ON A CONVOLUTION OF SEQUENCES IN A COMPACT GROUP

BY

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Let X be a (not necessarily commutative) compact Hausdorff topological group with unit e and let $C(X)$ be the Banach space (under uniform norm) of continuous complex-valued functions on X. We denote by V the set of normed non-negative regular Borel measures on X, in particular by μ the normed Haar measure on X. Identifying measures and corresponding functionals on $C(X)$, we shall write

$$\nu(f) = \int_X f(x) d\nu(x) \quad \text{for } \nu \in V, \ f \in C(X).$$

A sequence $\{x_n\} \subset X$ $(0 \leqq n < \infty)$ will be called *summable* (*ν-summable*) if $\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(x_n)$ exists $\left(\text{if } \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(x_n) = \nu(f)\right)$ for all $f \in C(X)$.
A μ-summable sequence will be called *uniformly distributed* (u.d.). For technical reasons and without loss of generality we shall always put $x_0 = e$.

In V convolution $(\star)$ is defined by

$$\nu_1 \star \nu_2(f) = \int_X \int_X f(xy)\, d\nu_1(x)\, d\nu_2(y) \quad \text{for all } f \in C(X).$$

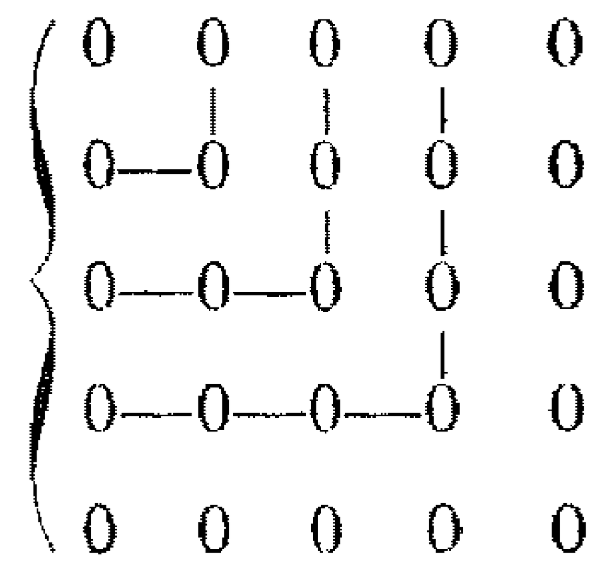
If $\{x_n\}$ and $\{y_n\}$ are ν_1-summable and ν_2-summable sequences respectively the problem arises whether they may be used in some sensible way to construct a $\nu_1 \star \nu_2$-summable sequence $\{z_n\}$. Of course one would expect that this can be done using in some way the set of all products $x_i y_j$ $(0 \leqq i, j < \infty)$:

$$(1) \qquad \begin{cases} e & y_1 & y_2 & y_3 & y_4 & \dots \\ x_1 & x_1y_1 & x_1y_2 & x_1y_3 & \dots \\ x_2 & x_2y_1 & x_2y_2 & \dots \\ x_3 & x_3y_1 & \dots \\ x_4 & \dots \\ \dots \end{cases}$$

In [6] it has been shown that indeed the sequence

$$(2) \qquad \{x_n\} \times \{y_n\} = \{e, x_1, y_1, x_1y_1, x_2, y_2, x_2y_1, x_1y_2, x_2y_2, \dots\}$$

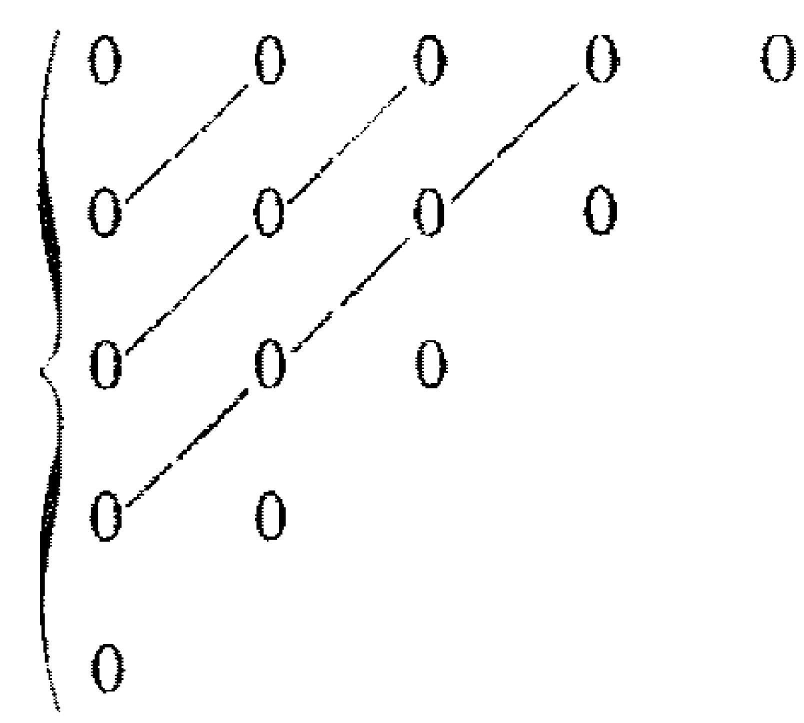
obtained by joining successively the finite sequences in (1) connected by broken line segments as in the following sketch (3) meets these requirements:

(3) 

On occasion of the colloquium on uniform distribution at the Mathematical Center, Amsterdam (1963/64) it has been pointed out to me by L. KUIPERS that it would seem more natural to consider the sequence

(4) $$\{x_n\} \star \{y_n\} = \{e, x_1, y_1, x_2, x_1y_1, y_2, x_3, x_2y_1, x_1y_2, y_3, \ldots\}$$

which is obtained by joining in the array (1) successively the finite diagonal sequences connected by line segments in the following sketch (5):

(5) 

As Prof. Kuipers has communicated to me, in an unpublished paper he has proved in the case of the additive group of reals mod 1 that if $x_n = n\alpha$ (α irrational), $y_n = n\beta$ (β arbitrary), then the sequence $\{x_n\} \star \{y_n\}$ as defined in (4) is uniformly distributed mod 1. He conjectured that a similar result would hold in the case of a general compact group X. In fact, a generalization of the just mentioned theorem to the case of k sequences $\{a_i{}^n\}$ $(1 \leq i \leq k)$ in a compact nonothetic group has been eastablished in the meantime by L. KUIPERS and P. A. J. SCHEELBEEK [10].

In the present paper for the case of two sequences the even sharper statements are proved that, if $\{x_n\}$ and $\{y_n\}$ are ν_1-summable and ν_2-summable sequences respectively, then $\{x_n\} \star \{y_n\}$ as defined in (4) is $\nu_1 \star \nu_2$-summable (theorem 1) and that uniformly distributed sequences may be characterized by the property of producing a summable (even uniformly distributed) sequence whenever composed by convolution (4) with any given sequence (theorem 3). It should be mentioned here that the corresponding statements are true if convolution is defined as in (2) ([6] theorem 1 and 2; the hypothesis of X being 2nd countable is superfluous for the proof of theorem 1 and may be replaced by the assumption of existence of u.d. sequences in theorem 2). Still, an initial segment of length approximately $2N$ of each of the sequences (2) or (4) is necessary in order to cover an initial segment of length N of the other sequence. Thus it seems not to be possible simply to deduce the statements for one of these sequences from the corresponding statements for the other sequence.

The proofs of theorem 1 and 3 follow the same line as in [6], but involve second order (in place of first order) Cesàro means. We shall therefore need a special case of the following lemma which is stated with the usual notation for Cesàro means ([4] 5.4).

Lemma 1: *Let $k>1$ be an integer, let $\{x_n\}$ be a sequence in X, and let $f \in C(X)$ be given. Then*

$$f(x_n) \to \alpha \qquad (C, k) \tag{6}$$

if and only if

$$f(x_n) \to \alpha \qquad (C, 1). \tag{7}$$

Proof: The implication (7) $\Rightarrow$ (6) is the well known statement that (C, k) includes $(C, 1)$ ([4] theorem 43). The implication (6) $\Rightarrow$ (7) follows from the fact that f is bounded ([4] theorem 70).

The sequence $\{x_n\}$ is ν-summable if $f(x_n) \to \nu(f)$ $(C, 1)$ for all $f \in C(X)$. According to lemma 1 we could also have defined ν-summability of $\{x_n\}$ by $f(x_n) \to \nu(f)$ (C, k) for all $f \in C(X)$ $(k \geqq 1)$. Thus, as far as continuous functions are concerned, all summation methods (C, k) $(k \geqq 1)$ are equivalent for sequences in a compact group. For the case of reals mod 1 this result is due to Cigler ([2] theorem 1).

For $k=2$ we obtain in particular:

Lemma 1′: *Let $\{x_n\}$ be a sequence in X and let $f \in C(X)$ be given. Then*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(x_n) = \alpha \tag{6'}$$

if and only if

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{(N+1-n)}{\binom{N+2}{2}} f(x_n) = \alpha.$$

Theorem 1: *Let $\{x_n\}$ and $\{y_n\}$ be ν_1-summable and ν_2-summable sequences in X respectively. Then $\{z_n\} = \{x_n\} \star \{y_n\}$ is $\nu_1 \star \nu_2$-summable.*

Proof: Making use of the well known criterion of Weyl ([9] theorem 4) we have to show that, under the hypotheses of the theorem, for every irreducible unitary representation D of X the equation

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} D(z_n) = \nu_1 \star \nu_2(D)$$

holds. We denote the degree of D by r.

Let $\varepsilon > 0$ be given. Using the well known euclidean matrix norm

$$\|A\| = [\sum_{k,e} |a_{k,l}|]^{\frac{1}{2}} = [\text{trace } (A^*A)]^{\frac{1}{2}} \quad (\text{see [8] § 1})$$

we choose N_1 such that

$$\|v_1(D) - \sum_{n=0}^{N} \frac{N+1-n}{\binom{N+2}{2}} D(x_n)\| < \varepsilon \qquad \text{for all } N \geqq N_1 \tag{8}$$

$$\|v_2(D) - \sum_{n=0}^{N} \frac{1}{N+1} D(y_n)\| < \varepsilon \qquad \text{for all } N \geqq N_1. \tag{9}$$

This is possible because of our hypothesis and lemma 1′.

If N is given we denote by N' the integer uniquely defined by the inequalities

$$\frac{(N'+1)(N'+2)}{2} \leqq N+1 < \frac{(N'+2)(N'+3)}{2}. \tag{10}$$

(Without loss of generality we may assume $N' > N_1$). As indicated by the following array (11)

$$\left\{\begin{array}{llll}
e & \cdots\cdots\cdots\cdots & & y_{N'} \\
\vdots & \quad (A_2) & \iddots & \\
x_{N'-N_1} & \cdots\cdots & x_{N'-N_1} y_{N_1} & \\
x_{N'-N_1+1} & \cdots\cdots & x_{N'-N_1+1} y_{N_1-1} & \\
\vdots & (A_1) \quad \iddots & x_{N'+1-M} y_M & \\
x_{N'} & (A_3) & & \\
x_{N'+1} & & &
\end{array}\right. \tag{11}$$

the sum $\sum_{n=0}^{N} D(z_n)$ decomposes as follows:

$$\left\{\begin{aligned}
\sum_{n=0}^{N} D(z_n) &= \sum_{i=0}^{N_1-1} \sum_{j=0}^{i} D(x_{N'-i} y_j) + \sum_{i=N_1}^{N'} \sum_{j=0}^{i} D(x_{N'-i} y_j) + \\
&\qquad + \sum_{j=0}^{M} D(x_{N'+1-j} y_j) = \\
&= \qquad A_1 \qquad + A_2 \qquad\qquad + A_3.
\end{aligned}\right. \tag{12}$$

$\left(\text{Note that } M \leqq N' \text{ and that } A_3 \text{ vanishes if } N+1 = \binom{N'+2}{2}\right)$. According to (12) we also decompose the difference

$$\nu_1 \star \nu_2(D) - \frac{1}{N+1} \sum_{n=0}^{N} D(z_n) = \frac{N_1(N_1+1)}{2(N+1)} \nu_1 \star \nu_2(D) - \tag{B_1}$$

$$- \frac{1}{N+1} \sum_{i=0}^{N_1-1} \sum_{j=0}^{i} D(x_{N'-i} y_j) + \tag{B_2}$$

$$+ \frac{1}{N+1} \sum_{i=N_1}^{N'} (i+1) \left[\nu_1 \star \nu_2(D) - \frac{1}{i+1} \sum_{j=0}^{i} D(x_{N'-i} y_j)\right] + \tag{B_3}$$

$$+ \frac{M+1}{N+1} \nu_1 \star \nu_2(D) - \tag{B_4}$$

$$- \frac{1}{N+1} \sum_{j=0}^{M} D(x_{N'+1-j} y_j) \tag{B_5}$$

and investigate the behaviour of every term B_k as $N \to \infty$:

$$\|B_1\| \to 0 \text{ since } \lim_{N\to\infty} \frac{N_1(N_1+1)}{2(N+1)} = 0$$

$$\|B_2\| \leqq \frac{N_1(N_1+1)}{2(N+1)} \sqrt{r} \to 0$$

$$\|B_4\| \to 0 \text{ since, by (10), } \frac{M+1}{N+1} \leqq \frac{(N'+1)\cdot 2}{(N'+1)(N'+2)} \to 0$$

$$\|B_5\| \leqq \frac{M+1}{N+1} \sqrt{r} \to 0.$$

We still have to investigate B_3. Because of

$$\nu_1 \star \nu_2(D) = \int_X \int_X D(xy) d\nu_1(x) d\nu_2(y) = \nu_1(D)\nu_2(D)$$

we can write

$$B_3 = \frac{1}{N+1} \sum_{i=N_1}^{N'} (i+1)[\nu_1(D)\nu_2(D) - D(x_{N'-i})\nu_2(D)] + \tag{B_6}$$

$$+ \frac{1}{N+1} \sum_{i=N_1}^{N'} (i+1) D(x_{N'-i}) \left[\nu_2(D) - \frac{1}{i+1} \sum_{j=0}^{i} D(y_j)\right] \tag{B_7}$$

where, since D is unitary,

$$\|B_7\| \leqq \frac{1}{N+1} \sum_{i=N_1}^{N'} (i+1) \left\| D(x_{N'-i}) \left[\nu_2(D) - \frac{1}{i+1} \sum_{j=0}^{i} D(y_j)\right]\right\| =$$

$$= \frac{1}{N+1} \sum_{i=N_1}^{N'} (i+1) \left\|\nu_2(D) - \frac{1}{i+1} \sum_{j=0}^{i} D(y_j)\right\| < \frac{1}{N+1} \sum_{i=N_1}^{N'} (i+1)\, \varepsilon \leqq \varepsilon$$

because of (9).

Furthermore, we have

$$B_6 = \frac{(N'+1)(N'+2)}{2(N+1)} \left[\nu_1(D) - \sum_{i=0}^{N'} \frac{i+1}{\binom{N'+2}{2}} D(x_{N'-i}) \right] \nu_2(D) - \tag{B_8}$$

$$- \frac{N_1(N_1+1)}{2(N+1)} \nu_1(D)\nu_2(D) + \tag{B_9}$$

$$+ \frac{1}{N+1} \sum_{i=0}^{N_1-1} (i+1) D(x_{N'-i}) \nu_2(D) \tag{B_{10}}$$

where

$$\|B_8\| \leqq \left\| \nu_1(D) - \sum_{i=0}^{N'} \frac{i+1}{\binom{N'+2}{2}} D(x_{N'-i}) \right\| \|\nu_2(D)\| < \varepsilon \|\nu_2(D)\| \text{ (because of (8))}$$

$$\|B_9\| \to 0 \text{ (as } B_1)$$

$$\|B_{10}\| < \frac{N_1(N_1+1)}{2(N+1)} \|\nu_2(D)\| \to 0.$$

Combining these estimates, we obtain

$$\left\| \nu_1 \star \nu_2(D) - \frac{1}{N+1} \sum_{n=0}^{N} D(z_n) \right\| \leqq \|B_1\| + \|B_2\| + \|B_4\| + \|B_5\| + \|B_7\| + \\ + \|B_8\| + \|B_9\| + \|B_{10}\| < (1 + \|\nu_2(D)\|)\, \varepsilon$$

for all N that are sufficiently large. Since ε was arbitrary this proves the theorem.

We note the following application, generalizing a theorem of Eckmann ([3] theorem 8):

Theorem 2: *Let the elements $a, b \in X$ be given. The sequence $\{a^n\} \star \{b^n\}$ is uniformly distributed if and only if the sequence $\{b^n\} \star \{a^n\}$ is uniformly distributed or, equivalently, if and only if, for every non-trivial irreducible unitary representation D of X,*

$$\text{(13)} \quad \begin{cases} \operatorname{rank} \{[D(e) - D(a)]\,[D(e) - D(b)]\} = \\ \qquad = \operatorname{rank} [D(e) - D(a)] + \operatorname{rank} [D(e) - D(b)] - \operatorname{degree} D. \end{cases}$$

Proof: Let Y and Z be the closed subgroups of X generated by a and b respectively. Let η and ζ be the corresponding normed Haar measures on Y and Z and define

$$\begin{aligned} \eta'(E) &= \eta(E \cap Y) \\ \zeta'(E) &= \zeta(E \cap Z) \end{aligned} \quad \text{for all Borel sets } E \subset X.$$

Then we have $\eta', \zeta' \in V$ and the sequences $\{a^n\}$, $\{b^n\}$ are η'-summable and ζ'-summable respectively. Thus, $\{a^n\} \star \{b^n\}$ is $\eta' \star \zeta'$-summable and $\{b^n\} \star \{a^n\}$ is $\zeta' \star \eta'$-summable. As shown in [7], $\eta' \star \zeta' = \mu$ is equivalent to $\zeta' \star \eta' = \mu$ and (13) is a necessary and sufficient condition for $\eta' \star \zeta' = \mu$ to hold.

We note that (13) is in particular satisfied, if at least one of the matrices $D(e)-D(a)$, $D(e)-D(b)$ has *degree* D as rank, i.e. if either $D(a)$ or $D(b)$ does not have Eigenvalue 1 (cf. [5] theorem 2).

Theorem 3: *Suppose that there exists a uniformly distributed sequence in X and let the sequence $\{x_n\} \subset X$ be given. The following statements are equivalent:*

a) *The sequence $\{x_n\}$ is uniformly distributed.*

b) *The sequence $\{x_n\} \star \{y_n\}$ is summable for every sequence $\{y_n\} \subset X$.*

c) *The sequence $\{x_n\} \star \{y_n\}$ is uniformly distributed for every sequence $\{y_n\} \subset X$.*

Proof: We shall use the notation as in the proof of theorem 1. a) $\Rightarrow$ c): Let $\{z_n\} = \{x_n\} \star \{y_n\}$. We have to show that, for any non-trivial irreducible unitary representation D of X,

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} D(z_n) = 0. \tag{14}$$

Since the contribution of the last (possibly incomplete) diagonal in (11) to the average $\frac{1}{N+1} \sum_{n=0}^{N} D(z_n)$ is small (B_5 in the proof of theorem 1), we may consider integers N of the form $\binom{N'+2}{2}-1$ only (see (10)). Given $\varepsilon<0$ we choose N_1 such that

$$\left\| \frac{1}{N+1} \sum_{n=0}^{N} D(x_n) \right\| < \varepsilon \qquad \text{for all } N \geqq N_1. \tag{15}$$

We consider the decomposition

$$\begin{aligned} \frac{1}{N+1} \sum_{n=0}^{N} D(z_n) &= \frac{1}{N+1} \sum_{i=0}^{N_1-1} \sum_{j=0}^{i} D(x_j y_{N'-i}) + \frac{1}{N+1} \sum_{i=N_1}^{N'} \sum_{j=0}^{i} D(x_j y_{N'-i}) = \\ &= \qquad C_1 \qquad\qquad + \qquad\qquad C_2 \qquad . \end{aligned}$$

We have, for $N' \to \infty$

$$\begin{aligned} &\|C_1\| \to 0 \qquad (\text{as } B_2) \\ &\|C_2\| \leqq \frac{1}{N+1} \sum_{i=N_1}^{N'} (i+1) \left\| \left[\frac{1}{i+1} \sum_{j=0}^{i} D(x_j) \right] D(y_{N-i}) \right\| = \\ &\qquad = \frac{1}{N+1} \sum_{i=N_1}^{N'} (i+1) \left\| \frac{1}{i+1} \sum_{j=0}^{i} D(x_j) \right\| < \varepsilon \qquad \text{by (15).} \end{aligned}$$

This proves (14).

c) $\Rightarrow$ b) trivial.

b) $\Rightarrow$ a) Suppose b) holds but $\{x_n\}$ is not uniformly distributed.

Two cases may happen:

1) The sequence $\{x_n\}$ is not summable at all. Select $f \in C(X)$ such that $\frac{1}{N+1}\sum_{n=0}^{N} f(x_n)$ diverges. Consider the sequence $\{z_n\} = \{x_n\} \star \{e\}$ ($y_n = e$ for all $n \geqq 0$). For $N+1 = \binom{N'+2}{2}$ we get

$$\frac{1}{N+1}\sum_{n=0}^{N} f(z_n) = \sum_{i=0}^{N'} \frac{i+1}{\binom{N'+2}{2}} f(x_{N'-i}).$$

The left member converges by hypothesis b), therefore so does the right member. By lemma 1′ the same has to be true for $\frac{1}{N+1}\sum_{n=0}^{N} f(x_n)$, a contradiction.

2) The sequence $\{x_n\}$ is ν-summable and $\nu \neq \mu$. Select $f \in C(X)$ such that $\nu(f) \neq \mu(f)$. Let the sequence $\{y_n\}$ be uniformly distributed. Then $\{x_n\} \star \{y_n\}$ is uniformly distributed by theorem 1 and $\{x_n\} \star \{e\}$ is ν-summable. If we join alternatingly finite sections (of appropriately increasing length) of the sequences $\{y_n\}$ and $\{e\}$ we obtain a new sequence $\{y_n'\}$ with the property that for $\{z_n'\} = \{x_n\} \star \{y_n'\}$ the sequence of the means $\frac{1}{N+1}\sum_{n=0}^{N} f(z_n')$ oscillates between $\nu(f)$ and $\mu(f)$, a contradiction.

We note that the existence of uniformly distributed sequences is in particular established under the hypothesis of the second axiom of countability ([8] theorem 7).

Convolutions of sequences have also been investigated by J. Popken [11] and recently by G. Brauer [1]. In his paper Brauer studies the possibilities of defining convolutions of sequences of real numbers such that for certain summability methods ϕ the functional equation

$$\phi\{s_n\} \star \{t_n\} = \phi\{s_n\} \cdot \phi\{t_n\} \tag{16}$$

is satisfied for all sequences in the domain of ϕ. In fact, (16) also makes sense in our present context if "functionals" ϕ_D on suitable domains S_D of sequences in X are considered which are of the form

$$\phi_D\{s_n\} = \lim_{N\to\infty} \frac{1}{N+1}\sum_{n=0}^{N} D(s_n)$$

(D a finite dimensional representation of X). Thus, (2) and (4) define convolutions of sequences in X with the property that (16) holds for all ϕ_D (D runs through all finite dimensional representations of X) and for all sequences $\{s_n\}$, $\{t_n\}$ in $\bigcap_D S_D$.

However, these sequences do not form a linear space (on which the ϕ_D act as linear functionals) without considerable amount of identification. Thus, the investigations in [1], [11] and in the present note aim in different directions and the definitions of convolution as given in [1] and [11] are not applicable in the situation we are concerned with.

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