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On the determinant of an asymmetric
hyperbolic region

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On the determinant of an asymmetric hyperbolic region.

by C. G. LEKKERKERKER (a Utrecht, Olanda).

dedicated to Prof. B. SEGRE.

Summary. - *A main problem in the geometry of numbers is the evaluation of the so-called determinant of various regions. The author derives a new estimate for the determinant of a certain two-dimensional region bounded by two hyperbolas and applies his result to a problem in the theory of automorphic star bodies.*

1. Introduction.

Let x, y be the Cartesian coordinates of a point in the plane. For given positive numbers a and b let $K_{a,b}$ denote the domain, determined by

$$K_{a,b} : -a \leq xy \leq b.$$

The main object of this note is to derive a new upper bound for the determinant of this domain.

We begin by recalling some of the usual definitions. Let M be an arbitrary domain in R_n and let Λ be any lattice in R_n , the determinant of which may be denoted by $d(\Lambda)$. The lattice Λ is called M -admissible, if no point of Λ , except for the origin, is an inner point of M . And the determinant of M , denoted by $\Delta(M)$, is defined as follows.

1°. if there exist M -admissible lattices, then $\Delta(M)$ is the lower bound of $d(\Lambda)$, taken over all M -admissible lattices.

2°. if no such lattice exists, then $\Delta(M) = \infty$.

The determinant of the two-dimensional domain $K_{a,b}$ will be denoted by $\Delta(a, b)$. For reasons of symmetry and homogeneity one has the obvious relations

$$(1) \quad \begin{aligned} \Delta(a, b) &= \Delta(b, a), \\ \Delta(ca, cb) &= c^2 \Delta(a, b) \quad \text{if } c > 0. \end{aligned}$$

Consequently, in order to evaluate or to estimate $\Delta(a, b)$, it is sufficient to consider the case

$$(2) \quad a = 1, \quad b \geq 1.$$

In the following we always suppose that (2) holds.

In the case $b = 1$ one has the classical result

$$(3) \quad \Delta(1, 1) = \sqrt{5}.$$

It is the merit of B. SEGRE to have first considered the more general, asymmetric case ⁽¹⁾. In the case (2) his result takes the following form.

THEOREM 1. - *If $b \geq 1$, then*

$$(4) \quad \Delta(1, b) \geq \sqrt{b^2 + 4b};$$

furthermore in (4) the equality sign holds if and only if b is a positive integer.

We shall prove here that the inequality (4) can be sharpened as follows.

THEOREM 2. - *Suppose $b \geq 1$. Let β be the smallest positive integer $\geq b$ and let σ denote the fraction $b/[\beta]$. Then we have*

$$(5) \quad \Delta(1, b) \geq \min \{ \sqrt{\beta^2 + 4b}, \sigma \sqrt{b^2 + 4b\sigma} \}.$$

The question whether or not the equality sign holds in (5) admits the following answer.

THEOREM 3. - *Let β and σ be defined as in theorem 2. Then in the relation (5) the equality sign holds if and only if either b satisfies the relation*

$$\sigma \sqrt{b^2 + 4b\sigma} \leq \sqrt{\beta^2 + 4b}$$

or the fraction $(\beta + 2)/(\beta + 1 - b)$ has an integral value.

We put $(\beta + 2)/(\beta + 1 - b) = q$, so that $b = \beta + 1 - \frac{1}{q}(\beta + 2)$. For a fixed value of β the number q is restricted to the interval $\frac{1}{2}(\beta + 2) < q \leq \beta + 2$. Consequently the condition that the fraction $(\beta + 2)/(\beta + 1 - b)$ is integral is equivalent with the following assertion:

there exist positive integers β and q such that

$$(6) \quad b = \beta + 1 - \frac{1}{q}(\beta + 2), \quad \frac{1}{2}(\beta + 2) < q \leq \beta + 2.$$

In particular b is of the form (6), if b is a positive integer. It is evident that the right hand sides of (4) and (5) are equal if and only if b is a positive integer, and that theorem 1 is included in theorems 2 and 3.

For the proof of theorems 2 and 3 it appears useful to consider a particular set of $K_{1,b}$ -admissible lattices. Let \mathfrak{B} be the boundary of $K_{1,b}$ and let $\mathfrak{B}_1, \mathfrak{B}_2$ be the parts of \mathfrak{B} which belong to the first and the second quadrant respectively. Further let P be the point $(-1, 1)$; clearly P lies on \mathfrak{B}_2 . Now we denote by $\mathcal{H}(P)$ the set of those $K_{1,b}$ -admissible lattices, which contain P as a lattice point. The principal part of the proof of theorems 2 and 3 consists in a proof of the following two lemmas (sections 2 and 3).

⁽¹⁾ B. SEGRE, *Lattice points in infinite domains and asymmetric diophantine approximations*, « Duke Mathem. Journal », 12 (1945), 337-365.

LEMMA 1. - Let $b \geq 1$ be arbitrary. If Λ is a lattice of the set $\mathcal{K}_b(P)$, then

$$(7) \quad d(\Lambda) \geq \sqrt{\beta^2 + 4b}.$$

LEMMA 2. - The set $\mathcal{K}_b(P)$ contains a lattice Λ_0 with

$$(7') \quad d(\Lambda_0) = \sqrt{\beta^2 + 4b},$$

if and only if the fraction $(\beta + 2)/(\beta + 1 - b)$ has an integral value.

The method of proof of lemma 1 is similar to a reasoning of K. OLLERENSHAW and C. A. ROGERS given in the symmetrical case $b = 1$ ⁽²⁾. We note that lemma 1 does not imply that the set $\mathcal{K}_b(P)$ is not empty. But this fact, which, by the way, is wellknown, is an immediate consequence of lemma 2. For this lemma implies that the set $\mathcal{K}_b(P)$ is not empty for some values of b , for instance all positive integral values. Now the set $\mathcal{K}_{b_1}(P)$ contains the set $\mathcal{K}_b(P)$ if b and b_1 are any positive numbers with $1 \leq b \leq b_1$. Hence $\mathcal{K}_b(P)$ is not empty for any value of $b \geq 1$ whatsoever.

Section 4 brings two further simple lemmas, whereafter in section 5 the proof of our theorems is given. In the last section we give an application to a problem in the theory of automorphic star bodies.

2. Proof of lemma 1.

Let Λ be an arbitrary lattice of the set $\mathcal{K}_b(P)$. The straight line through $O = (0, 0)$ and $P = (-1, 1)$ contains an infinity of lattice points with mutual distances $\sqrt{2}$. Consider the straight lines, which are parallel to this line and pass through a lattice point. Exactly one of these lines passes through the first quadrant and has a minimal distance to O . Call this line L . For any two points P_1 and P_2 of L we shall denote by $\delta(P_1, P_2)$ the difference between the abscissae of P_1 and P_2 . For points of L , which also belong to Λ , this quantity takes all integral values.

Let $y = \lambda - x$ be the equation of L and let S and T be the points of intersection of L and the hyperbola $xy = -1$ (see fig. 1). Clearly $\delta(S, T) > 2$. Hence, by the above remark, the interior of the line-segment \overline{ST} contains at least two points of Λ . Now, since Λ is $K_{1,b}$ -admissible, these points do not belong to the interior of $K_{1,b}$. As a consequence, L intersects the hyperbola $xy = b$, in two different points, Q and R say (see fig. 1). Furthermore the line segment \overline{QR} contains at least two lattice points. So we have

$$(8) \quad \delta(Q, R) \geq 1.$$

⁽²⁾ KATHLEEN OLLERENSHAW, *On the minima of indefinite quadratic forms*, • Journal London Math. Soc., 23 (1948), 148-153.

Solving for x and y the pair of equations

$$xy = -1, \quad y = \lambda - x,$$

and also the pair of equations

$$xy = b, \quad y = \lambda - x,$$

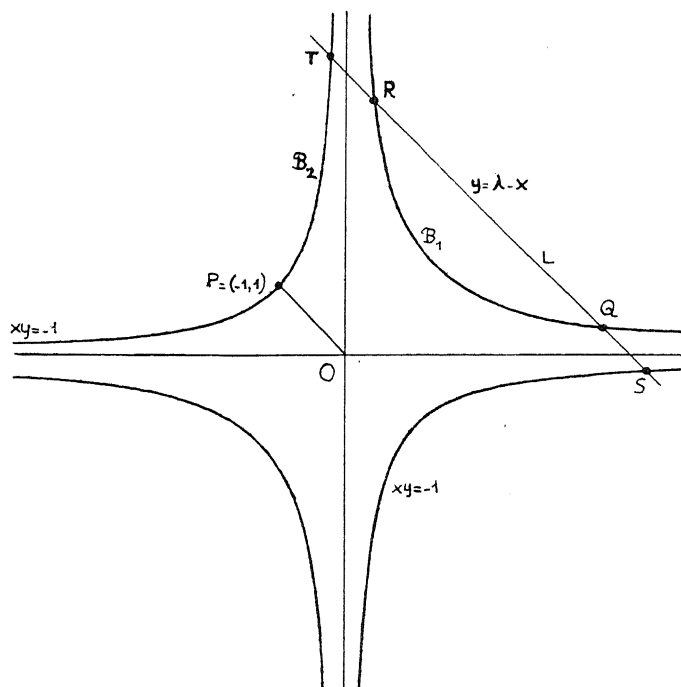


Fig. 1.

we find for the coordinates of the points S, T, Q, R

$$(9) \quad x_S = y_T = \frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 + 4}, \quad y_S = x_T = \frac{1}{2}\lambda - \frac{1}{2}\sqrt{\lambda^2 + 4},$$

$$(10) \quad x_Q = y_R = \frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 - 4b}, \quad y_Q = x_R = \frac{1}{2}\lambda - \frac{1}{2}\sqrt{\lambda^2 - 4b},$$

with obvious notations. Consequently

$$(11) \quad \delta(S, T) = \sqrt{\lambda^2 + 4}$$

$$(12) \quad \delta(Q, R) = \sqrt{\lambda^2 - 4b}.$$

Let us now suppose that k is a positive integer with the following properties:

- 1° $1 \leq k < \beta$
- 2° $\delta(Q, R) \geq k$.

Then it follows from (12) that we have

$$\lambda^2 \leq k^2 + 4b.$$

From the definition of β and the requirement 1° we learn that:

$$k < b;$$

hence, using (11), we deduce

$$\delta(S, T) \geq \sqrt{k^2 + 4b + 4} > \sqrt{k^2 + 4k + 4},$$

and so

$$\delta(S, T) > k + 2.$$

Consequently, the interior of the line-segment \overline{ST} contains at least $k + 2$ points of \wedge . Since \wedge is $K_{1,b}$ -admissible, these points must all belong to the segment \overline{QR} .

This leads to

$$\delta(Q, R) \geq k + 1.$$

Clearly, in virtue of (8), the requirements 1° and 2° are fulfilled with $k = 1$, unless $b = 1$. So, by a repeated application of the above reasoning, we conclude that:

$$(13) \quad \delta(Q, R) \geq \beta,$$

the closed line-segment \overline{QR} containing at least $\beta + 1$ lattice points. In view of (12), the last formula is equivalent with

$$(14) \quad \lambda \geq \sqrt{\beta^2 + 4b}.$$

The determinant $d(\wedge)$ is easily found to be equal to λ . So, finally, (14) is equivalent with (7). This proves lemma 1.

3. Proof of lemma 2.

We begin with some preliminary remarks. Let λ be determined by

$$(15) \quad \lambda = \sqrt{\beta^2 + 4b},$$

and let L be the line with equation $y = \lambda - x$. Denote by S , T and Q , R the points of intersection of L and the hyperbola $xy = -1$, $xy = b$ respectively (see fig. 2). Further let P^* be the point $(1, -1)$, which is the reflection of P in the origin and also lies on \mathfrak{B} . As in the proof of lemma 1, the coordinates of the points S , T , Q , R are given by (9) and (10), with λ given by (15). In particular we have

$$\begin{aligned} 0 < x_R &= \frac{1}{2} \sqrt{\beta^2 + 4b} - \frac{1}{2} \beta \\ &< \frac{1}{2} \sqrt{\beta^2 + 4\beta + 4} - \frac{1}{2} \beta = 1, \end{aligned}$$

hence

$$0 < x_R < x_{P^*}.$$

Consequently, the straight line through P and R , M say, intersects \mathcal{B}_1 one further point and \mathcal{B}_2 in exactly one point; call these points R_1 and successively (see fig. 2).

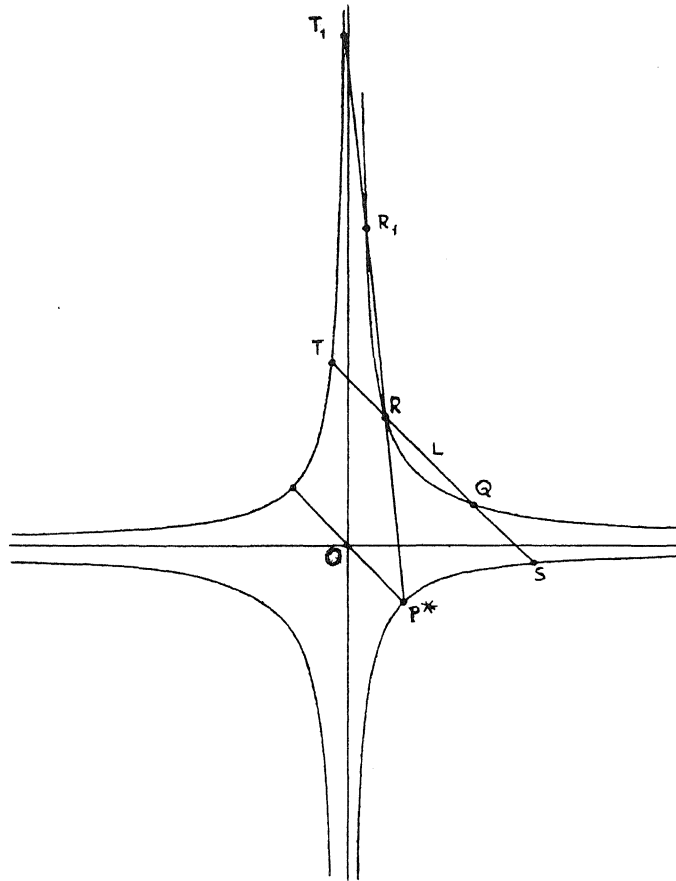


Fig. 2.

We wish to determine these points R_1 and T_1 . An arbitrary point Z on the line M can be written as

$$(16) \quad Z = P^* + t(R - P^*),$$

where t runs through all real numbers. This point lies on the hyperbola $xy = b$, if and only if t satisfies the equation

$$\{1 + t(x_R - 1)\} \cdot \{-1 + t(y_R + 1)\} = b.$$

On account of (10) and (15) this relation reduces to

$$\left\{ 1 + t \left(\frac{1}{2} \sqrt{\beta^2 + 4b} - \frac{1}{2} \beta - 1 \right) \right\} \cdot \left\{ -1 + t \left(\frac{1}{2} \sqrt{\beta^2 + 4b} + \frac{1}{2} \beta + 1 \right) \right\} = b,$$

i. e.

$$(\beta + 1 - b)t^2 - (\beta + 2)t + (b + 1) = 0.$$

One solution of this equation is given by $t = 1$, corresponding with the point $P^* + (R - P^*) = R$; the other solution is

$$t = (b + 1)/(\beta + 1 - b).$$

Clearly this number is greater than 1. We find that R_i is given by

$$(17) \quad R_i = P^* + \frac{b + 1}{\beta + 1 - b} \cdot (R - P^*)$$

and that R_i and P^* lie on opposite sides of R .

As a fact of elementary geometry, the line-segments \overline{PR} and $\overline{R_iT_i}$, cut off from the line M by the hyperbolas $xy = -1$ and $xy = b$, have equal lengths. Hence we find

$$(18) \quad T_i = R_i + (R - P^*) = P^* + \frac{\beta + 2}{\beta + 1 - b} \cdot (R - P^*).$$

Now suppose that there exists a lattice Λ_0 in the set $\mathcal{K}_b(P)$, for which (7') holds. Then, on account of (15) and (7'), this lattice is generated by P and some point of L . The lattice points on L have mutual distances $\sqrt{2}$. For the points Q, R, S, T we have, on account of (9), (10) and (15),

$$\delta(Q, R) = \beta, \quad \delta(S, T) = \sqrt{\beta^2 + 4b + 4} > \beta + 1.$$

By the same argument as used in the proof of lemma 1, it follows that the closed line-segment \overline{QR} contains at least $\beta + 1$ lattice points. Consequently Q and R are lattice points. Now, turning our attention to the line M , we first remark that the open line-segments \overline{PR} and $\overline{R_iT_i}$ belong entirely to the interior of $K_{1,b}$ and so are free from lattice points. But P and R belong to Λ_0 . It follows that a point Z of the form (16) belongs to Λ_0 if (and only if) t is integral. From this we infer, taking into account the first half of the relation (18), that R_i and T_i are points of Λ_0 . Henceforth, by (18), the fraction $(\beta + 2)/(\beta + 1 - b)$ has an integral value.

Conversely suppose that $q = (\beta + 2)/(\beta + 1 - b)$ is integral. Consider the lattice Λ_0 , generated by P and R . We shall prove that Λ_0 is $K_{1,b}$ -admissible.

Since β is integral, from $\delta(Q, R) = \beta$ it follows that Q is a point of Λ_0 . On account of (17) and (18) also R_i and T_i are points of Λ_0 . Further P^* is a point of Λ_0 . Now we have

$$\delta(S, T) = \sqrt{\beta^2 + 4b + 4} \leq \sqrt{\beta^2 + 4\beta + 4} = \beta + 2,$$

hence

$$\delta(R, T) = \frac{1}{2} \{ \delta(S, T) - \delta(Q, {}^3R) \} \leq 1;$$

so T does not belong to the interior of $K_{1,b}$.

Let Σ_0, Σ_1 be the closed line-segments $\overline{PQ}, \overline{R_1T_1}$ respectively, and denote by \mathcal{K}_0 the closed part of $K_{1,b}$ contained between Σ_0 and Σ_1 . We split \mathcal{K}_0 into two parts, viz. the intersections of \mathcal{K}_0 and the triangles PQT and TRT_1 respectively. Since the pair of points R, P and likewise the pair R, P^* constitute a basis for Λ_0 , it is clear that the interior of each of these strips is free from lattice points. The points of Λ_0 which lie on the boundary of one of these strips are easily identified. One arrives at the conclusion that the only lattice points which belong to Λ_0 are lying on the boundary \mathcal{B} of $K_{1,b}$.

Next, let t be the ordinate of T_1 and let Ω be the affine transformation determined by

$$(19) \quad \Omega : x' = t^{-1}x, \quad y' = ty.$$

The domain $K_{1,b}$ is left invariant under this transformation, as well as the boundary \mathcal{B} . We further have

$$\Omega P = T_1.$$

We denote for a moment by R_1^*, T_1^*, M^* the reflections into the line $y=x$ of R_1, T_1, M respectively. Clearly $\Omega^{-1}P^* = T_1^*$, i. e. $\Omega T_1^* = P^*$. Hence the line M^* is transformed into M by the transformation Ω . Then the same is true for the points of intersection of these lines and the hyperbola $xy=b$. In particular, we find

$$\Omega Q = R_1.$$

As a consequence the line-segment Σ_0 is transformed by Ω into Σ_1 . We now put

$$\mathcal{K}_m = \Omega^m \mathcal{K}_0,$$

where m runs through all integers and the right hand side is defined in an obvious way. Clearly the union of all domains \mathcal{K}_m coincides with the part of $K_{1,b}$ above the x -axis.

The pair of points P, R , as well as the pair of points P, Q , constitutes a basis for Λ_0 . The same is true for the pair of points $\Omega P = T_1, \Omega Q = R_1$ of Λ_0 . For, on account of (18) and the definition of the positive integer q , we have

$$-T_1 + q(T_1 - R_1) = -T_1 + q(R - P^*) = -P^* = P,$$

$$T_1 - (q-1)(T_1 - R_1) = T_1 - (q-1)(R - P^*) = P^* + (R - P^*) = R.$$

Consequently, not only the domain $K_{1,b}$ and its boundary \mathcal{B} , but also the lattice Λ_0 , is left invariant under the transformation Ω . Hence, since

the lattice points, which belong to \mathcal{K}_0 , are lying on \mathcal{B} , the same is true for each domain \mathcal{K}_m . Hence the region bounded by \mathcal{B}_1 , \mathcal{B}_2 and the x -axis is free from lattice points.

Finally we remark that $K_{1,b}$ as well as Λ_0 are symmetric with respect to the origin and also with respect to the line $y=x$. It follows that the origin is the only point of Λ_0 interior to $K_{1,b}$, i. e. that Λ_0 is $K_{1,b}$ -admissible.

This completes the proof of lemma 2.

4. Some further lemmas.

The set of $K_{1,b}$ -admissible lattices is not empty, as we have proved that even the set $\mathcal{K}_b(P)$ is not empty. It follows from the general theory of star bodies that there exists a so-called critical lattice of $K_{1,b}$, i. e. a $K_{1,b}$ -admissible lattice Λ' with determinant $d(\Lambda') = \Delta(1, b)$ ⁽³⁾. An analogous property holds for certain subsets of the set of $K_{1,b}$ -admissible lattices. In fact we shall prove the following

LEMMA 3. - Let $\mathcal{K}_b(\mathcal{B}_2)$ be the set of $K_{1,b}$ -admissible lattices Λ with

$$\inf_{xy < 0, (x, y) \in \Lambda} |xy| = 1.$$

Put

$$(20) \quad \Delta^*(1, b) = \inf_{\Lambda \in \mathcal{K}_b(\mathcal{B}_2)} d(\Lambda).$$

Then there exists a $K_{1,b}$ -admissible lattice Λ_0 with

$$P = (1, 1) \in \Lambda_0, \quad d(\Lambda_0) = \Delta^*(1, b).$$

REMARK. - Clearly a lattice Λ_0 with the above property belongs to the set $\mathcal{K}_b(\mathcal{B}_2)$. In general, however, a lattice of this set may not contain any point of the curve \mathcal{B}_2 .

Proof. - Each transformation Ω of the type (19) leaves $K_{1,b}$ invariant and so transforms any $K_{1,b}$ -admissible lattice into another $K_{1,b}$ -admissible lattice. By suitable choice of t we can obtain that an arbitrarily chosen point S of any lattice Λ is transformed by Ω into some point of the line $y = -x$. So, on account of the definitions of the quantity $\Delta^*(1, b)$ and the set $\mathcal{K}_b(\mathcal{B}_2)$, it is clear that there exist a sequence of lattices Λ_n and a sequence of positive numbers a_n ($n = 1, 2, \dots$) such that:

- a) $\Lambda_n \in \mathcal{K}_b(\mathcal{B}_2)$ for $n = 1, 2, \dots$ and $d(\Lambda_n) \rightarrow \Delta^*(1, b)$ as $n \rightarrow \infty$,
- b) $a_n \geq 1$ for $n = 1, 2, \dots$ and $a_n \rightarrow 1$ as $n \rightarrow \infty$,
- c) $A_n = (-a_n, a_n) \in \Lambda_n$.

For each $n = 1, 2, \dots$ there certainly exists a point B_n in the first quadrant, such that B_n is a point of Λ_n and furthermore A_n and B_n constitute a basis of Λ_n . The sequence of points B_n is restricted to some bounded part of the plane. Hence we can select a subsequence $\{B_{n_k}\}$ which converges to some point B . On the other hand, the points A_n tend to the point $P = (-1, 1)$.

Let Λ_0 be the lattice generated by P and B . It follows from a wellknown theorem of MAHLER ⁽³⁾ that Λ_0 is $K_{1,b}$ -admissible, whereas we have $d(\Lambda_0) = \lim_{n \rightarrow \infty} d(\Lambda_n)$. The lattice Λ_0 possesses therefore the desired properties.

It may be remarked that lemma 3 can easily be generalized in the following way.

LEMMA 3'. - *Let S be a n -dimensional automorphic star body and let Γ be its group of automorphisms. Let \mathcal{B}^* be a closed part of its boundary, which is left invariant under the transformations of Γ . Further let $\mathcal{K}(\mathcal{B}^*)$ be the set of S -admissible lattices which have points arbitrarily near to \mathcal{B}^* and suppose that $\mathcal{K}(\mathcal{B}^*)$ is not empty. Then there exists a lattice Λ_0 with*

$$d(\Lambda_0) = \inf_{\Lambda \in \mathcal{K}(\mathcal{B}^*)} d(\Lambda),$$

which is S -admissible and contains a point of \mathcal{B}^* .

We conclude this section with the following

LEMMA 4. - *Let b be a real number ≥ 1 . For $z > 0$ denote by $\psi(z)$ the least positive integer $\geq z$. Then in the interval $1 \leq x \leq b$ the function $f(x)$, defined by*

$$f(x) = x^2 \sqrt{\{\psi(b/x)\}^2 + 4b/x},$$

takes its minimal value in one or both points $x=1$, $x=b/[b]$ and not elsewhere.

Proof. - Let x_1, x_2, \dots, x_k be the points of the interval $1 \leq x \leq b$, where b/x is integral. The function $f(x)$ is positive in the whole interval $1 \leq x \leq b$ and continuous and monotonously increasing in each point $x \neq x_\nu$ ($\nu = 1, 2, \dots, k$). In each point $x = x_\nu$ it undergoes a negative jump and there takes as value the right hand limit in that point. Furthermore, if $k \geq 2$ and $x_\nu > x_\mu$, we have

$$\begin{aligned} f(x_\nu)^2 &= x_\nu^2 \cdot (b^2 + 4bx_\nu) \\ &> x_\mu^2 \cdot (b^2 + 4bx_\mu) = f(x_\mu)^2, \end{aligned}$$

hence $f(x_\nu) > f(x_\mu)$.

It follows that $f(x)$ attains its lower bound in one or both points $x=1$, $x=b/[b]$, and not elsewhere.

We remark that, if we put $b/[b] = \sigma$ and denote by β the least positive integer $\geq b$, the values $f(1)$ and $f(b/[b])$ are given by

$$(21) \quad \begin{cases} f(1) = \sqrt{\beta^2 + 4b} \\ f(b/[b]) = (b/[b])^2 \sqrt{[b]^2 + 4[b]} = \sigma \sqrt{b^2 + 4b\sigma}. \end{cases}$$

⁽³⁾ K. MAHLER, *On lattice points in n -dimensional star bodies I. Existence theorems*, Proc. Royal Soc., A, 187 (1946), 151-187, in particular theorem 8, p. 159 and theorem 19, p. 173.

5. Proof of theorems 2 and 3.

We begin with a preliminary consideration. Let Λ be a $K_{1,b}$ -admissible lattice. Put

$$\alpha = \inf_{xy < 0, (x, y) \in \Lambda} |xy|.$$

Clearly $\alpha \geq 1$.

First suppose $\alpha \leq b$. By the transformation

$$x' = \alpha^{-1}x, \quad y' = \alpha y$$

Λ is transformed into a lattice Λ' with the following properties:

a) Λ' is admissible for the domain $K_{1,b/\alpha}$,

b) $\inf_{xy < 0, (x, y) \in \Lambda'} |xy| = 1$,

c) $d(\Lambda') = \alpha^{-2}d(\Lambda)$.

Hence Λ' belongs to the set $\mathcal{H}_{b/a}(\mathcal{B}_2)$, defined with respect to the domain $K_{1,b/a}$. In virtue of lemma 3, there exists a $K_{1,b/a}$ -admissible lattice Λ_0 which contains $P = (-1, 1)$ and has determinant $d(\Lambda_0) = \Delta^*(1, b/a)$, hence $d(\Lambda_0) \leq d(\Lambda')$. To the lattice Λ_0 we can apply lemma 1. So we get, recalling the definitions of $\psi(z)$ and $f(z)$,

$$d(\Lambda_0) \geq \sqrt{\{\psi(b/a)\}^2 + 4b/a},$$

hence

$$(22) \quad d(\Lambda) = \alpha^2 d(\Lambda') \geq \alpha^2 \sqrt{\{\psi(b/a)\}^2 + 4b/a} = f(\alpha).$$

Next suppose $\alpha > b$. Then Λ is also $K_{b,b}$ -admissible. Now by (3) (which result could be obtained with the help of lemmas 1 and 2) we have $\Delta(1, 1) = \sqrt{5}$. So by (1) we have

$$\Delta(b, b) = b^2 \Delta(1, 1) = b^2 \sqrt{5},$$

hence

$$(23) \quad d(\Lambda) \geq \Delta(b, b) \geq b\sigma\sqrt{5} \geq \sigma\sqrt{b^2 + 4b\sigma}.$$

Theorem 2 now follows at once. For, consider any $K_{1,b}$ -admissible lattice Λ , and define α as above. In the case $\alpha \leq b$, on account of (22) and lemma 4, the number $d(\Lambda)$ can be minorized as follows:

$$d(\Lambda) \geq \min \{ f(1), f(b/[b]) \};$$

hence, from (21),

$$d(\Lambda) \geq \min \{ \sqrt{\beta^2 + 4b}, \sigma\sqrt{b^2 + 4b\sigma} \}.$$

Clearly, on account of (23), the last relation also holds in the case $\alpha > b$. The same estimate holds for $\Delta(1, b)$, Λ being arbitrary. This proves theorem 2.

In order to prove theorem 3, we first treat the case

$$(24) \quad \sigma\sqrt{b^2 + 4b\sigma} \leq \sqrt{\beta^2 + 4b}.$$

Since b/σ is integral, by lemma 2 there certainly exists a $K_{1,b/\sigma}$ -admissible lattice Λ_0 with determinant $d(\Lambda_0) = \sqrt{(b/\sigma)^2 + 4b/\sigma}$. Hence there exists a $K_{\sigma,b}$ -admissible lattice Λ' with determinant $d(\Lambda') = \sigma^2 \sqrt{(b/\sigma)^2 + 4b/\sigma} = \sigma\sqrt{b^2 + 4b\sigma}$. This lattice is also $K_{1,b}$ -admissible. It follows from theorem 2 that, in the actual case, theorem 3 is valid.

Next we consider the case

$$(25) \quad \sigma\sqrt{b^2 + 4b\sigma} > \sqrt{\beta^2 + 4b}.$$

This implies, on account of lemma 4 and the relations (21),

$$f(x) > f(1) \quad \text{if } x > 1.$$

Assume that in (5) the equality sign holds, and consider a critical lattice Λ of $K_{1,b}$. Let a be defined as in the beginning of this section. If we had $a > b$, then we might apply (23); so by (25) we should have $d(\Lambda) > \sqrt{\beta^2 + 4b}$, contrary to our assumption. Hence we have $a \leq b$ and we may apply (22). Hence $d(\Lambda) \geq f(a)$. But, by assumption, we have $d(\Lambda) = \sqrt{\beta^2 + 4b} = f(1)$. Hence, in virtue of the above remark, a is equal to 1. Then lemma 3 learns us that there even exists a $K_{1,b}$ -admissible lattice Λ_0 with

$$P = (-1, 1) \in \Lambda_0, \quad d(\Lambda_0) = \Delta(1, b) = \sqrt{\beta^2 + 4b}.$$

Applying lemma 2 we conclude that the fraction $(\beta + 2)/(\beta + 1 - b)$ is integral.

Conversely, assume that this fraction is integral. Then by lemma 2 there exists a certain $K_{1,b}$ -admissible lattice Λ with determinant $d(\Lambda) = \sqrt{\beta^2 + 4b}$. Hence in (5) the equality sign holds.

This completes the proof of theorem 3.

6. Application to a problem of Mahler.

Let Γ be the group of the affine transformations Ω of the type (17). Let us denote by $|X|$ the distance of any point X from the origin. The group Γ has the following properties:

- a) $K_{1,b}$ is left invariant under each transformation Ω in Γ ;
- b) if X is any point of the domain $K_{1,b}$, then there exists a transformation Ω in Γ such that $|\Omega X|$ is smaller than a fixed number c (for instance $c = b$);
- c) if d is any positive number and X is any point of $K_{1,b}$, then there exists a transformation Ω in Γ with $|\Omega X| > d$.

With the usual terminology, we can therefore say that $K_{1,b}$ is a *full automorphic star body* ⁽⁴⁾.

We consider now the set of numbers b with

$$(26) \quad 1 < b < 2, \quad \sigma\sqrt{b^2 + 4b\sigma} < \sqrt{\beta^2 + 4b}.$$

Since in this case we have $\sigma = b$, $\beta = 2$, the last inequality reduces to $b^2\sqrt{5} < 2\sqrt{1+b}$. Consequently, the above set is the interval $1 < b < b_0$, where b_0 is determined by

$$(27) \quad b_0^4 = \frac{4}{5}(1 + b_0), \quad b_0 > 1.$$

Further let s be any positive number and let $K_{1,b}^s$ be the set of points X with

$$X \in K_{1,b}, \quad |X| \leq s.$$

Let us consider a critical lattice \wedge and let a be defined as in section 5. From theorem 3 and (26) it follows that we have $d(\wedge) = \sigma\sqrt{b^2 + 4b\sigma} = b^2\sqrt{5}$. Suppose $a > b$. Then we have $d(\wedge) \geq \Delta(a, b)$, hence, on account of the relations (1), $d(\wedge) \geq b^2\Delta(1, a/b)$. Here a/b is > 1 . Now the right hand side of the inequality (5) is a strictly increasing function of b . Consequently we have $d(\wedge) > b^2\Delta(1, 1) = b^2\sqrt{5}$. This is a contradiction, and so we must have $a \leq b$. Then we can apply the relation (22). This gives $d(\wedge) \geq f(a)$. On account of lemma 4 and the relations (21) and (26) the function $f(x)$, defined in lemma 4, takes its minimal value in the interval $1 \leq x \leq b$ in the point $x = b/[b]$ and not elsewhere. Since in our case $[b] = 1$, it follows $a = b$. This result may be stated in the following equivalent form:

If $1 < b < b_0$, then each critical lattice of $K_{1,b}$ is also a critical lattice of $K_{b,b}$.

In particular, the part \mathfrak{B}_2 of the boundary of $K_{1,b}$ does not contain a point of any critical lattice of $K_{1,b}$. Consequently, for each critical lattice of $K_{1,b}$ and each value of $s > 0$ there exist lattices with a smaller determinant which are infinitely near to it and are admissible for the domain $K_{1,b}^s$. In particular, we see that:

$$\text{If } 1 < b < b_0, \text{ then} \\ \Delta(K_{1,b}^s) < \Delta(K_{1,b}) = \Delta(1, b) \text{ for each } s > 0.$$

Using the appropriate terminology, we can say that $K_{1,b}$ is *boundedly irreducible* if $1 < b < b_0$. So we have found a star body, which at the same time is full automorphic and boundedly irreducible. Thus is answered one of the problems raised by MAHLER ⁽⁵⁾.

⁽⁴⁾ H. DAVENPORT-C. A. ROGERS, *Diophantine inequalities with an infinity of solutions*, « Phil. Transactions Royal Soc. London », 242 (1950), 311-344.

⁽⁵⁾ K. MAHLER, *Lattice points in n -dimensional star bodies II. Reducibility theorems*, « Proc. Kon. Ned. Akad. v. Wet. », 49 (1946), in particular p. 629.

We remark that another example of a star body of the above type is given by J. W. S. CASSELS ⁽⁶⁾. CASSELS also states in a footnote of the paper cited that ROGERS-DAVENPORT have answered the problem, considering the domain $K_{1,b}$ for positive irrational b (for $b = b_0$, however, $K_{1,b}$ is boundedly reducible, as is immediately deduced from our considerations).

One could easily establish an analogous result for other b -intervals. But, since our knowledge of the quantity $\Delta(1, b)$ is yet imperfect, we can not determine them all and so we content ourselves with the result given above.

⁽⁶⁾ J. W. S. CASSELS, *On two problems of Mahler*, «Proc. Kon. Ned. Akad. v. Wet.», 51 (1948), 282-285.

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