \in S$, $x \neq y$, that all these mappings f commute. Hence they generate an uncountable commutative semigroup of transformations $I \longrightarrow I$, and this semigroup obviously contains M.