STRENGTHENING ALEXANDER'S SUBBASE THEOREM

BY G. E. STRECKER, E. WATTEL, H. HERRLICH, AND J. DE GROOT

Historically, compactness has been introduced within the framework of topology. However, it is illuminating and convenient to define compactness set-theoretically.

Let X be a non-empty set and E a family of subsets of X. If $A \subset X$, then A will be said to be *compact relative to* $\mathfrak{E}$, or equivalently $A \in \rho\mathfrak{E}$, provided that for every $\mathfrak{B} \subset \mathfrak{E}$ such that $\mathfrak{B} \cup \{A\}$ has f.i.p., $(\bigcap \mathfrak{B}) \cap A \neq \emptyset$. (Although compactness is usually defined in terms of coverings (by open sets), we prefer, for reasons of simplicity and convenience, to work within the complementary framework of collections with the finite intersection property (f.i.p.).) Thus in the case that $\mathfrak{E}$ is the collection of all closed subsets of a topological space, denoted $(X, \mathfrak{E})$, $\rho\mathfrak{E}$ is by this definition the collection of all compact subsets of the space. Observe that $\rho\mathfrak{E}$ contains all finite subsets of X, but $\rho\mathfrak{E}$ need not contain $\mathfrak{E}$. $\rho^n\mathfrak{E}$ is defined inductively; $\rho^n\mathfrak{E} = \rho(\rho^{n-1}\mathfrak{E})$. Furthermore, we let $\gamma\mathfrak{E}$ denote the collection of all (arbitrary) intersections of finite unions of members of $\mathfrak{E}$. Observe that γ is idempotent; $\gamma^2\mathfrak{E} = \gamma\mathfrak{E} \supset \mathfrak{E}$. Also the convention $\bigcap\emptyset = X$ is used. Thus $\mathfrak{E} = \gamma\mathfrak{E}$ if and only if $(X, \mathfrak{E})$ is a topological space.

In terms of these operators, Alexander's Subbase Theorem can be stated as follows:

THEOREM (Alexander). *For every* $\mathfrak{E} \subset 2^X$, $\rho\mathfrak{E} = \rho\gamma\mathfrak{E}$; *i.e. the family of sets compact relative to* $\mathfrak{E}$ *is the same as the family of sets compact relative to the larger collection* $\gamma\mathfrak{E}$.

In the course of the paper the theorem is strengthened by establishing the existence of an even larger collection, namely $\gamma(\mathfrak{E} \cup \rho^2\mathfrak{E})$, with the same compact sets. Also, necessary and sufficient conditions are obtained which determine whether or not $\gamma\mathfrak{E}$ is the largest collection $\mathfrak{D}$ for which $\rho\mathfrak{E} = \rho\mathfrak{D}$, or indeed whether or not there exists a collection $\mathfrak{D}$ maximal with respect to the property that $\rho\mathfrak{E} = \rho\mathfrak{D}$. (As usual, the term "largest" implies comparability with all other elements, whereas "maximal" does not necessarily carry that connotation.) For a Hausdorff space, $(X, \gamma\mathfrak{E})$, there is always a maximal collection—precisely $\gamma(\mathfrak{E} \cup \rho^2\mathfrak{E})$, and $\gamma\mathfrak{E}$ is maximal if and only if $(X, \gamma\mathfrak{E})$ is a k-space.

1. An extension of Alexander's Theorem. Throughout the paper we will assume that X is a non-empty set and that $\mathfrak{E}$ and $\mathfrak{D}$ are subsets of 2^X. $\mathfrak{E} \wedge \mathfrak{D}$ will denote $\{E \cap D \mid E \in \mathfrak{E}, D \in \mathfrak{D}\}$.

LEMMA 1. *If* $\mathfrak{E} \subset \mathfrak{D}$, *then* $\rho\mathfrak{D} \subset \rho\mathfrak{E}$.

Received January 23, 1967.

Proof. Trivial.

LEMMA 2. *If* $\mathfrak{C} \subset \gamma\mathfrak{C} \cap \rho\mathfrak{C}$ *and if* $\mathfrak{C}$ *has f.i.p., then* $\cap\mathfrak{C} \neq \emptyset$.

Proof. If $\mathfrak{C} = \emptyset$, then $\cap\mathfrak{C} = X \neq \emptyset$. If $\mathfrak{C} \neq \emptyset$, let $C \in \mathfrak{C}$; then $C \in \rho\mathfrak{C}$. By Alexander's Theorem, $C \in \rho(\gamma\mathfrak{C})$. Also $\mathfrak{C} \subset \gamma\mathfrak{C}$ and $\mathfrak{C} \cup \{C\}$ has f.i.p.; thus by the definition of compactness, $\cap\mathfrak{C} = (\cap\mathfrak{C}) \cap. C \neq \emptyset$.

LEMMA 3. $\gamma\mathfrak{C} \wedge \rho^n\mathfrak{C} \subset \rho^n\mathfrak{C}$, *for* $n \geq 1$.

Proof. If $n = 1$, this is merely a restatement of the well known result that in a topological space the intersection of a closed set and a compact set is again compact. To complete the proof by induction, assume that $\gamma\mathfrak{C} \wedge \rho^n\mathfrak{C} \subset \rho^n\mathfrak{C}$ and that $E \in \gamma\mathfrak{C}$, $C \in \rho^{n+1}\mathfrak{C}$, and $\mathfrak{B} \subset \rho^n\mathfrak{C}$ such that $\mathfrak{B} \cup \{E \cap C\}$ has f.i.p. Clearly $\mathfrak{B}' = \{E\} \wedge \mathfrak{B} \subset \gamma\mathfrak{C} \wedge \rho^n\mathfrak{C} \subset \rho^n\mathfrak{C}$ and $\mathfrak{B}' \cup \{C\}$ has f.i.p. Thus by the definition of compactness, $(\cap\mathfrak{B}) \cap E \cap C = (\cap\mathfrak{B}') \cap C \neq \emptyset$; and so $\gamma\mathfrak{C} \wedge \rho^{n+1}\mathfrak{C} \subset \rho^{n+1}\mathfrak{C}$.

LEMMA 4. $\rho\mathfrak{C} \wedge \rho^2\mathfrak{C} = \rho\mathfrak{C} \cap \rho^2\mathfrak{C}$.

Proof. Clearly $\rho\mathfrak{C} \cap \rho^2\mathfrak{C} \subset \rho\mathfrak{C} \wedge \rho^2\mathfrak{C}$, and by applying the statement of Lemma 3 with $n = 1$ to $\rho\mathfrak{C}$ we have

$$\rho\mathfrak{C} \wedge \rho^2\mathfrak{C} \subset \gamma\rho\mathfrak{C} \wedge \rho^2\mathfrak{C} \subset \rho^2\mathfrak{C}.$$

Thus we need only show that $\rho\mathfrak{C} \wedge \rho^2\mathfrak{C} \subset \rho\mathfrak{C}$. Let $C \in \rho\mathfrak{C}$, $D \in \rho^2\mathfrak{C}$ and $\mathfrak{B} \subset \mathfrak{C}$ be such that $\mathfrak{B} \cup \{C \cap D\}$ has f.i.p. Now $\mathfrak{B}' = \mathfrak{B} \wedge \{C\} \subset \gamma\mathfrak{C} \wedge \rho\mathfrak{C}$; so that by Lemma 3, $\mathfrak{B}' \subset \rho\mathfrak{C}$. Also $\mathfrak{B}' \cup \{D\}$ has f.i.p. so that by the definition of ρ^2, $(\cap\mathfrak{B}) \cap C \cap D = (\cap\mathfrak{B}') \cap D \neq \emptyset$. Therefore $C \cap D \in \rho\mathfrak{C}$.

Remark. Note that by the inductive definition of ρ^n, Lemma 4 may be stated more generally as: $\rho^n\mathfrak{C} \wedge \rho^{n+1}\mathfrak{C} = \rho^n\mathfrak{C} \cap \rho^{n+1}\mathfrak{C}$, $n \geq 1$. For a more extensive treatment of the ρ and γ operators, see [2].

THEOREM 1. *For every* $\mathfrak{C} \subset 2^X$, $\rho\mathfrak{C} = \rho\gamma(\mathfrak{C} \cup \rho^2\mathfrak{C})$.

Proof. By Alexander's Theorem, $\rho\gamma(\mathfrak{C} \cup \rho^2\mathfrak{C}) = \rho(\mathfrak{C} \cup \rho^2\mathfrak{C})$ and by Lemma 1, $\rho(\mathfrak{C} \cup \rho^2\mathfrak{C}) \subset \rho\mathfrak{C}$. Thus we must show that $\rho\mathfrak{C} \subset \rho(\mathfrak{C} \cup \rho^2\mathfrak{C})$.

Let $C \in \rho\mathfrak{C}$ and let $\mathfrak{B} \subset \mathfrak{C} \cup \rho^2\mathfrak{C}$ be such that $\mathfrak{B} \cup \{C\}$ has f.i.p. Furthermore let $\mathfrak{B}_1 = \mathfrak{B} \cap \mathfrak{C}$ and $\mathfrak{B}_2 = \mathfrak{B} \cap \rho^2\mathfrak{C}$.

Case 1. $\mathfrak{B}_2 = \emptyset$. Clearly $\mathfrak{B} \subset \mathfrak{C}$, so that by the definition of ρ, $(\cap\mathfrak{B}) \cap C \neq \emptyset$.

Case 2. $\mathfrak{B}_1 = \emptyset$. Then $\mathfrak{B}' = \{C\} \wedge \mathfrak{B} \subset \rho\mathfrak{C} \wedge \rho^2\mathfrak{C}$, so that by Lemma 4, $\mathfrak{B}' \subset \rho\mathfrak{C} \cap \rho^2\mathfrak{C}$. But since $\mathfrak{B}'$ has f.i.p., we have that by Lemma 2 applied to $\mathfrak{B}'$ and $\rho\mathfrak{C}$, $\cap(\mathfrak{B} \cap C) = \cap\mathfrak{B}' \neq \emptyset$.

Case 3. $\mathfrak{B}_1 \neq \emptyset$; $\mathfrak{B}_2 \neq \emptyset$. Then $\mathfrak{B}_1 \wedge \mathfrak{B}_2 \subset \gamma\mathfrak{C} \wedge \rho^2\mathfrak{C}$, so that by Lemma 3, $\mathfrak{B}_1 \wedge \mathfrak{B}_2 \subset \rho^2\mathfrak{C}$. Thus we have a situation essentially the same as that of Case 2.

Next we give an example of a Hausdorff space for which, by an application of Theorem 1, we obtain a strictly stronger (i.e., finer) space with the same compact subsets.

Example 1. Let X be an uncountable set and let p be a distinguished element of X. Let

$$\mathfrak{C} = \{E \subset X \mid E \text{ uncountable implies } p \in E\}.$$

Then $(X, \mathfrak{C})$ is Hausdorff, $\rho\mathfrak{C}$ is the collection of all finite subsets of X, and $\rho^2\mathfrak{C} = 2^X$. Thus $\mathfrak{C}$ is properly contained in $\mathfrak{C} \cup \rho^2\mathfrak{C}$.

2. Spaces maximal with respect to the compactness operator. Having found a general strengthening of Alexander's Subbase Theorem, we now seek a classification of those spaces for which no strengthening is possible, i.e. those spaces $(X, \mathfrak{C})$ for which there is never a collection strictly larger than $\mathfrak{C}$ with the same compact sets.

If $\mathfrak{C} \subset 2^X$, we let $\alpha\mathfrak{C}$ denote $\{A \subset X \mid \{A\} \wedge \rho\mathfrak{C} \subset \rho\mathfrak{C}\}$.

N.B. By Lemma 3 (with $n = 1$), $\gamma\mathfrak{C} \subset \alpha\mathfrak{C}$; by Lemma 4, $\rho^2\mathfrak{C} \subset \alpha\mathfrak{C}$; and by Alexander's Theorem, $\alpha\mathfrak{C} = \alpha\gamma\mathfrak{C}$.

LEMMA 5. $\rho\mathfrak{C} = \rho\mathfrak{D}$ implies that $\alpha\mathfrak{C} = \alpha\mathfrak{D}$.

Proof. Trivial.

LEMMA 6. If $A \in \alpha\mathfrak{C}$, then $\rho(\{A\} \cup \mathfrak{C}) = \rho\mathfrak{C}$.

Proof. By Lemma 1, $\rho(\{A\} \cup \mathfrak{C}) \subset \rho\mathfrak{C}$. To show the reverse containment suppose that $C \in \rho\mathfrak{C}$ and $\mathfrak{B} \subset \{A\} \cup \mathfrak{C}$ such that $\mathfrak{B} \cup \{C\}$ has f.i.p. If $A \notin \mathfrak{B}$, then by the definition of ρ, $(\bigcap\mathfrak{B}) \cap C \neq \emptyset$. If $A \in \mathfrak{B}$, then $A \cap C \in \rho\mathfrak{C}$ and $(\mathfrak{B}\backslash\{A\}) \cup \{A \cap C\}$ has f.i.p. Therefore $(\bigcap\mathfrak{B}) \cap C = \bigcap(\mathfrak{B}\backslash\{A\}) \cap (A \cap C) \neq \emptyset$.

THEOREM 2. *In order that a space $(X, \mathfrak{C})$ have the property that $\mathfrak{C}$ is the largest collection $\mathfrak{D}$ such that $\rho\mathfrak{D} = \rho\mathfrak{C}$ it is necessary and sufficient that $\mathfrak{C} = \alpha\mathfrak{C}$.*

Note that for a Hausdorff space, $(X, \mathfrak{C})$, $\mathfrak{C} = \alpha\mathfrak{C}$ is precisely the statement that $(X, \mathfrak{C})$ is a k-space, [3; 230].

Proof. The necessity is established by Lemma 6 and the fact that $\mathfrak{C} \subset \alpha\mathfrak{C}$. To show the sufficiency, suppose that $\mathfrak{C} = \alpha\mathfrak{C}$ and that for some $\mathfrak{D}$, $\rho\mathfrak{D} = \rho\mathfrak{C}$. Let $A \in \mathfrak{D}$ and let $C \in \rho\mathfrak{C}$. By Lemma 3, $A \cap C \in \rho\mathfrak{D} = \rho\mathfrak{C}$; so that $A \in \alpha\mathfrak{C} = \mathfrak{C}$. Thus $\mathfrak{D} \subset \mathfrak{C}$.

COROLLARY. *A Hausdorff space $(X, \mathfrak{C})$ is a k-space if and only if there exists no collection $\mathfrak{D}$ strictly larger than $\mathfrak{C}$ such that $\rho\mathfrak{D} = \rho\mathfrak{C}$.*

3. Existence of collections maximal with respect to compactness. Having characterized those spaces which are maximal with respect to their collection of compact subsets, we now investigate those sets for which there is a maximal collection (or indeed even a largest collection) with the same compact sets. Notice that the space of Example 1 satisfies the condition that there exists a largest collection with the same compact sets, yet it is not itself the largest collection. The following theorem yields the needed characterization.

THEOREM 3. *If* $\mathfrak{C} \subset 2^X$, *then the following are equivalent*:

(i) *there exists a largest collection* $\mathfrak{D}$ *such that* $\rho\mathfrak{D} = \rho\mathfrak{C}$.
(ii) *there exists a collection* $\mathfrak{D}$ *maximal with respect to the property* $\rho\mathfrak{D} = \rho\mathfrak{C}$.
(iii) $\rho\mathfrak{C} = \rho\alpha\mathfrak{C}$.

Remark. If the maximal collection $\mathfrak{D}$ exists, it must be $\alpha\mathfrak{C}$.

Proof of Theorem 3. Clearly (i) implies (ii). To show that (ii) implies (iii), assume that $\mathfrak{D}$ is maximal with respect to the property that $\rho\mathfrak{D} = \rho\mathfrak{C}$. By Lemma 5, $\alpha\mathfrak{D} = \alpha\mathfrak{C}$; and by Lemma 6, if $A \in \alpha\mathfrak{D}$, then $\rho(\{A\} \cup \mathfrak{D}) = \rho\mathfrak{C}$, so that by the maximality of $\mathfrak{D}$, $A \in \mathfrak{D}$. Thus $\alpha\mathfrak{C} = \mathfrak{D}$, so that $\rho\alpha\mathfrak{C} = \rho\mathfrak{D} = \rho\mathfrak{C}$. To show that (iii) implies (i), assume that $\rho\mathfrak{C} = \rho\alpha\mathfrak{C}$. Thus if $A \in \alpha\alpha\mathfrak{C}$ and $C \in \rho\mathfrak{C}$, $A \cap C \in \rho\alpha\mathfrak{C} = \rho\mathfrak{C}$, so that $A \in \alpha\mathfrak{C}$. This, together with the fact that for any $\mathfrak{A}$, $\mathfrak{A} \subset \gamma\mathfrak{A} \subset \alpha\mathfrak{A}$, yields:

$$\gamma\alpha\mathfrak{C} \subset \alpha\alpha\mathfrak{C} \subset \alpha\mathfrak{C} \subset \gamma\alpha\mathfrak{C}.$$

Therefore $(X, \alpha\mathfrak{C})$ is a space, and $\alpha\mathfrak{C} = \alpha(\alpha\mathfrak{C})$. Hence by Theorem 2, $\alpha\mathfrak{C}$ is the largest collection $\mathfrak{D}$ such that $\rho\mathfrak{D} = \rho\mathfrak{C}$.

DEFINITION. A space is called *hereditarily compact* if every subset is compact (cf. [4]). Such a space is called *maximal hereditarily compact* if there exists no strictly stronger hereditarily compact topology on its underlying set.

COROLLARY. *Every maximal hereditarily compact space is finite.*

Proof. If $(X, \mathfrak{C})$ is hereditarily compact, then $\rho\mathfrak{C} = 2^X$. Thus $\alpha\mathfrak{C} = 2^X$, so that $\rho\alpha\mathfrak{C} = \{A \subset X \mid A \text{ is finite}\}$.

Remark. Spaces for which there is no maximal collection with the same compact sets exist in profusion since there are numerous infinite hereditarily compact spaces. An example is the co-infinite topology on an infinite set.

Since both Theorems 1 and 3 provide general "strengthenings" of Alexander's Subbase Theorem, it is worth noting that there are specific instances where one of them gives a better result than the other. Example 2 below illustrates an instance where an application of Theorem 3 and the remark following it yield a larger collection with the same compact sets than is obtained by an application of Theorem 1. Example 3 shows that there are spaces for which Theorem 1 provides a proper strengthening even though no maximal strengthening exists.

Example 2. Let I be the closed unit interval [0, 1]. We will say that x is an *I-limit point* of $A \subset I$ if x is a limit point of A in the topology induced from the real line. Let $X = I \times \{1, 2\}$ and let $\pi: X \to I$ be the projection $\pi(x, n) = x$. Let $\mathfrak{C} = \{\pi^{-1}[a, b] \mid 0 \leq a \leq b \leq 1\} \cup \{\{c\} \mid c \in X\}$; then $(X, \gamma\mathfrak{C})$ is a T_1 space,

$$\rho\mathfrak{C} = \{C \subset X \mid x \text{ an } I\text{-limit point of } \pi C \text{ implies } \pi^{-1}x \cap C \neq \emptyset\},$$

$$\alpha\mathfrak{C} = \{A \subset X \mid x \text{ an } I\text{-limit point of } \pi A \text{ implies } \pi^{-1}x \subset A\} \text{ and}$$

$$\rho^2\mathfrak{C} = \{E \subset X \mid E \text{ is finite}\}.$$

Thus $\gamma\mathfrak{C} = \gamma(\mathfrak{C} \cup \rho^2\mathfrak{C})$ but $\rho\mathfrak{C} = \rho\alpha\mathfrak{C}$ and $\gamma\mathfrak{C}$ is properly contained in $\alpha\mathfrak{C}$.

Example 3. Let $X = A \cup B$, where A and B are disjoint uncountable sets. Let

$$\mathfrak{C} = \{F \subset X \mid B \subset F\} \cup \{F \subset X \mid F \cap B \text{ is finite and } F \cap A \text{ is countable}\}.$$

Then $(X, \mathfrak{C})$ is a T_1 space,

$$\rho\mathfrak{C} = \{C \subset X \mid C \cap A \text{ is finite}\},$$

$$\rho^2\mathfrak{C} = \{E \subset X \mid E \cap B \text{ is finite}\},$$

$$\alpha\mathfrak{C} = 2^X, \text{ and}$$

$$\rho\alpha\mathfrak{C} = \{D \subset X \mid D \text{ is finite}\}.$$

Thus there is no maximal collection $\mathfrak{D}$ for which

$$\rho\mathfrak{D} = \rho\mathfrak{C}, \quad \text{but} \quad \mathfrak{C} \text{ is properly contained in } (\mathfrak{C} \cup \rho^2\mathfrak{C}) \quad \text{and} \quad \rho(\mathfrak{C} \cup \rho^2\mathfrak{C}) = \rho\mathfrak{C}.$$

We now consider a large class of spaces for which there is always a strongest space with the same compact sets. It should be noted that for any member of the class obtained, an application of Theorem 1 yields this strongest space.

PROPOSITION 1. *If $\mathfrak{C} \subset 2^X$ and $X \in \rho^2\mathfrak{C}$, then $\rho\mathfrak{C} = \rho\alpha\mathfrak{C}$ and there exists a largest collection $\mathfrak{D} = \alpha\mathfrak{C} = \gamma(\mathfrak{C} \cup \rho^2\mathfrak{C}) = \rho^2\mathfrak{C}$ such that $\rho\mathfrak{D} = \rho\mathfrak{C}$.*

Proof. Clearly, by Lemma 1, $\rho\alpha\mathfrak{C} \subset \rho\mathfrak{C}$. Let $C \in \rho\mathfrak{C}$ and $\mathfrak{S} \subset \alpha\mathfrak{C}$ be such that $\mathfrak{S} \cup \{C\}$ has f.i.p. Thus $\mathfrak{S}' = \mathfrak{S} \wedge \{C\}$ has f.i.p. and is contained in $\rho\mathfrak{C}$. Therefore since $X \in \rho^2\mathfrak{C}$, it follows that $(\cap\mathfrak{S}) \cap C = \cap\mathfrak{S}' \neq \emptyset$. Thus $\rho\mathfrak{C} = \rho\alpha\mathfrak{C}$. By Theorem 3 and the remark following it, $\alpha\mathfrak{C} = \gamma\alpha\mathfrak{C}$ is the largest collection $\mathfrak{D}$ such that $\rho\mathfrak{D} = \rho\mathfrak{C}$. Since for any $\mathfrak{C}$, $\mathfrak{C} \cup \rho^2\mathfrak{C} \subset \alpha\mathfrak{C}$, we have that $\rho\mathfrak{C} \subset \gamma(\mathfrak{C} \cup \rho^2\mathfrak{C}) \subset \gamma\alpha\mathfrak{C} = \alpha\mathfrak{C}$. To show that $\alpha\mathfrak{C} \subset \rho^2\mathfrak{C}$, suppose that $A \in \alpha\mathfrak{C}$ and $\mathfrak{C} \subset \rho\mathfrak{C}$ such that $\mathfrak{C} \cup \{A\}$ has f.i.p. Then

$$(\cap\mathfrak{C}) \cap A = \cap(\mathfrak{C} \wedge \{A\}) \cap X \neq \emptyset,$$

since $\mathfrak{C} \wedge \{A\} \subset \rho\mathfrak{C}$ and $X \in \rho^2\mathfrak{C}$.

COROLLARY. *If $\mathfrak{C} \subset 2^X$ and $\rho\mathfrak{C} \subset \gamma\mathfrak{C}$, then $\rho\mathfrak{C} = \rho\alpha\mathfrak{C}$.*

Proof. If $\mathfrak{M} \subset \rho\mathfrak{C}$, then $\mathfrak{M} \subset \gamma\mathfrak{C} \cap \rho\mathfrak{C}$; so that if $\mathfrak{M}$ has f.i.p., we have by Lemma 2 that $\cap\mathfrak{M} \neq \emptyset$. Thus $X \in \rho^2\mathfrak{C}$.

COROLLARY. *If $(X, \mathfrak{C})$ is Hausdorff, then $\rho\mathfrak{C} = \rho\alpha\mathfrak{C}$.*

4. Problems.

(A) Given $\mathfrak{E} \subset 2^X$, what are necessary and sufficient conditions for the existence of some $\mathfrak{D}$ such that $\mathfrak{E} = \rho\mathfrak{D}$?

(B) Given $\mathfrak{E} \subset 2^X$, what are necessary and sufficient conditions for the existence of some $\mathfrak{D}$ such that $\mathfrak{E} = \gamma\rho\mathfrak{D}$?

(C) Given $\rho\mathfrak{E}$, characterize $\{\mathfrak{D} \subset 2^X \mid \rho\mathfrak{D} = \rho\mathfrak{E}\}$.

Note that (A) and (B) are actually different questions since the collection of all countable subsets of an uncountable space can be $\gamma\rho\mathfrak{D}$ for some $\mathfrak{D}$ but cannot be $\rho\mathfrak{D}$ for any $\mathfrak{D}$. Also note that answering question (C) in the case $\rho\mathfrak{E} = 2^X$ is a central topic of [4].

References

1. J. Alexander, *Ordered sets, complexes, and the problem of compactification*, Proc. Nat. Acad. Sci. U.S.A., vol. 25(1939), pp. 296–298.
2. J. de Groot, H. Herrlich, G. E. Strecker and E. Wattel, *Compactness as an operator*, to appear.
3. J. L. Kelley, *General Topology*, New York, 1955.
4. A. H. Stone, *Hereditarily compact spaces*, Amer. J. Math., vol. 61(1955), pp. 197–278.

University of Florida
and
Mathematisch Centrum, Amsterdam