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J. van de LUNE

THE TRUNCATED AVERAGE LIMIT AND SOME OF ITS APPLICATIONS IN ANALYTIC NUMBER THEORY

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The Truncated Average Limit and some of its applications in analytic number theory

by

J. van de Lune

Abstract

In this report the concept "Truncated Average Limit" is introduced as follows. Let $\{a_n\}_{n=1}^{\infty}$ be a non-negative sequence and define the functions ϕ and Φ by

$$\phi(A) = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \min(a_k, A)$$

and

$$\Phi(A) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \min(a_k, A).$$

If $\lim_{A\to\infty} \phi(A) = \lim_{A\to\infty} \Phi(A) = L$, then L is called the truncated average limit of the sequence $\{a_n\}_{n=1}^{\infty}$.

This concept can be extended to arbitrary complex sequences. Subsequently this limiting process is applied to various number theoretic sequences involving the sequence $\{\lambda_n\}_{n=1}^{\infty}$, where $\lambda_1 = 1$ and $\lambda_n = \frac{\log n}{\log g_n}$, $(n\geq 2)$, g_n being the largest prime divisor of n.

1. The truncated average limit and some of its elementary properties

Notation: If a and A are real numbers, then we will write (a,A) instead of $\min(a,A)$.

Let $\{a_n\}_{n=1}^{\infty}$ be a given sequence such that $a_n \geq 0$ for all n. Also let $A \geq 0$ be given. Then obviously

$$0 \leq \frac{1}{n} \sum_{k=1}^{n} (a_k,A) \leq A.$$

We now define

$$\Phi(A) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (a_k,A)$$

and

$$\phi(A) = \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (a_k,A).$$

It is clear that $0 \leq \phi(A) \leq \Phi(A) \leq A$ and that $\phi(A)$ and $\Phi(A)$ are both monotonically non-decreasing functions of A. If both $\lim_{A\to\infty} \phi(A)$ and $\lim_{A\to\infty} \Phi(A)$ exist and are equal to, say, L then we write

$$L = \mathrm{tal}(a_n),$$

and L is called the *truncated average limit* of the sequence $\{a_n\}_{n=1}^{\infty}$.

Proposition 1.1. If $a_n \geq 0$ for $n = 1,2,3,\ldots$ and $\mathrm{tal}(a_n) = a$ and $\lambda \geq 0$, then $\mathrm{tal}(\lambda a_n) = \lambda a$.

Proof. The case $\lambda = 0$ is trivial. If $\lambda > 0$ the result follows immediately from the identity

$$\frac{1}{n} \sum_{k=1}^{n} (\lambda a_k,A) = \lambda \cdot \frac{1}{n} \sum_{k=1}^{n} (a_k,\frac{A}{\lambda}), \qquad (A \geq 0). \quad \square$$

Proposition 1.2. If a, b and A are all ≥ 0 then

$$(a,\frac{A}{2}) + (b,\frac{A}{2}) \leq (a+b,A) \leq (a,A) + (b,A).$$

Proof. Straightforward verification by cases. □

Proposition 1.3. If $a_n \geq 0$ and $b_n \geq 0$ for $n = 1,2,3,\ldots$ and $\mathrm{tal}(a_n)$ and $\mathrm{tal}(b_n)$ exist, then $\mathrm{tal}(a_n+b_n) = \mathrm{tal}(a_n) + \mathrm{tal}(b_n)$.

Proof. This follows easily from the inequalities $(A \geq 0)$

$$\frac{1}{n}\sum_{k=1}^{n} \{(a_k,\frac{A}{2}) + (b_k,\frac{A}{2})\} \leq \frac{1}{n}\sum_{k=1}^{n} (a_k+b_k,A) \leq$$

$$\leq \frac{1}{n}\sum_{k=1}^{n} \{(a_k,A) + (b_k,A)\}. \quad □$$

Proposition 1.4. Suppose $\{a_n\}_{n=1}^{\infty}$ is a real sequence such that $a_n = P_n - N_n = Q_n - M_n$, where P_n, N_n, Q_n and M_n are ≥ 0 for $n = 1,2,3,\ldots$. Suppose also that all the sequences $\{P_n\}_{n=1}^{\infty}$, $\{N_n\}_{n=1}^{\infty}$, $\{Q_n\}_{n=1}^{\infty}$ and $\{M_n\}_{n=1}^{\infty}$ have a finite tal, called P, N, Q and M respectively. Then we have

$$P - N = Q - M.$$

Proof. $P_n - N_n = Q_n - M_n$, hence $P_n + M_n = Q_n + N_n$. Using proposition 1.3 it follows that $\mathrm{tal}(P_n) + \mathrm{tal}(M_n) = \mathrm{tal}(P_n+M_n) = \mathrm{tal}(Q_n+N_n) = \mathrm{tal}(Q_n) + \mathrm{tal}(N_n)$ and the proposition follows. □

If $\{a_n\}_{n=1}^{\infty}$ is such that $a_n = P_n - N_n$ and P_n and N_n are ≥ 0, both sequences $\{P_n\}_{n=1}^{\infty}$ and $\{N_n\}_{n=1}^{\infty}$ having finite tal's (say P and N respectively) then we write $\mathrm{TAL}(a_n) = P - N$ and $P - N$ is called the *Truncated Average Limit* of the sequence $\{a_n\}_{n=1}^{\infty}$. From the previous proposition it is clear that this definition makes sense.

Proposition 1.5. If $a_n \geq 0$ for $n = 1,2,3,\ldots$ and $TAL(a_n)$ exists then also $tal(a_n)$ exists and equals $TAL(a_n)$.

Note. It is trivial that if $tal(a_n)$ exists then also $TAL(a_n)$ exists and equals $tal(a_n)$.

Proof. Suppose $TAL(a_n) = a$. Then there is a decomposition $a_n = P_n - N_n$ such that $P_n \geq 0$, $N_n \geq 0$, $tal(P_n) = P$, $tal(N_n) = N$ and $P - N = a$. From proposition 1.2 it follows that

$$\frac{1}{n}\sum_{k=1}^{n}(a_k,A) \leq \frac{1}{n}\sum_{k=1}^{n}\{(a_k+N_k,2A) - (N_k,A)\} =$$

$$= \frac{1}{n}\sum_{k=1}^{n}(P_k,2A) - \frac{1}{n}\sum_{k=1}^{n}(N_k,A).$$

Hence

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(a_k,A) \leq \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(P_k,2A) - \liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(N_k,A) \stackrel{def}{=}$$

$$\stackrel{def}{=} \Phi_1(2A) - \phi_2(A).$$

On the other hand we have

$$\frac{1}{n}\sum_{k=1}^{n}(a_k,A) \geq \frac{1}{n}\sum_{k=1}^{n}\{(a_k+N_k,A) - (N_k,A)\} =$$

$$= \frac{1}{n}\sum_{k=1}^{n}(P_k,A) - \frac{1}{n}\sum_{k=1}^{n}(N_k,A)$$

and it follows that

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(a_k,A) \geq \liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(P_k,A) - \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(N_k,A) =$$

$$\stackrel{def}{=} \phi_1(A) - \Phi_2(A).$$

Since

$$\lim_{A\to\infty} \phi_1(A) = \lim_{A\to\infty} \Phi_1(A) = P$$

and

$$\lim_{A\to\infty} \phi_2(A) = \lim_{A\to\infty} \Phi_2(A) = N$$

it follows that $\text{tal}(a_n)$ exists and equals $P - N = a = \text{TAL}(a_n)$. □

Proposition 1.6. If $\text{TAL}(a_n) = a$ and $\text{TAL}(b_n) = b$ then

$$\text{TAL}(a_n+b_n) = a + b.$$

Proof. Write $a_n = P_n - N_n$ where P_n and N_n are ≥ 0 for $n = 1,2,\ldots$ and let $\text{tal}(P_n) = P$, $\text{tal}(N_n) = N$ and $P - N = A$. Also write $b_n = Q_n - M_n$ where Q_n and M_n are ≥ 0 for $n = 1,2,\ldots$ and let $\text{tal}(Q_n) = Q$, $\text{tal}(M_n) = M$ and $Q - M = b$. Then $a_n + b_n = (P_n-N_n) + (Q_n-M_n) = (P_n+Q_n) - (N_n+M_n)$. Since $\text{tal}(P_n+Q_n) = P + Q$ and $\text{tal}(N_n+M_n) = N + M$, we obtain

$$\text{TAL}(a_n+b_n) = (P+Q) - (N+M) = (P-N) + (Q-M) = a + b.$$ □

Proposition 1.7. If $\text{TAL}(a_n)$ exists and $\lambda \in \mathbb{R}$ then

$$\text{TAL}(\lambda a_n) = \lambda \cdot \text{TAL}(a_n).$$

Proof. The case $\lambda = 0$ is trivial. If $\lambda > 0$, write $\lambda a_n = \lambda P_n - \lambda N_n$ and if $\lambda < 0$, write $\lambda a_n = (-\lambda)N_n - (-\lambda)P_n$, where P_n and N_n have the same meaning as as in the previous proof, and the proposition follows easily. □

Proposition 1.8. If $\{a_n\}_{n=1}^{\infty}$ is a non-negative sequence with finite limit a, then $\text{tal}(a_n) = a$.

Proof. Since $\lim_{n\to\infty} a_n$ exists, there is a number G such that $a_n \leq G$ for all n. Take $A \geq G$ and observe that then

$$\frac{1}{n}\sum_{k=1}^{n}(a_k,A) = \frac{1}{n}\sum_{k=1}^{n} a_k.$$

Hence $\Phi(A) = \phi(A) = a$ for $A \geq G$, and the proposition follows. □

Remark. Actually we proved that if $0 \leq a_n \leq G$ and the Césaro limit of $\{a_n\}_{n=1}^{\infty}$ is a, then also $\mathrm{tal}(a_n) = a$.

Proposition 1.9. If $\lim_{n\to\infty} a_n = a$ then also $\mathrm{TAL}(a_n) = a$.

Proof. Since a_n is bounded, say $|a_n| \leq G$, we may write $a_n = (a_n+G_n) - G_n$ where $G_n = G$ for all n. Then $a_n + G_n \geq 0$ and $\lim_{n\to\infty} (a_n+G_n) = a + G$. Also $G_n \geq 0$ and $\lim_{n\to\infty} G_n = G$. Hence $\mathrm{tal}(a_n+G_n) = a + G$ and $\mathrm{tal}(G_n) = G$. Consequently $\mathrm{TAL}(a_n) = (a+G) - G = a$. □

Proposition 1.10. If $a_n \leq b_n$, for all $n \in \mathbb{N}$, then

$$\mathrm{TAL}(a_n) \leq \mathrm{TAL}(b_n)$$

provided that both sides exist.

Proof. Write $a_n = P_n - N_n$ and $b_n = Q_n - M_n$ where P_n, N_n, Q_n and M_n have their usual meaning. Then $P_n - N_n \leq Q_n - M_n$ or $P_n + M_n \leq Q_n + N_n$. Thus, since the proposition is evidently true for non-negative sequences, we obtain $\mathrm{tal}(P_n+M_n) \leq \mathrm{tal}(Q_n+N_n)$ or $\mathrm{tal}(P_n) + \mathrm{tal}(M_n) \leq \mathrm{tal}(Q_n) + \mathrm{tal}(N_n)$ or $\mathrm{TAL}(a_n) = \mathrm{tal}(P_n) - \mathrm{tal}(N_n) \leq \mathrm{tal}(Q_n) - \mathrm{tal}(M_n) = \mathrm{TAL}(b_n)$, proving the proposition. □

Remarks.

1) If $\{z_n\}_{n=1}^{\infty}$ is a complex sequence then we can define

$$\mathrm{TAL}(z_n) = \mathrm{TAL}(\mathrm{Re}\ z_n) + i\ \mathrm{TAL}(\mathrm{Im}\ z_n)$$

provided that the last two Truncated Average Limits exist.

2) It is easy to construct sequences for which the TAL exists whereas the Cesàro limit does not exist; it turns out to be much more difficult to construct a sequence for which the converse is true. However, an example for such a sequence has been given by P. van Emde Boas.

2. Number theoretical background for the applications

For a proper understanding of the applications, we will have to digress on a topic from analytic number theory.

Let g_n be the largest prime divisor of the natural number $n(\geq 2)$, whereas $g_1 = 1$. Let $G(n,\alpha)$ be the number of positive integers m such that $m \leq n$ and $g_m < m^\alpha$, where α is any given real number. It can be shown [2], [4] that the limit

$$G(\alpha) = \lim_{n\to\infty} \frac{1}{n} G(n,\alpha)$$

exists and is continuous on the real line. Furthermore $G(\alpha)$ satisfies the equation

$$G'(\alpha) = \frac{1}{\alpha} G\left(\frac{\alpha}{1-\alpha}\right)$$

for $0 < \alpha < 1$.

If we define $y: [0,\infty) \to \mathbb{R}$ by

$$\begin{aligned} y(x) &= 1 && \text{if } 0 \leq x \leq 1 \\ y(x) &= G\left(\frac{1}{x}\right) && \text{if } x > 1, \end{aligned}$$

then y is continuous and satisfies the difference-differential equation

$$y'(x) = -\frac{1}{x}\, y(x-1), \qquad (x > 1).$$

From this it is easily deduced that y also satisfies the relation

$$x \cdot y(x) = \int_{x-1}^{x} y(t)\, dt, \qquad (x \geq 1).$$

From this equation it follows that $y(x) > 0$ for all $x \geq 0$ and using the difference-differential equation once more, it follows that y is monotonically decreasing on the interval $[1,\infty)$.

The Laplace transform $\hat{y}$ of y can be computed explicitely as

$$\hat{y}(s) = \int_0^\infty e^{-sx}\, y(x)dx = \exp\{\gamma + \int_0^s \frac{e^{-z}-1}{z}\, dz\}$$

which is an entire function (γ is Euler's constant). Using the inversion formula for Laplace transforms it can be shown [4] that

$$y(x) = \frac{e^{x\gamma(x)}}{(x \log x)^x}, \qquad (x > 1)$$

where $\limsup_{x\to\infty} \gamma(x) \leq 1$. Informally speaking: $y(x)$ tends exceedingly fast to zero when $x\to\infty$. We have for example [5]

$$y(1000) = 0.4 \cdot 10^{-3463}.$$

This concludes our excursion into number theory.

We now define the sequence $\{\lambda_k\}_{k=1}^\infty$ as follows:

$$\lambda_1 = 1$$
$$\lambda_k = \frac{\log k}{\log g_k}, \text{ (implicitly } g_k^{\lambda_k} = k\text{), if } k \geq 2.$$

In the remainder of this report we will consider a number of sequences (involving the above λ_k's), which have Truncated Average Limits, whereas in several cases it may be very difficult or even impossible to show that these sequences have Cesàro limits. In the next section we first derive some general theorems concerning this matter.

3. Applications

Proposition 3.1. If the function $f:[1,\infty) \to \mathbb{R}$ is such that the integral

$$\int_1^A f(x)dy(x), \qquad (A>1)$$

exists as an ordinary Riemann-Stieltjes integral, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{k\le n\\ \lambda_k<A}} f(\lambda_k) = -\int_1^A f(x)dy(x).$$

Proof. On the interval [1,A] we construct a partition $1 = a_0 < a_1 < a_2 < \ldots < a_{r-1} < a_r = A$ and define

$$M_\nu = \sup_{a_{\nu-1}\le x\le a_\nu} f(x),$$

and

$$m_\nu = \inf_{a_{\nu-1}\le x\le a_\nu} f(x).$$

Since $\int_1^A f(x)dy(x)$ exists we may choose the subdivision of [1,A] such that

$$\sum_{\nu=1}^{r} M_\nu \{y(a_{\nu-1})-y(a_\nu)\} < -\int_1^A f(x)dy(x) + \varepsilon$$

and

$$\sum_{\nu=1}^{r} m_\nu \{y(a_{\nu-1})-y(a_\nu)\} > -\int_1^A f(x)dy(x) - \varepsilon.$$

We now write

$$\frac{1}{n} \sum_{\substack{k\le n\\ \lambda_k<A}} f(\lambda_k) = \frac{1}{n} \sum_{\nu=1}^{r} \sum_{\substack{a_{\nu-1}\le\lambda_k<a_\nu\\ k\le n}} f(\lambda_k)$$

and observe that

$$\frac{1}{n}\sum_{\nu=1}^{r}\sum_{\substack{a_{\nu-1}\le\lambda_k<a_\nu\\ k\le n}} f(\lambda_k) \le \frac{1}{n}\sum_{\nu=1}^{r}\sum_{\substack{a_{\nu-1}\le\lambda_k<a_\nu\\ k\le n}} M_\nu =$$

$$= \frac{1}{n}\sum_{\nu=1}^{r} M_\nu \{G(n,\frac{1}{a_{\nu-1}}) - G(n,\frac{1}{a_\nu})\}$$

which follows from the fact that for all ν the number of natural numbers k satisfying the conditions $k \le n$ and $a_{\nu-1} \le \lambda_k < a_\nu$ is equal to

$$G(n,\frac{1}{a_{\nu-1}}) - G(n,\frac{1}{a_\nu}).$$

Since $$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{r} M_\nu \{G(n,\frac{1}{a_{\nu-1}}) - G(\frac{1}{a_\nu})\} = \sum_{\nu=1}^{r} M_\nu \{G(\frac{1}{a_{\nu-1}}) - G(\frac{1}{a_\nu})\} =$$

$$= \sum_{\nu=1}^{r} M_\nu \{y(a_{\nu-1}) - y(a_\nu)\} <$$

$$< -\int_1^A f(x)dy(x) + \varepsilon$$

we obtain that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{\substack{k\le n\\ \lambda_k<A}} f(\lambda_k) < -\int_1^A f(x)dy(x) + \varepsilon.$$

In a similar way one also proves that

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{\substack{k\le n\\ \lambda_k<A}} f(\lambda_k) > -\int_1^A f(x)dy(x) - \varepsilon.$$

Since this holds for all $\varepsilon > 0$, the proposition follows. □

Proposition 3.2. If the function $f(x)$ is bounded for $x \ge 1$ and the integral $\int_1^\infty f(x)dy(x)$ exists as a Riemann-Stieltjes integral, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\lambda_k) = - \int_1^{\infty} f(x)dy(x).$$

Proof. Let $|f(x)| \le M$ for $x \ge 1$. We write

$$\frac{1}{n} \sum_{k=1}^{n} f(\lambda_k) = \frac{1}{n} \sum_{\substack{k\le n \\ \lambda_k < A}} f(\lambda_k) + \frac{1}{n} \sum_{\substack{k\le n \\ \lambda_k \ge A}} f(\lambda_k)$$

and fix $A > 1$ such that $M\cdot y(A)$ is small. We then have

$$\left|\frac{1}{n} \sum_{k=1}^{n} f(\lambda_k) + \int_1^{\infty} f(x)dy(x)\right| \le \left|\frac{1}{n} \sum_{\substack{k\le n \\ \lambda_k < A}} f(\lambda_k) + \int_1^{A} f(x)dy(x)\right| +$$

$$+ \frac{1}{n} \sum_{\substack{k\le n \\ \lambda_k \ge A}} |f(\lambda_k)| + \left|\int_A^{\infty} f(x)dy(x)\right|.$$

According to proposition 3.1 the first of these right-hand terms can be made arbitrarily small by taking n large enough. The second term is not larger than $\frac{M}{n} G(n,\frac{1}{A})$ which tends to $M\cdot y(A)$ as $n \to \infty$, whereas the third term is at most

$$- M \int_A^{\infty} dy(x) = M\cdot y(A).$$

From this it is clear that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\lambda_k) = - \int_1^{\infty} f(x)dy(x). \quad \square$$

THEOREM 3.1. Let the non-negative continuous function $f:[1,\infty) \to \mathbb{R}$ be such that

$$\int_1^{\infty} f(x)dy(x)$$

exists as a Riemann-Stieltjes integral and let $\lim_{x\to\infty} f(x)y(x) = 0$. Then

$$\operatorname{tal} f(\lambda_n) = -\int_1^\infty f(x)dy(x).$$

Proof. If f(x) is bounded, the assertion follows from proposition 3.2. Suppose therefore that f(x) is not bounded. Since f(x) is bounded on bounded intervals we can define

$$\mu(A) = \sup\{a \mid f(x) \le A \text{ for } 1\le x\le a\}$$

for every sufficiently large $A > 0$. The following statements are easily proved to be true:

1. $\mu(A)$ is an increasing function of A such that $\lim_{A\to\infty} \mu(A) = \infty$.
2. $f(x) \le A$ for $1 \le x < \mu(A)$.
3. $f(\mu(A)) = A$.

Now construct a subdivision $1 = a_0 < a_1 < \dots < a_n = \mu(A)$ of the interval $1 \le x \le \mu(A)$. Then we have

$$\frac{1}{N}\sum_{k=1}^{N} (f(\lambda_k),A) \ge \frac{1}{N}\sum_{\nu=1}^{n} \sum_{\substack{a_{\nu-1}\le\lambda_k<a_\nu \\ k\le N}} (f(\lambda_k),A) =$$

$$= \frac{1}{N}\sum_{\nu=1}^{n} \sum_{\substack{a_{\nu-1}\le\lambda_k<a_\nu \\ k\le N}} f(\lambda_k) \ge$$

$$\ge \frac{1}{N}\sum_{\nu=1}^{n} m_\nu \{G(N,\frac{1}{a_{\nu-1}}) - G(N,\frac{1}{a_\nu})\},$$

where $m_\nu \overset{\text{def}}{=} \inf_{a_{\nu-1}\le x\le a_\nu} f(x)$, and consequently

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (f(\lambda_k),A) \ge \sum_{\nu=1}^{n} m_\nu \{y(a_{\nu-1}) - y(a_\nu)\}.$$

Since the mesh of the subdivision may be taken as small as we want, it follows that

$$\phi(A) \overset{\text{def}}{=} \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (f(\lambda_k),A) \ge - \int_{1}^{\mu(A)} f(x)dy(x).$$

Hence

$$\lim_{A\to\infty} \phi(A) \ge - \int_{1}^{\infty} f(x)dy(x).$$

On the other hand we have

$$\frac{1}{N} \sum_{k=1}^{N} (f(\lambda_k),A) \le \frac{1}{N} \sum_{\nu=1}^{n} \sum_{\substack{a_{\nu-1}\le\lambda_k<a_\nu \\ k\le N}} f(\lambda_k) + \frac{1}{N} \sum_{\substack{\lambda_k\ge a_n=\mu(A) \\ k\le N}} A \le$$

$$\le \frac{1}{N} \sum_{\nu=1}^{n} M_\nu \{G(N,\frac{1}{a_{\nu-1}}) - G(N,\frac{1}{a_\nu})\} + \frac{A}{N} G(N,\frac{1}{a_n}),$$

where $M_\nu = \sup\limits_{a_{\nu-1}\le x\le a_\nu} f(x)$. Letting $N \to \infty$, it follows that

$$\Phi(A) \overset{\text{def}}{=} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (f(\lambda_k),A) \le$$

$$\le \sum_{\nu=1}^{n} M_\nu \{y(a_{\nu-1}) - y(a_\nu)\} + Ay(a_n) =$$

$$= \sum_{\nu=1}^{n} M_\nu \{y(a_{\nu-1}) - y(a_\nu)\} + f(\mu(A))\cdot y(\mu(A)).$$

From this it is clear that

$$\Phi(A) \le - \int_{1}^{\mu(A)} f(x)dy(x) + f(\mu(A))\cdot y(\mu(A)).$$

Hence

$$\lim_{A\to\infty} \Phi(A) \leq - \int_1^\infty f(x)dy(x).$$

Consequently $\lim_{A\to\infty} \phi(A) = \lim_{A\to\infty} \Phi(A) = - \int_1^\infty f(x)dy(x)$, proving the theorem. □

THEOREM 3.2. If $f(x)$ can be written as $f(x) = p(x) - q(x)$ where $p(x)$ and $q(x)$ are continuous and non-negative for $x \geq 1$ and $\lim_{x\to\infty} p(x)y(x) = \lim_{x\to\infty} q(x)y(x) = 0$ while the integrals

$$\int_1^\infty p(x)dy(x) \text{ and } \int_1^\infty q(x)dy(x)$$

exist as Riemann-Stieltjes integrals, then

$$\text{TAL } f(\lambda_k) = - \int_1^\infty f(x)dy(x).$$

Proof. This is an immediate consequence of theorem 3.1 and the definition of the Truncated Average Limit. □

In order to obtain some concrete results concerning theorems 3.1 and 3.2 we will derive a number of integrals involving the function $y(x)$. The corresponding results concerning Truncated Average Limits will only be stated occasionally.

First we give a few simple examples.

Example 1. Note that

$$1 = - \int_1^\infty dy(x) = - \int_1^\infty y'(x)dx = - \int_1^\infty \frac{-1}{x} y(x-1)dx = \int_0^\infty \frac{y(x)}{x+1} dx$$

so that

$$- \int_1^\infty \log(1+x)dy(x) = - \int_0^\infty \log(1+x)dy(x) =$$

$$= - \log(1+x)y(x) \Big|_0^\infty + \int_0^\infty \frac{y(x)}{x+1}\, dx = 1.$$

Hence

$$- \int_1^\infty \log(1+x)dy(x) = 1 \text{ or } \operatorname{tal}(\log(1+\lambda_n)) = 1.$$

Example 2. $- \int_1^\infty (\frac{1}{x} + \frac{1}{x+1})dy(x) = 1$ or $\operatorname{tal}(\frac{1}{\lambda_n} + \frac{1}{\lambda_n+1}) = 1.$

For the proof see theorem 3.3, taking $s = -1$.

Example 3. $- \int_1^\infty x\, dy(x) = e^\gamma$ or $\operatorname{tal}(\lambda_n) = e^\gamma$,

where γ is Euler's constant. The proof will be given on page 17.

Remark. It seems that the integral $- \int_1^\infty \frac{1}{x}\, dy(x)$ can not be evaluated elementary. However, it will be shown that the value of the integral is larger than γ, disproving a conjecture that its value is γ. We have (compare section 1)

$$- \int_1^\infty \frac{1}{x}\, dy(x) = \int_0^1 u\, dG(u) =$$

$$= u\, G(u) \Big|_0^1 - \int_0^1 G(u)du = 1 - \int_0^1 G(u)du.$$

Since $G(u)$ is convex on $0 \le u \le 1$ (cf.[5]) we also have, using table 1 in [5],

$$\int_0^1 G(u)du = (\int_0^{1/3} + \int_0^{1/2} + \int_{1/2}^1)G(u)du <$$

$$< \frac{1}{3} \cdot \frac{G(\frac{1}{3})}{2} + (\frac{1}{2} - \frac{1}{3}) \frac{G(\frac{1}{2}) + G(\frac{1}{3})}{2} + \frac{1}{2} \log 2 <$$

$$< \frac{0.05}{6} + \frac{0.4}{12} + 0.35 < 0.4.$$

Hence

$$-\int_1^\infty \frac{1}{x}\,dy(x) > 1 - 0.4 = 0.6 > \gamma.$$

Definition.

$$\alpha(s) = \int_0^\infty x^s y(x)dx, \quad (\text{Re } s > -1)$$

and

$$\beta(s) = -\int_0^\infty x^s dy(x), \quad (s \in \mathbb{C}).$$

Proposition 3.3. $\beta(s+1) = (s+1)\alpha(s)$, $(\text{Re } s > -1)$.

Proof. If Re $s > -1$, then

$$\alpha(s) = \int_0^\infty x^s y(x)dx = \int_0^\infty y(x)\; d\,\frac{x^{s+1}}{s+1}$$

$$= y(x)\,\frac{x^{s+1}}{s+1}\Bigg|_0^\infty - \int_0^\infty \frac{x^{s+1}}{s+1}\,dy(x) =$$

$$= -\frac{1}{s+1}\int_0^\infty x^{s+1}dy(x) = \frac{\beta(s+1)}{s+1}. \qquad \square$$

Note. This formula also provides the analytic continuation of $\alpha(s)$.

Proposition 3.4. For every $s \in \mathbb{C}$ we have

$$\int_0^\infty (x+1)^s y(x)dx = (s+1)\int_0^\infty x^s y(x)dx.$$

Proof. $\beta(s+1) = -\int_0^\infty x^{s+1}dy(x) = -\int_1^\infty x^{s+1}dy(x) =$

$$= -\int_1^\infty x^{s+1} \cdot \frac{-1}{x} y(x-1)dx = \int_1^\infty x^s y(x-1)dx =$$

$$= \int_0^\infty (x+1)^s y(x)dx.$$

Also $\beta(s+1) = (s+1)\alpha(s) = (s+1)\int_0^\infty x^s y(x)dx$ and the proposition follows. □

Proposition 3.5. $\alpha(n) = \frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k}\alpha(k)$, $n=1,2,3,\ldots$.

Proof. $\int_0^\infty (x+1)^n y(x)dx = \int_0^\infty y(x)(\sum_{k=0}^{n}\binom{n}{k}x^k)dx = \sum_{k=0}^{n}\binom{n}{k}\alpha(k).$

On the other hand $\int_0^\infty (x+1)^n y(x)dx = (n+1)\alpha(n)$, and it follows that

$$n\alpha(n) = \sum_{k=0}^{n-1}\binom{n}{k}\alpha(k).$$ □

Proposition 3.6. For $n=0,1,2,3,\ldots$, $\alpha(n)$ is a rational multiple of e^γ.

Proof. $\alpha(0) = \int_0^\infty y(x)dx = \hat{y}(0) = e^\gamma$ (see section 1). The assertion now follows from proposition 3.5 by induction. □

Proposition 3.7. $\beta(n+1) = \sum_{k=0}^{n}\binom{n}{k}\alpha(n)$, $n=0,1,2,3,\ldots$.

Proof. In (the proof of) proposition 3.5 we saw that

$$(n+1)\alpha(n) = \sum_{k=0}^{n}\binom{n}{k}\alpha(k).$$

Since $(n+1)\alpha(n) = \beta(n+1)$ the proposition follows. □

Proposition 3.8. $\beta(0) = 1$ and for $n=1,2,3,\ldots$, $\beta(n)$ is a rational multiple of e^γ.

Proof. Clearly $\beta(0) = -\int_0^\infty dy(x) = 1$. The remaining part follows from propositions 3.6 and 3.7. □

Examples. $\mathrm{tal}(\lambda_n) = e^\gamma$; $\mathrm{tal}(\lambda_n^2) = 2e^\gamma$; $\mathrm{tal}(\lambda_n^3) = \frac{9}{2} e^\gamma$.

Proposition 3.9. The number e^γ is rational if and only if there exist two integers a and b, both different from zero such that

$$\int_0^\infty (ax+b)dy(x) = 0.$$

Proof. This follows easily from proposition 3.8 and the fact that $\beta(1) = \alpha(0) = e^\gamma$. □

THEOREM 3.3. For all $s \in \mathbb{C}$ we have

$$\int_1^\infty \{sx^s - (x+1)^s + 1\}\, dy(x) = 0.$$

Proof. According to proposition 3.4 we have

$$\int_0^\infty (x+1)^{s-1}\, y(x)dx = s\int_0^\infty x^{s-1}\, y(x)dx, \quad (\mathrm{Re}\ s > 0).$$

The left-hand side can be rewritten as

$$\int_0^\infty (x+1)^{s-1}\, y(x)dx = \frac{1}{s}\int_0^\infty y(x)d(x+1)^s =$$

$$= \frac{1}{s}\{y(x)(x+1)^s \Big|_0^\infty - \int_0^\infty (x+1)^s dy(x)\} =$$

$$= -\frac{1}{s} - \frac{1}{s}\int_0^\infty (x+1)^s dy(x) =$$

$$= -\frac{1}{s} - \frac{1}{s}\int_1^\infty (x+1)^s dy(x).$$

Also

$$s\int_0^\infty x^{s-1}y(x)dx = \int_0^\infty y(x)dx^s =$$

$$= y(x)x^s \Big|_0^\infty - \int_0^\infty x^s dy(x) = -\int_1^\infty x^s dy(x).$$

Hence

$$1 + \int_1^\infty (x+1)^s dy(x) = s\int_1^\infty x^s dy(x)$$

and observing that $-\int_1^\infty dy(x) = 1$, we obtain

$$\int_1^\infty \{sx^s - (x+1)^s + 1\}\, dy(x) = 0, \ (\mathrm{Re}\ s > 0).$$

Since the left-hand side is an entire function of s, the proposition follows by analytic continuation. □

Proposition 3.10. $\beta(n+1) = \frac{1}{n}\sum_{r=1}^{n}\binom{n+1}{r}\beta(r)$, $(n=1,2,3,\dots)$.

Proof. In theorem 3.3, take $s = n+1$. This yields

$$(n+1)\beta(n+1) = -1 - \int_1^\infty (x+1)^{n+1}dy(x) =$$

$$= -1 - \int_1^\infty \{\sum_{r=0}^{n+1}\binom{n+1}{r}x^r\}\, dy(x) =$$

$$= -1 + \sum_{r=0}^{n+1}\binom{n+1}{r}\beta(r) =$$

$$= -1 + \beta(0) + \sum_{r=1}^{n}\binom{n+1}{r}\beta(r) + \beta(n+1).$$

Hence

$$n\ \beta(n+1) = \sum_{r=1}^{n} \binom{n+1}{r} \beta(r). \quad \square$$

From theorem 3.3 we obtain

$$\int_1^\infty \{c_n n x^n - c_n(x+1)^n + c_n\}\ dy(x) = 0$$

where c_n is a complex constant. Summing over n we get (formally)

$$\int_1^\infty \sum_{n=-\infty}^{\infty} \{c_n n x^n - c_n(x+1)^n + c_n\}\ dy(x) = 0$$

or, writing $f(x) = \sum_{n=-\infty}^{\infty} c_n x^n$,

$$\int_1^\infty \{xf'(x) - f(x+1) + f(1)\}\ dy(x) = 0.$$

In order to study to what extent this formula is valid we first prove the following:

THEOREM 3.4. Let the following conditions be satisfied

1. $F(x)$ is R-integrable over every interval of the form $[a,T]$ where $a < T$, a fixed
2. $F(x) \geq 0$ for $x \geq a$
3. for every $n = 0,1,2,3,\ldots$, $u_n(x)$ is R-integrable over $[a,T]$
4. $u_n(x) > 0$ for $x > a$
5. $\dfrac{u_{n+1}(x)}{u_n(x)}$ is monotonically non-decreasing for $x > a$, $(n=0,1,2,3,\ldots)$
6. $\sum_{n=0}^{\infty} c_n\ u_n(x)$ converges uniformly on every interval $[a,T]$
7. $0 < \gamma_n \overset{def}{=} \int_a^\infty F(x)u_n(x)dx < \infty$, $(n=0,1,2,3,\ldots)$

8. $\sum_{n=0}^{\infty} c_n \gamma_n$ converges.

Then we have (compare [3, pp.81-83] and [6, p.47]) that

$$\int_a^{\infty} F(x) \{ \sum_{n=0}^{\infty} c_n u_n(x)\} \, dx \overset{\text{def}}{=} \lim_{T\to\infty} \int_a^{T} F(x) \{ \sum_{n=0}^{\infty} c_n u_n(x)\} \, dx$$

exists and is equal to $\sum_{n=0}^{\infty} c_n \gamma_n$.

Proof. $\displaystyle\int_a^{T} F(x) \{ \sum_{n=0}^{\infty} c_n u_n(x)\} \, dx = \sum_{n=0}^{\infty} c_n \int_a^{T} F(x)u_n(x)dx =$

$$= \sum_{n=0}^{\infty} c_n \left\{ \int_a^{\infty} F(x)u_n(x)dx - \int_T^{\infty} F(x)u_n(x)dx \right\} =$$

$$= \sum_{n=0}^{\infty} c_n \gamma_n - \sum_{n=0}^{\infty} c_n \int_T^{\infty} F(x)u_n(x)dx.$$

Hence it suffices to show that

$$\lim_{T\to\infty} \sum_{n=0}^{\infty} c_n \int_T^{\infty} F(x)u_n(x)dx = 0.$$

Summation by parts yields

$$\sum_{n=0}^{\infty} c_n \int_T^{\infty} F(x)u_n(x)dx =$$

$$= \frac{A_0}{\gamma_0} \int_T^{\infty} F(x)u_0(x)dx + \sum_{n=0}^{\infty} A_{n+1} \{ \frac{1}{\gamma_{n+1}} \int_T^{\infty} F(x)u_{n+1}(x)dx - \frac{1}{\gamma_n} \int_T^{\infty} F(x)u_n(x)dx\},$$

where $A_n = \sum_{k=n}^{\infty} c_k \gamma_k$. First we observe that

$$\lim_{T\to\infty} \frac{A_0}{\gamma_0} \int_T^{\infty} F(x)u_0(x)dx = 0.$$

Now we show that for $n = 0,1,2,\ldots,$

$$\frac{1}{\gamma_{n+1}} \int_T^\infty F(x)u_{n+1}(x)dx - \frac{1}{\gamma_n} \int_T^\infty F(x)u_n(x)dx \ge 0.$$

This inequality is equivalent to the following one

$$\gamma_n \int_T^\infty F(x)u_{n+1}(x)dx - \gamma_{n+1} \int_T^\infty F(x)u_n(x)dx \ge 0.$$

The left-hand side can be rewritten as

$$\int_a^\infty F(x)u_n(x)dx \cdot \int_T^\infty F(x)u_{n+1}(x)dx - \int_a^\infty F(x)u_{n+1}(x)dx \cdot \int_T^\infty F(x)u_n(x)dx =$$

$$= \left\{\int_a^T F(x)u_n(x)dx + \int_T^\infty F(x)u_n(x)dx\right\} \cdot \int_T^\infty F(x)u_{n+1}(x)dx +$$

$$- \left\{\int_a^T F(x)u_{n+1}(x)dx + \int_T^\infty F(x)u_{n+1}(x)dx\right\} \cdot \int_T^\infty F(x)u_n(x)dx =$$

$$= \int_a^T F(x)u_n(x)dx \cdot \int_T^\infty F(x)u_{n+1}(x)dx - \int_a^T F(x)u_{n+1}(x)dx \cdot \int_T^\infty F(x)u_n(x)dx =$$

$$= \int_a^T F(x)u_n(x)dx \cdot \int_T^\infty \frac{u_{n+1}(x)}{u_n(x)} F(x)u_n(x)dx +$$

$$- \int_a^T \frac{u_{n+1}(x)}{u_n(x)} F(x)u_n(x)dx \cdot \int_T^\infty F(x)u_n(x)dx \ge$$

$$\ge \frac{u_{n+1}(T)}{u_n(T)} \int_a^T F(x)u_n(x)dx \cdot \int_T^\infty F(x)u_n(x)dx +$$

$$- \frac{u_{n+1}(T)}{u_n(T)} \int_a^T F(x)u_n(x)dx \cdot \int_T^\infty F(x)u_n(x)dx = 0.$$

Hence

$$\limsup_{T\to\infty} \left| \sum_{n=0}^{\infty} A_{n+1} \left\{ \frac{1}{\gamma_{n+1}} \int_T^{\infty} F(x)u_{n+1}(x)dx - \frac{1}{\gamma_n} \int_T^{\infty} F(x)u_n(x)dx \right\} \right| \le$$

$$\le \limsup_{T\to\infty} \sum_{n=0}^{r} \left| A_{n+1} \right| \cdot \left\{ \frac{1}{\gamma_{n+1}} \int_T^{\infty} F(x)u_{n+1}(x)dx - \frac{1}{\gamma_n} \int_T^{\infty} F(x)u_n(x)dx \right\} +$$

$$+ \limsup_{T\to\infty} \sum_{n=r+1}^{\infty} \left| A_{n+1} \right| \left\{ \frac{1}{\gamma_{n+1}} \int_T^{\infty} F(x)u_{n+1}(x)dx - \frac{1}{\gamma_n} \int_T^{\infty} F(x)u_n(x)dx \right\} \le$$

$$\le 0 + \limsup_{T\to\infty} M_r \cdot \sum_{n=r+1}^{\infty} \left\{ \frac{1}{\gamma_{n+1}} \int_T^{\infty} F(x)u_{n+1}(x)dx - \frac{1}{\gamma_n} \int_T^{\infty} F(x)u_n(x)dx \right\} =$$

$$= \limsup_{T\to\infty} M_r \cdot \frac{1}{\gamma_{r+2}} \int_T^{\infty} F(x)u_{r+2}(x)dx \le M_r,$$

where $M_r = \sup_{n>r} \left| A_{n+1} \right|$. Since $\lim_{r\to\infty} M_r = 0$, the theorem follows. □

THEOREM 3.5. Let $\{\sigma_n\}_{n=1}^{\infty}$ be a non-negative strictly increasing sequence of real numbers such that the two series

$$\sum_{n=1}^{\infty} c_n \sigma_n \int_1^{\infty} x^{\sigma_n} dy(x) \text{ and } \sum_{n=0}^{\infty} c_n \int_1^{\infty} (1+x)^{\sigma_n} dy(x),$$

(where the coefficients c_n are complex) are convergent. Then, defining

$$f(x) = \sum_{n=0}^{\infty} c_n x^{\sigma_n},$$

assuming that this series converges uniformly on all intervals [1,T], we have

$$\int_1^{\infty} \{xf'(x) - f(x+1) + f(1)\}dy(x) = 0.$$

Proof. In theorem 3.4 take $a = 1$, $F(x) = -y'(x)$, $u_n(x) = x^{\sigma_n}$,

$(n = 0,1,2,3,\ldots)$, $\gamma_n = \int_1^\infty x^{\sigma_n} dy(x)$. Then it is easily verified that all conditions are satisfied. Thus we get

$$\int_1^\infty xf'(x)dy(x) = \sum_{n=0}^\infty c_n\sigma_n \int_1^\infty x^{\sigma_n} dy(x).$$

Now apply theorem 3.4 with $a = 1$, $F(x) = -y'(x)$, $u_n(x) = (x+1)^{\sigma_n}$, $(n = 0,1,2,3,\ldots)$, $\gamma_n = \int_1^\infty (x+1)^{\sigma_n} dy(x)$. Again the conditions are satisfied and we obtain

$$\int_1^\infty f(x+1)dy(x) = \sum_{n=0}^\infty c_n \int_1^\infty (x+1)^{\sigma_n} dy(x).$$

Also

$$\int_1^\infty f(1)dy(x) = -f(1) = -\sum_{n=0}^\infty c_n \int_1^\infty dy(x).$$

Combining these results we obtain

$$\int_1^\infty \{xf'(x) - f(x+1) + f(1)\}dy(x) =$$

$$= \sum_{n=0}^\infty \left\{c_n\sigma_n \int_1^\infty x^{\sigma_n} dy(x) - c_n \int_1^\infty (x+1)^{\sigma_n} dy(x) + c_n \int_1^\infty dy(x)\right\} =$$

$$= \sum_{n=0}^\infty c_n \int_1^\infty \{\sigma_n x^{\sigma_n} - (x+1)^{\sigma_n} + 1\}dy(x) = 0,$$

and the proof is complete. □

4. Miscellany

In the remainder of this report we derive some integrals involving $y(x)$, which are of a somewhat different nature.

Proposition 4.1. For all $s \in \mathbb{C}$,

$$\hat{y}(s) = \int_0^\infty e^{-sx}y(x)dx = \exp\{\gamma + \int_0^s \frac{e^{-z}-1}{z}\,dz\}$$

and

$$\frac{d\hat{y}(s)}{ds} = \frac{e^{-s}-1}{s}\,\hat{y}(s).$$

Proof. See [4] or [1 , pp. 16-22]. □

Proposition 4.2. For all $s \in \mathbb{C}$,

$$-\int_0^\infty e^{-sx}dy(x) = -\int_1^\infty e^{-sx}dy(x) = -1 + s\cdot\hat{y}(s).$$

Proof. Integrate by parts and use proposition 4.1. □

Proposition 4.3. For Re $z > -1$,

$$\beta(-z) = \frac{1}{\Gamma(1+z)}\int_0^\infty e^{-s}s^z\hat{y}(s)ds.$$

Proof. For Re $z > -1$ we have

$$\Gamma(1+z)\beta(-z) = \left\{\int_0^\infty e^{-u}u^z du\right\}\left\{-\int_1^\infty x^{-z}dy(x)\right\} =$$

$$= -\int_1^\infty x^{-z}\left\{\int_0^\infty e^{-u}u^z du\right\}dy(x) =$$

$$= -\int_1^\infty\left\{\int_0^\infty e^{-u}\left(\frac{u}{x}\right)^z du\right\}dy(x) =$$

$$= -\int_1^\infty\left\{\int_0^\infty e^{-sx}s^z x\,ds\right\}dy(x) =$$

$$= -\int_0^\infty s^z \left\{ \int_1^\infty e^{-sx} x\,dy(x) \right\} ds = \int_0^\infty s^z e^{-s} \hat{y}(s)\,ds.$$

For the last step above use the second part of proposition 4.1 and proposition 4.2. □

Proposition 4.4. For Re z > -1,

$$\int_1^\infty \frac{1}{x+z}\,dy(x) = \int_0^\infty \frac{e^{-sz}-1}{z}\, e^{-s}\hat{y}(s)\,ds.$$

Proof.

$$\int_1^\infty \frac{1}{x+z}\,dy(x) = \int_1^\infty \left\{ \int_0^\infty e^{-s(x+z)} ds \right\} dy(x) =$$

$$= \int_0^\infty e^{-sz} \left\{ \int_1^\infty e^{-sx} dy(x) \right\} ds =$$

$$= \int_0^\infty e^{-sz}(-1+s\hat{y}(s))\,ds =$$

$$= \frac{e^{-sz}}{-z}\,(-1+s\hat{y}(s)) \Big|_0^\infty - \int_0^\infty \frac{e^{-sz}}{-z}\left(s\,\frac{d\hat{y}(s)}{ds} + \hat{y}(s)\right) ds =$$

$$= -\frac{1}{z} + \frac{1}{z}\int_0^\infty e^{-sz} e^{-s}\hat{y}(s)\,ds.$$

According to proposition 4.3 taking z = 0, we have

$$\int_0^\infty e^{-s}\hat{y}(s)\,ds = \beta(0) = 1,$$

and the proposition follows. □

Remark. In the proof of the last proposition we saw that

$$\int_0^\infty e^{-sz}(-1+s\hat{y}(s))ds = -\frac{1}{z} + \frac{1}{z}\int_0^\infty e^{-s(z+1)}\hat{y}(s)ds,$$

or, equivalently, for Re $z > 0$,

$$\int_0^\infty e^{-sz}s\hat{y}(s)ds = \frac{1}{z}\int_0^\infty e^{-s(z+1)}\hat{y}(s)ds.$$

Defining

$$\phi(z) = \int_0^\infty e^{-zs}\hat{y}(s)ds \text{ for Re } z > 0$$

we find that $\phi(z)$ satisfies the difference-differential equation

$$\phi'(z) = -\frac{1}{z}\,\phi(z+1), \qquad \text{Re } z > 0$$

Related to this fact is the

OPEN QUESTION: Is there any meromorphic function $h(z) \not\equiv 0$ satisfying the equation

$$h'(z) = -\frac{1}{z}\,h(z-1)\ ?$$

Define the continuous function $k(x)$ by

$$\begin{cases} k(x) = 1 & \text{for } 0 \le x \le 1 \\ k'(x) = \frac{1}{x}\,k(x-1) & \text{for } x > 1. \end{cases}$$

Proposition 4.5. $0 < k(x) \le 1+x, \qquad x \ge 0.$

Proof. It is almost trivial that $k(x) > 0$ for $x \ge 0$. For $0 \le x \le 1$ we have $k(x) = 1 \le 1+x$.

For $1 \le x \le 2$ we have $k(x) = 1 + \log x < 1+x$. Assume that $k(x) \le 1+x$ for $0 \le x \le n$. Then for $n < x \le n+1$ one has

$$k(x) - k(x-1) = \int_{x-1}^{x} \frac{1}{t} k(t-1)dt$$

or

$$k(x) = k(x-1) + \int_{x-1}^{x} \frac{1}{t} k(t-1)dt \le x + \int_{x-1}^{x} \frac{1}{t} \cdot t\, dt = 1 + x$$

and the proposition follows by induction. □

Proposition 4.6. The Laplace-transform $\hat{k}$ of k converges absolutely for Re s > 0 and

$$\hat{k}(s) = \exp\{ \int_{s}^{\infty} \frac{e^{-z}}{z} dz - \log s\}.$$

Proof. Similar to the proof of proposition 4.1 (compare [1 , pp. 16-22]). □

Proposition 4.7. $y * k = i$, where $*$ denotes convolution and i is the identity function: $i(x) = x$.

Proof. According to proposition 4.1 we have

$$\hat{y}(s) = \exp\{\gamma + \int_{0}^{s} \frac{e^{-z}-1}{z} dz\}.$$

Observe that

$$\gamma + \int_{0}^{s} \frac{e^{-z}-1}{z} dz = - \int_{s}^{\infty} \frac{e^{-z}}{z} dz - \log s,$$

because both sides have the same derivative, and that for s > 0,

$$\int_{s}^{\infty} \frac{e^{-z}}{z} dz + \log s = e^{-z} \log z \Big|_{s}^{\infty} + \int_{s}^{\infty} e^{-z} \log z\, dz + \log s =$$

$$= (1-e^{-s})\log s + \int_s^\infty e^{-z} \log z \, dz =$$

$$= \frac{1-e^{-s}}{s} \, s \log s + \int_s^\infty e^{-z} \log z \, dz$$

which, for $s \to 0$, tends to

$$\int_0^\infty e^{-z} \log z \, dz = - \gamma.$$

Hence

$$\hat{y}(s) = \exp\{- \int_s^\infty \frac{e^{-z}}{z} dz - \log s\}$$

and it follows that

$$\hat{y}(s) \cdot \hat{k}(s) = \exp(-2 \log s) = \frac{1}{s^2} = \hat{\imath}(s).$$

Thus $\hat{y} \cdot \hat{k} = \hat{\imath}$, and the remarkable relation $y * k = i$ follows easily. □

Final remarks. The differential equation for $y(x)$ is obtained, see [4], using the Prime Number Theorem. The question arises if conversely the PNT also follows from the properties of $y(x)$. The sequence $\{\gamma_k\}_{k=1}^\infty$ is strongly related to $y(x)$ and indeed some properties of this sequence would have a certain influence on the distribution of the primes. For example, we have shown that $\mathrm{tal}(\lambda_n) = e^\gamma$. Suppose we could show that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \lambda_n = e^\gamma$, or even the weaker statement $\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \lambda_n < \infty$. Then it is an easy excercise to show that there are infinitely many primes.

Numerical calculations, performed by C. DE VREUGD at the Mathematical Centre, indicate that indeed

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \lambda_k = e^\gamma.$$

However, we are not able to prove this.

The notion of Truncated Average Limit can easily be generalized to more general measure spaces. However, we will not pursue this subject here.

Addendum

Just before the printing of this report it was shown by P.van Emde Boas that the truncated average and the Cesàro limiting processes are independent, i.e. the existence of one of the limits does not imply the existence of the other and from the existence of both it does not follow that the two limits are equal. For details see

P. van Emde Boas, *The truncated-average limit and the Cesàro limit are independent,* Mathematical Centre Report ZW 21/74, Amsterdam.

Finally J. van Rongen recently proved that indeed

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \lambda_k = e^{\gamma} .$$

For details see

J. van Rongen, Mathematical Centre Report ZW 22/74, Amsterdam.

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