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# Graph representation of semigroups

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# Graph representation of semigroups

by

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## 1.1 Notations and definitions

Throughout this paper $(S;.)$ will denote a semigroup with unit element u.

$\varphi$ will be the mapping defined as follows:

$$\varphi(a) = a.S \ ; \quad a \in S.$$

Since $\varphi$ maps every element $a \in S$ onto the principal right ideal generated by a, we shall call $\varphi$ the principal right ideal-(p.r.i) function.

Similarly we can define the principal left ideal function $\psi$: $\psi(a)=S.a \quad a \in S$.

It is evident that $\varphi$ and $\psi$ must fulfil the following conditions:

1) $a \in \varphi(a) \ ; \ a \in \psi(a)$ for every $a \in S$.
2) $b \in \varphi(a) \Longrightarrow \varphi(b) \subset \varphi(a)$.
3) $b \in \psi(a) \Longrightarrow \psi(b) \subset \psi(a)$.
4) $\varphi(a) \cap \psi(b) \neq \emptyset$ for any pair $a, b \in S$.

If $(S;.)$ is commutative then $\varphi = \psi$.

We now ask which mappings $\varphi$ and $\psi$ can be obtained by this proces.

Let X be an arbitrary set and $\varphi, \psi$ mappings of X into P(X),

where P(X) denotes the system of all subsets of X.
We shall say that $\varphi$ and $\psi$ are realised by a semigroup (X;.) if we can introduce a binary operation on X such that (X;.) is a semigroup and such that $\varphi$ is the p.r.i.-function and $\psi$ the p.l.i.-function of (X;.).
If $\varphi$ and $\psi$ are realised by some semigroup, then they evidently must fulfil conditions (1),(2),(3).
If $\varphi$ is the p.r.i.-function of a semigroup (X;.), then $\varphi$ is the p.l.i.-function of the semigroup $(X,*)$, where $x * y = y.x$ for any pair $x,y \in X$.
Therefore we shall restrict ourself to the principal right ideal function $\varphi$.

## 1.2 Graph of a semigroup

To simplify our considerations we shall represent the mapping $\varphi$ by an oriented graph.
An oriented graph is a pair $(X,\nu)$ consisting of a non-empty set X, together with a relation $\nu$ on X, i.e. together with a subset $\nu$ of $X \times X$.
If $(a,b) \in \nu$, where $a,b \in X$, we shall write $a \nu b$. We shall, if possible, represent the elements of the set X by points in $E^2$. Two points a and b will be joined by an oriented line going from a to b if $a \nu b$. A subgraph $(X', \nu')$ of $(X, \nu)$ is a graph such that $X' \subset X$ and $\nu' = \nu | X'$.
$(X', \nu')$ will be called the subgraph generated by X'. By a path $W = a_1 a_2 .. a_k$ is meant any subgraph of $(X,\nu)$ generated by the points $a_1, a_2 .. a_k$ with $a_i \nu a_{i+1}$. $(i=1,2,...k-1)$

We now define an equivalence relation on the set X by putting $x \sim y$ if and only if there exists a path $W_1 = a_1 a_2 .. a_k$ and a path $W_2 = b_1 b_2 .. b_r$ with $a_1 = x$, $a_k = y$, $b_1 = y$, $b_r = x$.
We shall call the equivalence class containing x, the cycle containing x and it will be denoted by $R_x$. On the set of all cycles we can now introduce a partial ordering by defining

$R_x \geq R_y$ if and only if there is a path $W=a_1 a_2 .. a_k$ with $a_1=x$, $a_k=y$. A minimal cycle of $(X,\nu)$ will be a cycle $R_x$, such that if $R_y \leq R_x$, then $R_y=R_x$.

Now let $\varphi$ be the p.r.i.-function of the semigroup $(S;.)$. We associate with $\varphi$ an oriented graph $(S,\varphi^*)$, where $a \varphi^* b$ if and only if $b \in \varphi(a)$ for $a,b \in S$. The relation $\varphi^*$ defined in this way is reflexive and transitive since $\varphi$ fulfils conditions (1) and (2) of 1.1. From the transitivity of $\varphi^*$ it follows that if $W=a_1 a_2 .. a_k$ is a path of $(S,\varphi^*)$ then $a_1 a_k$ is also a path. The skeleton of the graph $(S,\varphi^*)$ will be the graph which can be constructed from $(S,\varphi^*)$ by identifying all cycles $R_x$ to a point.

Lemma 1

The cycles $R_x$ of the graph $(S,\varphi^*)$ are precisely the $\mathcal{R}$-classes of the semigroup $(S;.)$

Proof

$y \in R_x \longleftrightarrow xy$ and $yx$ are both paths in $(S,\varphi^*) \Longleftrightarrow x \varphi^* y$ and $y \varphi^* x \Longleftrightarrow y \in \varphi(x)$ and $x \in \varphi(y) \Longleftrightarrow y \in xS$ and $x \in yS \longleftrightarrow x$ and $y$ are in the same $\mathcal{R}$-class.

Theorem 1

If the graph $(S,\varphi^*)$ has a minimal cycle, then the semigroup $(S;.)$ has a kernel K and K is the union of all minimal cycles. Furthermore, the minimal cycles of $(S,\varphi^*)$ are precisely the minimal right ideals of $(S;.)$.

Proof

Let $R_x$ be a minimal cycle of $(S,\varphi^*)$ and let $a \in R_x$.
Then since for every $b \in aS$ $R_b \leq R_a=R_x$, we have $R_b=R_x$ and $aS \subset R_x$.
Hence $R_x$ is a right ideal of S.
Furthermore we have for every $a,b \in R_x$, $a \in \varphi(b) \Longrightarrow a \in bS$ and hence $R_x$ is a minimal right ideal.
Now let R be a minimal right ideal of $(S;.)$ and $a \in R$.
Then $aS=R$ and R is an $\mathcal{R}$-class of $(S;.)$.

From lemma 1 it then follows that R is a cycle of $(S,\varphi^*)$.
Now let $R_b \leq R$ and $a \in R$.
Then $a\varphi^* b$, which implies $b \in \varphi(a) \Longrightarrow b \in R \Longrightarrow R_b = R$ and R is minimal.
The first statement of the theorem follows from the fact that a semigroup containing a minimal right ideal has a kernel K, K is the union of all minimal right ideals.
Since $(S,.)$ has a unit element u the graph $(S,\varphi^*)$ has a maximal cycle $R_u$ and $(S,\varphi^*)$ is connected $R_u$ is also an upperbound for the partial ordered set of all cycles, since for any $x \in S$, $x \in \varphi(u)$ which implies $R_u \geqslant R_x$.
We can now state the problem in the following way:
Let there be given a graph $(X,\nu)$ with $\nu$ reflexive and transitive and with an upper bound $R_u$ for the set of all cycles.
Under what conditions does there exists a semigroup $(S,.)$ with unit element and with p.l.i.-function $\varphi$ , such that $\varphi$ is represented by the graph $(X,\nu)$ (i.e. $S=X$, $\nu=\varphi^*$).
$(S;.)$ will then be called the realisation of $(X,\nu)$.
In the following we shall only consider graphs $(X,\nu)$ with $\nu$ reflexive and transitive and with an upper bound for the set of all cycles.

## 2. Graphs of commutative semigroups

### 2.1 Realisable graphs

If $(S;.)$ is a commutative semigroup then the p.r.i.-function $\varphi$ and the p.l.i-function $\psi$ must be equal. Then the condition (3) can be written in the form

$$(3')\ \varphi(a) \cap \varphi(b) \neq \emptyset \quad \text{for any pair } a,b \in S.$$

Lemma 2

The graph $(S;\varphi^*)$ of a commutative semigroup $(S;.)$ has at most one minimal cycle.

Proof

In this case there exists at most one minimal right ideal. The rest follows from theorem 1.

Theorem 2

Let $(X;\nu)$ be a graph such that the set of all cycles is a lower semi lattice according to the ordering of the cycles. Then there exists a binary operation, on X, such that $(X;\nu)$ is the graph of the commutative semigroup $(X;.)$.

Proof

Let $R_x$ be an arbitrary cycle. We can introduce on $R_x$ a binary operation, such that $(R_x;.)$ is a commutative group. Let us denote by $R_x \cap R_y$ the cycle, which is the meet of $R_x$ and $R_y$ in the lower semilattice of cycles. By $j(R_x)$ we shall denote the unit element of the group $(R_x;.)$. Moreover, we shall denote by $R_x(y)$ the following function on X

$$R_x(y)=y \qquad \text{if} \quad R_x \geqslant R_y$$

$$R_x(y)=j(R_x) \qquad \text{otherwise.}$$

Let us define

$$x.y=(R_x \cap R_y)(x).(R_x \cap R_y)(y),$$

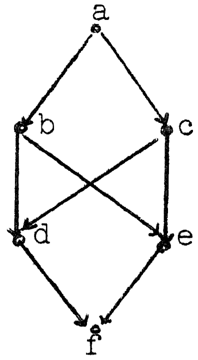
where the point means the multiplication in $(R_x \cap R_y;.)$. It can be elementary proved that $(x;.)$ is a commutative semigroup and that $(X;\nu)$ is the graph of this semigroup.

Corollary

Let X have no more than 6 points, $\varphi$ be a mapping X into P(X) fulfilling conditions (1),(2) and (3'). Let for some point a, $\varphi(a)=X$. Then $\varphi$ is a p.r.i.-function of a commutative semigroup $(X;.)$.

Proof

According to the previous theorem it is enough to consider the graphes the skeletons of which are not lower semilattice. But there is only one skeleton of this type, namely



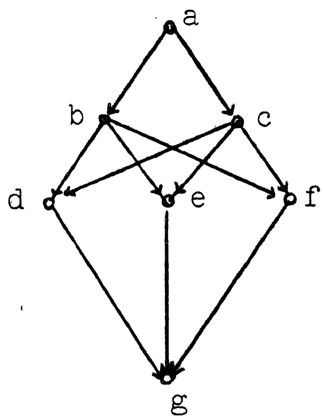

But this graph can be realised by the semigroup

| | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| a | a | b | c | d | e | f |
| b | b | d | e | f | f | f |
| c | c | e | d | f | f | f |
| d | d | f | f | f | f | f |
| e | e | f | f | f | f | f |
| f | f | f | f | f | f | f |

.

2.2.*) Now we shall exhibit an example of a graph, fulfilling the necessary conditions for $\varphi$, which cannot be realised by a commutative semigroup. We know that such a graph must have at least 7 points. The graph looks as follows: $X=\{a,b,c,d,e,f,g\}$

$\varphi(a) = X$
$\varphi(b) = \{b,d,e,f,g\}$,
$\varphi(c) = \{c,d,e,f,g\}$, or
$\varphi(d) = \{d,g\}$ ,
$\varphi(e) = \{e,g\}$ ,
$\varphi(f) = \{f,g\}$ ,
$\varphi(g) = \{g\}$ ,




*) This example was constructed first by P.C. Baayen.

Let us assume that this graph can be realised by $(X;.)$. Then there must exist $x \in X$ such that $x.a=b$.
Let us assume that $x.b=g$. Then also $x.d=x.e=x.f=g$. Evidently $\{d,e,f\} \subset xX$, but this is impossible as only $x.c$ can belong to $\{d,e,f\}$. Therefore we have $x.b \neq g$.
Let $x.b \in \{d,e,f\}$. As the points d,e,f play the same role we can assume that $x.b=d$. But $\{e,f\} \in x.X$, and therefore either $x.e=e$ or $x.f=f$, as only $x.c$ could belong to $\{e,f\}$. Let $x.f=f$. But there must exist $z \in X$ such that $z.b=f$. We have

$$z.x.b=z.d=x.z.b=x.f=f.$$

But this is not possible as $f \notin \varphi(d)$.
Therefore $x.b=b$.
Let $x.c=c$. There exists $y \in X$ such that $y.a=c$.
Then

$$x.y.a=x.c=c=y.x.a=y.b.$$

This is not possible as $c \notin \varphi(b)$.
Let $x.c=g$. As $x.b=b$ we have also $x.d=d$, $x.e=e$, $x.f=f$.
Evidently for some $z \in X$ $z.c=d$. Then

$$x.z.c=x.d=d=z.x.c=z.g,$$

while is impossible as $d \notin \varphi(g)$.
In the same way we can prove that $x.c \notin \{d,e,f\}$.
Hence, the graph cannot be realised by a commutative semigroup.

## 3. Non-commutative semigroups

### 3.1

Let X be a set consisting of 6 points and let $\varphi$ and $\psi$ be mappings of X into P(X), with $\varphi(a)=X$ all $a \in X$ and $\psi(a)=X$ all $a \in X$.
Then $\varphi$ and $\psi$ can be realised by the cycle group of order 6 but also by the symmetric group of degree 3. Hence they can be realised both by a commutative and a non-commutative semigroup.

Hence $\varphi$ and $\psi$ are in general not enough to caracterise the commutative semigroups.
(N.B. $\varphi = \psi$ for every normal semigroup).
On the other hand there are mappings which can be realised only by a commutative semigroup. For example $X=\{a,b\}$ $\varphi(a)=\psi(a)=X$, $\varphi(b)=X$ can only be realised by the cyclic group of order 2.

In 2.2. we have given an example of a graph which cannot be realised by a commutative semigroup.
This graph however is the graph of the following non-commutative semigroup.

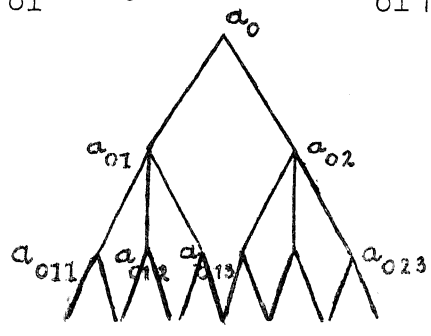
| | a | b | c | d | e | f | g |
|---|---|---|---|---|---|---|---|
| a | a | b | c | d | e | f | g |
| b | b | b | b | d | e | f | g |
| c | c | c | c | d | e | f | g |
| d | d | d | d | d | d | d | g |
| e | e | e | e | e | e | e | g |
| f | f | f | f | f | f | f | g |
| g | g | g | g | g | g | g | g |

We will now give an example of a class of graphs which can't be realised by semigroups.
It is known that if a semigroup S contains a minimal left and a minimal right ideal, then the kernel K of S is the disjoint union of isomorphic groups $L_\alpha \cap R_\beta$ $\quad K = \bigcup_{\substack{L \in \mathcal{L}(S) \\ R \in \mathcal{R}(S)}} L_\alpha \cap R_\beta$ , where $\mathcal{L}(S)$ and $\mathcal{R}(S)$ are the sets of minimal left and right ideals of S.
From this fact follows the following theorem

Theorem 3
Let there be given a graph $(X,\nu)$ and suppose that $(X,\nu)$ contains a minimal cycle $R_a$ and a minimal cycle $R_b$ with $|R_a|=n$, $|R_b| \neq n$ n finite. Then $(X,\nu)$ cannot be realised by a semigroup $(X;.)$

Proof
Let $(X;.)$ be a realisation of $(X,\nu)$.

Then $R_a$ is a minimal right ideal of $(X,.)$. Since $R_a$ has only a finite number of points, $R_a$ contains an idempotent.
Hence K contains an idempotent and also a minimal left ideal.
From this it follows that $R_a = \bigcup_{L \in \mathcal{L}(S)} R_a \cap L_\alpha$ and $R_b = \bigcup_{L_\alpha \in \mathcal{L}(S)} R_b \cap L_\alpha$.
Since $R_a \cap L_\alpha \cong R_b \cap L_\alpha$, we have $|R_a \cap L_\alpha| = |R_b \cap L_\alpha| \Longrightarrow |R_a| = |R_b|$ a contradiction.
Let there be given any sequence of positive natural numbers $m_1, m_2, m_3 \ldots$ .
Let X be the set of points $\{a_{n_o n_1 \ldots n_k}\}$ with $k=0,1,2,\ldots$ and $n_o=0$, $n_k=1,\ldots m_k$ $k \geq 1$.
We now make X into a graph $(X,\nu)$ by defining $a_{n_o n_1 \ldots n_k} \nu\, a_{h_o h_1 \ldots h_s}$ if and only if $s=k+1$ and $n_o=h_o,\ldots,n_k=h_{s-1}$.
This means that $(X,\nu)$ is a tree with top point $a_o$ and such that $a_o$ is joined with $a_{o1}, a_{o2}, \ldots, a_{on_1}$ by an arc leading from $a_o$ to $a_{oi}$. Each of the $a_{oi}$ is joined with $a_{oi1}, a_{oi2}, \ldots a_{oin_2}$ and so on.




From $(X,\nu)$ we define the graph $(X,\nu^*)$ by $b\, \nu^* c$ if and only if there exists a path in $(X,\nu)$ going from b to c. $\nu^*$ is then reflexive and transitive and $(X,\nu^*)$ has an upper bound for the set of all cycles.
Here the cycles consist of only one point. We will call such a graph $(X,\nu^*)$ a homogeneous tree.

<u>Theorem 4</u>
Let $(X,\nu^*)$ be a homogeneous tree. Then $(X,\nu^*)$ can be realised by a semigroup $(X,\cdot)$.

Proof

We define a multiplication on X in the following way

$$a_{n_0 n_1 \ldots n_k} \cdot a_{h_0 h_1 \ldots h_r} \begin{cases} = a_{n_0 \ldots n_k} & \text{if } k \geq 1 \\ = a_{n_0 \ldots n_k\, h_{k+1}\, h_r} & \text{if } k < r. \end{cases}$$

It is easy to see that this multiplication is associative and hence $(X, \cdot)$ as a semigroup.

$(X, .)$ is a realisation of $(X, \nu^*)$, since it follows from the multiplication that

$$\varphi(a_{n_0 \ldots n_k}) = a_{n_0 \ldots n_k} X = \{ a_{n_0 \ldots n_k\, h_{k+1} \ldots h_{k+r}} \} \begin{matrix} r = 0, 1, \ldots \\ h_{k+1} = 1, 2, \ldots m_{k+1} \end{matrix}$$

and hence if $b \in \varphi(a)$ then $a \, \nu^* \, b$.

Analoguously we can prove that if $a \, \nu^* \, b$ then $b \in \varphi(a)$. Thus $(X, .)$ is a realisation of $(X, \nu^*)$.