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COMPINGENT LATTICES AND ALGEBRAS

MATHEMATICS

BY

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1. *Introduction*

In this paper we discuss the relations between R-lattices as introduced by T. SHIROTA and compingent algebras as defined by H. DE VRIES. The concept of an R-lattice was applied by SHIROTA in a paper dealing with I. Kaplansky's theorem on the characterization of a compact space X by its function lattice $C(X)$; it enabled him to give elegant proofs of several generalizations of this theorem. DE VRIES, on the other hand, set himself the task of generalizing the Stone theory about the duality between zero-dimensional compact Hausdorff spaces and Boolean algebras to the wider field of arbitrary compact Hausdorff spaces. The concept of a compingent algebra, developed by him in the course of these investigations, turned out to be a useful tool for the study of compactifications of completely regular spaces and related subjects.

We show in section 3 that there is a close connection between the two concepts. Every compingent algebra is an R-lattice, and an R-lattice is a compingent algebra as soon as it is complemented. Moreover, every R-lattice S can be considered as an unfinished compingent algebra: it can be isomorphically embedded in a compingent algebra B in such a way that B is generated by S (theorems 3.4 and 3.7). For these reasons, we prefer the designation "compingent lattice" to Shirota's "R-lattice".

In order to put into light the respective merits of compingent lattices and algebras, we thought it useful to give an idea – of necessity in a very short and incomplete way – in what manner these concepts have been used up to now (sections 2 and 4).

In an appendix it is shown that the axiom system for compingent lattices as presented here is logically independent.

2. *Compingent algebras*

In a recent thesis, entitled "Compact spaces and compactifications" [12], H. DE VRIES introduced the notion of a compingent algebra. Such an algebra can be considered as a Boolean algebra provided with a supplementary relation. The exact definition reads as follows.

2.1. Definition. A *compingent algebra* B is a Boolean algebra, in which there is defined a binary relation $\ll$ satisfying the following conditions:

$P1.$ $0 \ll 0$;

$P2.$ $a \ll b \Rightarrow a \leqslant b$;

$P3.$ $a \leqslant a' \ll b \Rightarrow a \ll b$;

$P4.$ $a \ll b,\ c \ll d \Rightarrow a \wedge c \ll b \wedge d$;

$P5.$ $a \ll b \Rightarrow b^c \ll a^c$;

$P6.$ $a \ll b \neq 0 \Rightarrow (\exists c \neq 0)\ (a \ll c \ll b)$.

(By x^c we mean the Boolean complement of x. We recall that by definition every Boolean algebra contains at least two distinct elements.)

Important examples of compingent algebras can be obtained in the following way. Let X be a completely regular space, and let $B(X)$ be the Boolean algebra of all regularly open subsets of X. Then $B(X)$ can be made a compingent algebra, defining the relation $\ll$ by

(1) $$O_1 \ll O_2 \Leftrightarrow O_1 \text{ and } O_2^c \text{ are functionally separated,}$$

for arbitrary $O_1, O_2 \in B(X)$. In case X is normal, $O_1 \ll O_2$ is equivalent to $\bar{O}_1 \subset O_2$.

The importance of the algebras $B(X)$ is put into light by the following representation theorem: "Every compingent algebra is isomorphic to a subalgebra of an algebra $B(X)$, X being a suitable compact Hausdorff space (hence a completely regular space).

Thus, there is a connection between topological spaces and compingent algebras. This connection works in both directions, as is suggested already by the representation theorem quoted above: from every compingent algebra B a topological space $\mathfrak{M}_B$ can be obtained in a way, similar to the usual method by which the dual space of a Boolean algebra is constructed.

In fact, a Boolean algebra B can be considered as a compingent algebra, in which the compingent relation coincides with the ordering:

$$a \ll b \Leftrightarrow a \leqslant b.$$

It is well-known from the work of M. H. Stone (see e.g. [11], or also [2]) that every Boolean algebra is isomorphic to an algebra $B(X)$, where X is a zero-dimensional compact Hausdorff space. For X one can take the space $\mathfrak{M}_B$ of all maximal filters of B, supplied with a suitable topology.

If B is an arbitrary compingent algebra, $\mathfrak{M}_B$ is defined, not as the set of *all* maximal filters of B, but as the set of all maximal *concordant* filters of B, the concordant filters being defined in the following way.

2.2. Definition. A filter F of a compingent algebra B is called *concordant* if it has the following property:

(2) $$a, b \in F \Rightarrow (\exists c \in F)\ (c \ll a \wedge b).$$

If the set $\mathfrak{M}_B$ of all maximal concordant filters is supplied with the topology generated by the sets $\omega(a) = \{F : a \in F \in \mathfrak{M}_B\}$, $a \in B$, it can be shown that $\mathfrak{M}_B$ is a compact Hausdorff space, and also that B is isomorphic to a subalgebra of $B(\mathfrak{M}_B)$.

Many more results have been obtained. In the first place *every* compact Hausdorff space X turns out to be homeomorphic to a space $\mathfrak{M}_B$, where B is a complete compingent algebra (a compingent algebra is called complete if the underlying Boolean algebra is complete). If B_1 and B_2 are complete compingent algebras, then B_1 and B_2 are isomorphic iff the spaces $\mathfrak{M}_{B_1}$ and $\mathfrak{M}_{B_2}$ are homeomorphic. Similarly, if X_1 and X_2 are compact Hausdorff spaces, then X_1 and X_2 are homeomorphic iff $B(X_1)$ and $B(X_2)$ are isomorphic.

As a matter of fact there is a fairly complete duality between the theory of (complete) compingent algebras and the theory of compact Hausdorff spaces. Continuous maps of $\mathfrak{M}_{B_1}$ into $\mathfrak{M}_{B_2}$ correspond to certain homomorphisms (the so-called chary ones; see [12] def. 1.5.1.) of B_2 into B_1; moreover, if B_1 is complete, this correspondence is one-to-one. If φ is a continuous map of $\mathfrak{M}_{B_1}$ *onto* $\mathfrak{M}_{B_2}$, the corresponding homomorphism h of B_2 into B_1 is an *isomorphism*, and conversely. Similarly, φ is topological iff the corresponding homomorphism h maps B_2 onto a dense subalgebra of B_1 (cf. [12], section 1.7).

This duality affords us a handy algebraic tool to cope with compact Hausdorff spaces. More generally, the theory of compingent algebras proves to be very useful for the study of compactifications.

We mentioned already that every completely regular space X leads to a compingent algebra $B(X)$. The space $\mathfrak{M}_B$, constructed from $B = B(X)$, is compact, whether X is compact or not. It is natural to ask if there is any connection between X and $\mathfrak{M}_B$, and it is not unnatural to hope that $\mathfrak{M}_B$ will be a compactification of X. And indeed this turns out to be the case.

The situation is even more delightful: as is shown by H. DE VRIES ([12], theorem 2.2.4), *every* possible compactification of X is topologically equivalent to a compactification $(\mu_B, \mathfrak{M}_B)$, constructed from a suitable subalgebra B of $B(X)$ (μ_B designates the canonical embedding of X into $\mathfrak{M}_B$). Thus by means of the theory of compingent algebras one can keep track of all possible compactifications of a given completely regular space, and these methods may be used to construct compactifications with prescribed properties in a rather easy way. In this manner H. DE VRIES derives some important results concerning compactifications of normal spaces (see e.g. [12] theorems 4.1.5., 4.2.1., 4.3.2. and 4.3.5.).

The considerations above might lead to the impression that the theory of compingent algebras will serve to solve all problems concerning compact spaces. There are some indications, however, that run counter to this optimistic point of view.

Relevant in this respect are some of the older theorems interrelating topology and algebra. In 1939 I. GEL'FAND and A. N. KOLMOGOROV proved that a compact Hausdorff space X is completely determined (up to homeomorphism) by the ring $C(X)$ of all real-valued continuous functions on X ([3]). I. KAPLANSKY [6] proved in 1947 that the lattice $C(X)$ does already determine X, whereas A. N. MILGRAM [7], on the other hand, showed in 1949 that X is determined by the multiplicative semigroup of the ring $C(X)$.

Let us try to prove, say, Milgram's theorem by means of the theory of compingent algebras. Then the following method presents itself: starting with the semigroup $C(X)$, to construct a compingent algebra B such that X is homeomorphic to $\mathfrak{M}_B$. By [12], theorem 1.4.4., $\mathfrak{M}_B$ and X will indeed be homeomorphic as soon as B is isomorphic to a subalgebra B' of $B(X)$, such that its elements form a base for the topology of X.

If $f \in C(X)$, let $P(f)$ be the set

$$P(f) = \{x \in X : f(x) \neq 0\},$$

and let $R(f) = \overline{P(f)}^0$, the regular interior of $P(f)$. The set $R(X) = \{R(f) : f \in C(X)\}$ is a base for the topology of X (this is true already if X is just a completely regular space; cf. [4] theorem 3.2). It is easily seen, moreover, that $R(X)$ is a sublattice of $B(X)$.

This is promising indeed. But it will be of no avail if we do not succeed in exhibiting an isomorphism between the lattice $R(X)$, and some structure, derived in a purely algebraic way from the semigroup $C(X)$. Fortunately, this can be done: in [10] T. SHIROTA proves that if f and g belong to $C_+(X)$ – the subsemigroup of $C(X)$ consisting of all squares h^2, $h \in C(X)$ – then

$$R(f) = R(g) \Leftrightarrow f < g \ \& \ g < f, \tag{3}$$

where $<$ is the – truly algebraical – relation

$$f < g \Leftrightarrow (\forall h \in C_+(X))\,(g \cdot h = 0 \Rightarrow f \cdot h = 0). \tag{4}$$

Indeed, the relation $<$ provides us not only with the equivalence relation (3), but determines also the order relation in the lattice $R(X)$:

$$R(f) \subset R(g) \Leftrightarrow f < g. \tag{5}$$

Moreover, Shirota shows that it is also possible to recognize from the algebraic properties of $C(X)$ whether $R(f) \ll R(g)$ or not (by $\ll$ we mean the compingent relation as defined in $B(X)$):

$$R(f) \ll R(g) \Leftrightarrow (\exists h \in C_+(X))\,(f \cdot h = f \ \& \ h < g). \tag{6}$$

Thus, Milgram's theorem will follow if it can be shown that $R(X)$ is not only a sublattice of $B(X)$, but in fact a subalgebra of the Boolean algebra $B(X)$.

It is at this state that serious difficulties arise. For from an example of R. S. PIERCE [8] it follows that the lattice $R(X)$ is *not* always a Boolean algebra: sometimes it contains elements that have no complement. There are classes of compact spaces for which there is no trouble; e.g. if X is fully normal (i.e. if every open subset of X is an F_σ-set), we have even $R(X) = B(X)$ (see e.g. [4] theorem 3.11.b). But Milgram's theorem deals with *arbitrary* compact Hausdorff spaces.

It may be true, of course, that $R(X)$ contains a sublattice $B = \{R(f) : f \in F\}$, $F \subset C(X)$, such that B is complemented, while the elements of B still constitute a base for the topology of X. If the subset F of $C(X)$ could be determined in a purely algebraic way, this would solve our problem and finish the proof of Milgram's theorem. However, the authors at present see no way to do so.

Another possible approach (pointed out to us by H. de Vries) is the following one. Let A be the free Boolean algebra generated by the set $R(X)$, and let φ be the Boolean homomorphism of A onto $B(X)$ such that an arbitrary generator $R(f)$ of A is mapped onto the element $R(f)$ of $B(X)$. The ideal $I = \{a \in A : \varphi(a) = \phi\}$ is readily characterized by means of the multiplication in $C_+(X)$; it follows that the Boolean algebra A/I, which is isomorphic to the Boolean subalgebra of $B(X)$ generated by the lattice $R(X)$, can be constructed from the semigroup by purely algebraic means.

Thus we have solved one problem: we do arrive now at a Boolean algebra. But this time we are confronted with a new difficulty: how to characterize the compingent relation in A/I. We are not without hope that this may be done; however, it looks as if it will be a complicated affair.

A third possibility would be to accept $R(X)$ for what it is, and to build up a theory taking into account such "compingent lattices". In fact, such a wider theory exists already: in 1952 T. Shirota has developed one with the express purpose to prove theorems such as those of Milgram and of Kaplansky.

3. *Compingent lattices and their relation to compingent algebras*

In his article "A generalization of a theorem of I. KAPLANSKY" [10] T. SHIROTA introduced certain structures, called by him R-lattices. We prefer to call them compingent lattices. The definition following below is a slightly simplified version of the one given in [10].

3.1. Definition. A *compingent lattice* is a distributive lattice S with at least two elements, having a zero element and satisfying the Wallman disjunction property, in which there is defined a binary relation $\ll$ with the following properties.

*S*1. $a \ll b' \leq b \Rightarrow a \ll b$;

*S*2. $a \ll b,\ c \ll d \Rightarrow a \wedge c \ll b \wedge d$;

$S3.\quad a \ll b \Rightarrow (\exists c)\,(a \ll c \ll b)$;

$S4.\quad (\forall a)\,(\exists c)\,(a \ll c)$;

$S5.\quad b \neq 0 \Rightarrow (\exists a \neq 0)\,(a \ll b)$;

$S6.\quad a \ll b \ll c \ll d \ll e \rightarrow (\exists x)\,(\exists y)\,(x \ll y \;\&\; a \wedge y = b \wedge x = 0 \;\&\; c \vee x = d \;\&\; b \vee y = e)$.

(A lattice L is said to satisfy Wallman's disjunction property if for any pair of distinct elements a, b of L there exists a $c \in L$ such that either $a \wedge c = 0$ & $b \wedge c \neq 0$, or $a \wedge c \neq 0$ and $b \wedge c = 0$; cf. [13] p. 115).

We shall now mention a number of algebraic properties of compingent lattices. The proofs will be included, as they are omitted in [10].

3.2. Proposition. *Let S be a compingent lattice. For arbitrary $a, b, c \in S$, the following assertions are true.*

(i) $a \ll b \ll c \Rightarrow (\exists y)\,(a \wedge y = 0$ *and* $b \vee y = c)$;

(ii) $a \ll b \Rightarrow a \leqslant b$;

(iii) $a \ll b \ll c \Rightarrow a \ll c$;

(iv) $0 \ll a$;

(v) $a \leqslant b \ll c \Rightarrow a \ll c$;

(vi) $a \ll c \;\&\; b \ll c \Rightarrow a \vee b \ll c$.

Proof.

Assertion (i): evident from $S6$ in conjunction with $S3$.

Assertion (ii): evident if $a=0$. If $a \neq 0$, let $a' \ll a \ll b$ ($S5$); by (i) $(\exists y)\,(a \vee y = b)$, i.e. $a \leqslant b$.

Assertion (iii) is an immediate consequence of (ii) and $S1$.

We now proceed to a proof of assertion (iv). On the ground of $S1$ we need only consider the special case $a=0$. If $0 = x \wedge y$, for some $x \neq 0$, $y \neq 0$, let $x' \ll x$ and $y' \ll y$ ($S5$); then $0 = x' \wedge y' \ll x \wedge y = 0$, by $S2$; if not, then $0 = x \wedge y$ implies $x=0$ or $y=0$. In that case we reason as follows. As by definition S contains at least two elements, there is a $d \neq 0$. By $S4$, $S5$, $e \ll d \ll f$, for some f and $e \neq 0$. By (i) $e \wedge y = 0$ and $d \vee y = f$, for a suitable y. As $0 = e \wedge y \Rightarrow e = 0$ or $y = 0$, and as $e \neq 0$, we find $y = 0$ and hence $d = f$. Apply $S6$ to $d \ll d \ll d \ll d \ll d$: there are x, y such that $x \ll y$, $d \wedge x = 0 = d \wedge y$. As $d \neq 0$, it follows by assumption that $x = y = 0$. Hence $0 \ll 0$.

Assertion (v): by $S4$ there is a d with $a \ll d$. Then $a \wedge b \ll c \wedge d$, by $S2$. As $a \leqslant b$, $a \wedge b = a$; and $a \ll c \wedge d \leqslant c \Rightarrow a \ll c$ ($S1$).

The proof of assertion (vi) is very tricky. It is not given in [10]; a sketch of the proof below (communicated by T. Shirota) can be found in [5].

By $S3$ and $S4$, there are d, e, p, r, s, t in S such that $a \ll d \ll c$, $b \ll e \ll c \ll p \ll r \ll s \ll t$. Applying $S6$ to the quintuples $a \ll d \ll c \ll p \ll r$ and $b \ll e \ll c \ll p \ll r$, we infer the existence of $x, y, u, v \in S$

such that $x \ll u$, $y \ll v$, $a \wedge u = d \wedge x = b \wedge v = e \wedge y = 0$, $c \vee x = c \vee y = p$, $d \vee u = e \vee v = r$. We put $k = x \wedge y$ and $n = u \wedge v$.

By $S2$, $k \ll n$. Moreover, $n = u \wedge v \leqslant u \leqslant r \ll s$, hence (by (v)) $n \ll s$. Let $k \ll m \ll n$ ($S3$); then $k \ll m \ll n \ll s \ll t$, hence ($S6$) there exist z, w such that $z \ll w$, $k \wedge w = m \wedge z = 0$, $n \vee z = s$, $m \vee w = t$.

By (ii), $a \vee b \leqslant c \leqslant s = n \vee z$; hence $a \vee b = (n \vee z) \wedge (a \vee b) = (n \wedge (a \vee b)) \vee \vee (z \wedge (a \vee b))$. Now $n \wedge a \leqslant u \wedge a = 0$ and $n \wedge b \leqslant v \wedge b = 0$, hence $n \wedge (a \vee b) = 0$. Thus $a \vee b = z \wedge (a \vee b)$, or $a \vee b \leqslant z$. As also $a \vee b \leqslant c$, we infer $a \vee b \leqslant z \wedge c \ll \ll w \wedge p$ (using $S2$). But $w \wedge p = w \wedge (c \vee k) = (w \wedge c) \vee (w \wedge k) = w \wedge c \leqslant c$. By $S1$, $a \vee b \ll w \wedge p \leqslant c \Rightarrow a \vee b \ll c$.

The next proposition is offered (without proof) because of its theoretical interest; it shows the existence of an axiom system for compingent lattices involving the compingent relation as the only fundamental notion. For compingent algebras this was proved by H. DE VRIES [12] theorem 1.1.4.).

3.3. Proposition. *Let S be a compingent lattice. Then for $a, b \in S$*

$$a \leqslant b \Leftrightarrow (\forall c \in S)\,(b \ll c \Rightarrow a \ll c).$$

We use these algebraic properties in order to prove the following theorem, interrelating the concepts compingent lattice and compingent algebra.

3.4. Theorem. *Every compingent algebra is a compingent lattice. Conversely, a compingent lattice is a compingent algebra iff it is complemented.*

Proof.

Let B be a compingent algebra. Then B is a Boolean algebra, and hence a distributive lattice satisfying Wallman's disjunction property; moreover, B contains at least two distinct elements. We prove that the axioms $S1-S6$ are valid.

S1: follows from $P3$ in conjunction with $P5$.

$S2$: is identical to $P4$.

$S3$: follows from $P6$ if $c \neq 0$, and from $P1$ and $P2$ if $c = 0$.

$S4$: $0 \ll 0 \leqslant a^c \Rightarrow 0 \ll a^c \Rightarrow a \ll 1$, by $P1$, $S1$ (already proved) and $P5$.

$S5$: follows from $P6$, as $0 \ll b$ by $P1$ and $S1$.

$S6$: Assume $a \ll b \ll c \ll d \ll e$. If $x = d \wedge c^c$ and $y = f \wedge a^c$, then clearly $a \wedge y = b \wedge x = 0$, and $c \vee x = c \vee d = d$, $b \vee y = b \vee f = f$ ($P2$). Moreover, as $d \ll e$ and as $a \ll c \Rightarrow c^c \ll a^c$ ($P5$), we have $x \ll y$ ($P4$).

Conversely, suppose S is a compingent lattice. If S is a compingent algebra, then it is a Boolean algebra, hence is complemented. Suppose now that S is complemented; we prove that $P1-P6$ hold in S.

Axiom $P1$ is contained in prop. 3.2 (iv), and $P2$, $P3$, $P4$ are identical with prop. 3.2 (ii) and (v) and with $S2$, respectively.

Axiom $P5$: let $a \ll b$. By $S4$ and prop. 3.2 (ii), $1 \ll 1$; hence $b \ll 1$, by prop. 3.2 (v). Let $a \ll c \ll b$ ($S3$); as $a \ll c \ll b \ll 1 \ll 1$, there are x, y

such that $x \ll y$, $a \wedge y = 0$ and $b \vee x = 1$ ($S6$). Then $b^c \leqslant x \ll y \leqslant a^c$; hence $b^c \ll a^c$, by $S1$ and prop. 3.2 (v).

Axiom $P6$: follows from $S5$ and prop. 3.2 (iv) if $a = 0$, and from $S3$ if $a \neq 0$.

In a compingent lattice S, concordant filters are defined by (2), exactly as in compingent algebras. Let $\mathfrak{M}_S$ designate the topological space of all maximal concordant filters in S with the topology generated by the sets $\omega(a) = \{F : a \in F \in \mathfrak{M}_S\}$, $a \in S$. In [10] it is shown that $\mathfrak{M}_S$ always is a locally compact Hausdorff space; in fact, for each $a \in S$ the set $\omega(a)$ is compact in $\mathfrak{M}_S$. Moreover, the map $a \to \omega(a)$ is a lattice-isomorphism of S into $B(\mathfrak{M}_S)$, and

$$a \ll b \Leftrightarrow \overline{\omega(a)} \subset \omega(b).$$

Now if X is any Hausdorff space, $\bar{U} \subset V \subset X$, V open in X and $\bar{V}$ compact, then U and $X \setminus V$ are functionally separated (the subspace $\bar{V}$ of X being normal). Hence

$$a \ll b \Leftrightarrow \omega(a) \ll \omega(b), \tag{7}$$

where the right hand side $\ll$ denotes the compingent relation in the compingent algebra $B(\mathfrak{M}_S)$. Thus we have shown:

3.5. Proposition. *Every compingent lattice is isomorphic to a compingent sublattice of a compingent algebra.*

The sets $\omega(a)$, $a \in S$, constitue a base for the topology of $\mathfrak{M}_S$, consisting of regularly open relatively compact sets. In [10] a converse is stated: if X is any locally compact Hausdorff space, and if S is a sublattice of $B(X)$ such that S is a base for the topology of X, consisting of regularly open relatively compact sets, and containing ϕ, then S is a compingent sublattice of $B(X)$. (For every locally compact Hausdorff space there exists at least one such a base S).

If S contains a unit element 1, then $\mathfrak{M}_S = \omega(1)$ is compact. Conversely, if $\mathfrak{M}_S$ is compact, there are finitely many $a_1, a_2, \ldots, a_n \in S$ such that

$$\mathfrak{M}_S = \omega(a_1) \cup \ldots \cup \omega(a_n) \subset \omega(a_1 \vee a_2 \vee \ldots \vee a_n);$$

it follows (cf. (5)) that $a_1 \vee a_2 \vee \ldots \vee a_n$ is a unit element in S. Thus:

3.6. Proposition (cf. [10] p. 124). $\mathfrak{M}_S$ *is compact* $\Leftrightarrow$ S *has a unit.*

In particular we see again that the dual space $\mathfrak{M}_B$ of a compingent algebra B is compact.

Now let S be an arbitrary compingent lattice. Let $X = \mathfrak{M}_S$ if S has a unit; else let $X = \mathfrak{M}_S \cup \{\infty\}$ be a one-point compactification of $\mathfrak{M}_S$. If B_0 is the Boolean subalgebra of the Boolean algebra $B(X)$, generated by $\omega(S)$, then B_0 is a base for the topology of X; for if $p \in \mathfrak{M}_S$, then $\omega(S)$ contains already a local base at p, and if $p = \infty$, the complement in X of the sets $\omega(a)$, $a \in S$, form a local base at p. For if C is compact in $\mathfrak{M}_S$, then $C \subset \omega(a)$ for some $a \in S$, by an argument similar to the one preceding prop. 3.5. It follows that B_0 is a compingent lattice, by the theorem quoted after prop. 3.4, and hence (theorem 3.4) a compingent subalgebra of $B(X)$. Moreover, $X = \mathfrak{M}_{B_0}$ ([12] theorem 1.4.4.).

Consequently, proposition 3.4 admits of the following refinement.

3.7. Theorem. *Every compingent lattice S can be embedded as a compingent sublattice in a compingent algebra B in such a way that*

(i) *the Boolean algebra B is generated by its subset S;*

(ii) *$\mathfrak{M}_B$ is topologically equivalent to $\mathfrak{M}_S$, if S has a unit element, and to the one-point compactification of $\mathfrak{M}_S$, if S has no unit.*

4. *Applications of compingent lattices*

We mentioned in section 3 that every regularly open relatively compact base of a locally compact Hausdorff space X which is a sublattice of $B(X)$ and contains ϕ, is a compingent sublattice of $B(X)$, and that every compingent lattice can be obtained in this way (up to isomorphism). In general, one locally compact space X leads to several compingent lattices. However, every one of these determines the space X up to homeomorphism, by the following important theorem of Shirota ([10], theorem 2).

4.1. Theorem. *Let X be a locally compact Hausdorff space. Let S be a sublattice of $B(X)$ such that S is a base for the topology of X, $\phi \in S$, and all elements of S are regularly open and relatively compact. Then $\mathfrak{M}_S$ is homeomorphic to X.*

This theorem enables us to finish the proof of Milgram's theorem, sketched in section 2. We saw there that it is possible to obtain from the semigroup $C(X)$, in a purely algebraic way, a lattice L isomorphic to $R(X)$; the relation $\ll$ in $R(X)$ is also mirrored in L by an algebraically defined binary relation (see (4)). We remarked that $R(X)$ is a base for the topology of X; as $R(X) \subset B(X)$, all elements of $R(X)$ are regularly open sets, and as X is compact they are relatively compact. Finally $\phi \in R(X)$; hence by the result of Shirota quoted just after prop. 3.5, $R(X)$ is a compingent lattice. Theorem 4.1. now yields that X is homeomorphic to $\mathfrak{M}_{R(X)}$, hence to $\mathfrak{M}_L$. Milgram's theorem follows.

The proof of Milgram's theorem as outlined here was first suggested by Shirota ([10] p. 127). In the paper referred to, Shirota also proved several generalizations of the theorem of Kaplansky mentioned in section 2, using compingent lattices. Similar methods have been used by F. W. Anderson [1] in order to prove the following theorem: "*Every completely regular G_δ-space X is characterized by the lattice $C(X)$*".

Concerning the usefulness of compingent lattices as compared to compingent algebras, we may remark the following. Since compingent lattices are more general, they can be applied in cases where compingent algebras can not. This is a real advantage of the former over the latter; the main disadvantage is of a technical nature: algebraically they are rather awkward to deal with.

The fact that in compingent lattices complements need not exist, makes it necessary to approximate the content of $P5$ by the complicated existential statement $S6$. The resulting axiom system for compingent lattices is essentially weaker than the one for compingent algebras, and consequently the proofs are as a rule much more involved and complicated (cf. the proof of prop. 3.2 (vi); in the case of compingent algebras, this assertion is an immediate consequence of $P4$ and $P5$).

Indeed, experience has taught us that clumsiness of manipulation is inherent to compingent lattices. As an illustration we may point to the fact that a number of important theorems in [12] are proved with the help of the following simple property ([12] 1.2.2.):

> "*A proper concordant filter F of a compingent algebra B is maximal iff $a \ll b$ always implies that either $b \in F$ or $a^c \in F$.*"

It is clear that for compingent lattices a similar proposition cannot even be formulated.

Where compingent algebras can be used, on the contrary, they often provide proofs of a lucid and elegant character. It is in this sense, we dare say, that for most topological uses compingent algebras, though less general, can be considered an improvement over compingent lattices.

APPENDIX: *Independence of the axiom system for compingent lattices*

We remark that the axiom system given in def. 3.1 is actually independent, in the sense that in a Wallman lattice each of the six postulates is independent from the other five. We shall prove this by exhibiting six models of Wallman lattices provided with a binary relation $\ll$, in such a way that the model headed Vj $(j=1, 2, \ldots, 6)$ fails to satisfy Sj, while the other axioms are valid (cf. [9]).

$V1$: Let L be a two-element Boolean algebra. We define: $a \ll b$ iff $a=b$. As simple inspection reveals, all axioms except $S1$ are satisfied.

$V2$: Let G be a non-denumberable set and L the class of all at most countable subsets of G. The set theoretic operations $\cup$ and $\cap$ convert L into a Wallman lattice. We define: $a \ll b$ iff there is an element $b' \in L$ such that i) $a \cup b' = b$, ii) $a \cap b' = \phi$, iii) b' is finite only if b is finite.

In this system $S2$ is not valid. A counter-instance is the case where p, q and r are disjoint countably infinite subsets of G. Then we have: $p \ll p \cup q$ and $p \ll p \cup r$, but not: $p \cap p \ll (p \cup q) \cap (p \cup r)$, as $p \ll p$ is false.

It is obvious that $S1$, $S3$, $S4$, $S5$ are valid in L. That $S6$ also holds may be seen as follows: let $a \ll b \ll c \ll d \ll e$. We can write b, c, d and e

as unions of disjunct elements of L: $b=a\cup b'$, $c=b\cup c'$, $d=c\cup d'$, $e=d\cup e'$. We take $x=d'$ and $y=c'\cup d'\cup e'$. It is easily seen that $x\ll y$, whether x is finite or not. Furthermore: $a\cap y\subset(b\cap c')\cup(c\cap d')\cup(d\cap e')=\phi$, $b\cap x\subset c\cap d'=\phi$, $c\cup x=c\cup d'=d$, $b\cup y=(b\cup c')\cup(d'\cup e)=c'\cup d'\cup e'=$ $=d\cup e'=e$. These formulae show that $S6$ is satisfied.

$V3$: (This example is due to H. de Vries – oral communication –.) Let T be the space of all positive real numbers under the (ordinary) euclidean topology. For every $\alpha\in T$ and all $n>|\alpha|^{-1}$ we define an interval $U(\alpha,n)=$ $=(\alpha-1/n,\ \alpha+1/n)$. These sets form a regularly open base for the topology of T. Furthermore we define: $P=\bigcup_{n=0}^{\infty}(4n+1,\ 4n+2)$, $Q=\bigcup_{n=0}^{\infty}(4n,\ 4n+3)$. Let L be the Boolean algebra generated by P, Q, and all $U(\alpha,n)$, under the operations of intersection ($\cap$), regular union ($\dot\cup$) and regular complementation (^{k}). For $A,B\in L$ we define, as usual: $A\ll B$ iff $\bar A\subset B$.

Then $S3$ is not satisfied: for $P\ll Q$, and any R such that $P\ll R\ll Q$ is bound to consist of infinitely many disjoint components. By the definition of L, R would have to coincide with one of the following elements: P, Q, P^k, Q^k, except perhaps on a finite initial segment of T. This is impossible however, because of the relations $\bar P\subset R$ and $\bar R\subset Q$.

The validity of $S1$, $S2$, $S4$ and $S5$ is easy to prove. To prove $S6$, we use the fact that $A\ll B$ implies $B^k\ll A^k$. Given $A\ll B\ll C\ll D\ll E$, then $X=D\cap C^k$ and $Y=E\cap B^k$ are elements of L such that $X\ll Y$. We have the following equalities: $A\cap Y\subset A\cap B^k=\phi$, $B\cap X\subset B\cap C^k=\phi$, $C\dot\cup X=(C\dot\cup D)\cap(C\dot\cup C^k)=D\cap T=D$, and similarly $B\dot\cup Y=E$.

$V4$: Let L be the Boolean algebra consisting of four elements: o, p, q, 1. We define: $a\ll b$ iff $1\neq a\leqslant b$.

Since there is no $c\in L$ such that $1\ll c$, $S4$ does not hold. The validity of the axioms $S1$, $S2$, $S3$ and $S5$ is obvious. To prove $S6$ it suffices to consider the case $a\ll a\ll a\ll a\ll 1$, where a is arbitrary in L, since the other cases are either perfectly analogous or trivial. We take $x=0$ and $y=a^c$; then $x\ll y$, $a\wedge y=0=a\wedge x$, $a\vee x=a$ and $a\vee y=1$.

$V5$: Let L be the Boolean algebra consisting of four elements: o, p, q, 1. We define: $a\ll b$ iff $a=0$ or $b=1$.

There is no $a\neq 0$ such that $a\ll p$, though $p\neq 0$; so $S5$ is not true. The other axioms are evidently satisfied.

$V6$: Let L be a two-element Boolean algebra with elements 0 and 1. We define: $a\ll b$ iff $b=1$.

Now $S6$ does not hold. We have: $1\ll 1\ll 1\ll 1\ll 1$. Since x and y have to satisfy the conditions $x\wedge 1=0$ and $y\wedge 1=0$, we can only take $x=y=0$. But $0\ll 0$ is not true in this system.

The other axioms are trivially valid.

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