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ON CERTAIN REPRESENTATIONS OF POSITIVE INTEGERS

BY

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In this paper we investigate some properties of positive integers n, which are representable in the form $n = ux + vy$, where u and v are two positive and relatively prime integers, and x and y are non negative integers; these integers are called representable or representable by u and v.

The following properties are well known [1]. (Confer the appendix).

All integers $\geq (u-1)(v-1)$ are representable by u and v. The integer $N = uv - u - v$ cannot be represented by u and v. If an integer n with $0 \leq n \leq N$ is representable, then $N - n$ is not, and conversely. Hence there are $\frac{1}{2}(u-1)(v-1)$ non negative integers[2] which cannot be represented by u and v.

In what follows P denotes the set of integers which are representable by u and v and which are $\leq N$; Q denotes the set of non negative integers which are not representable by u and v. Then $P \cup Q$ is the set $0, 1, \ldots, N$. Further U denotes the set $1, \ldots, u-1$ and V denotes the set $1, \ldots, v-1$.

In order to deduce properties of the elements of P and Q we define for any c and any set M the set $M + c$ as the set of all elements $m + c$ where $m \in M$; further we define the set cM as the set of all elements cm where $m \in M$. Finally we shall denote the sum of the k^{th} powers of the elements of a set M by M^k.

In this paper we derive a formula for Q^k; confer formula (11).

We now prove two lemma's.

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[1]) Confer A. BRAUER, On a problem of partitions, Amer. J. Math. 64 (1942), 299—312.

[2]) This property was already mentioned by J. J. SYLVESTER, Math. Questions with their solutions from the educational times, 41 (1884), 21.

L e m m a 1. If $q \in Q$ and $q \notin Q + u$, we have $q \in U$ and $v \nmid q$, and conversely.

P r o o f. Since $q \notin Q + u$, either $q - u$ is representable or $q - u < 0$. If $q - u$ is representable, so is q, which contradicts $q \in Q$. Hence $q < u$. From $q \in Q$ it follows that $q > 0$, so $0 < q < u$, i.e. $q \in U$. Further since $q \in Q$ we have $v \nmid q$.

Conversely if $q \in U$ and $v \nmid q$ the integer q is not representable, so $q \in Q$. Further $q - u < 0$, so $q - u \notin Q$, hence $q \notin Q + u$.

L e m m a 2. If $q \in Q + u$ and $q \notin Q$, we have $q \in vW$, where W denotes the set $\left[\frac{u}{v}\right] + 1, \ldots, u - 1$, and conversely.

P r o o f. Since $q \in Q + u$ we have $q > 0$ and since $q \notin Q$, two non negative integers x and y exist with $q = ux + vy$. Further from $q \in Q + u$ it follows that $q - u \in Q$, so $q - u = u(x - 1) + vy$ is not representable. Now $y \geq 0$, so $x - 1 < 0$, hence $x = 0$ and $q = vy$. Finally from $q \in Q + u$ it follows that $0 < q - u \leq uv -$ $- u - v$, so $u < vy \leq (u - 1)v$. Thus $\left[\frac{u}{v}\right] + 1 \leq y \leq u - 1$ and $q \in vW$.

Conversely since $q \in vW$ obviously $q \notin Q$ and further $q = vy$ with $\left[\frac{u}{v}\right] + 1 \leq y \leq u - 1$. The positive integer $q - u$ is not representable, for otherwise non negative integers x' and y' would exist with $q - u = vy - u = ux' + vy'$, hence $u(x' + 1) = v(y - y')$. This would imply $u | y - y'$ which is impossible, since $0 < y - y' \leq$ $\leq y \leq u - 1$. Hence $q \in Q + u$.

Applying lemma 1 and 2 we get

$$Q \cup (vW) = (Q + u) \cup Z, \tag{1}$$

where Z denotes the set of all elements of U, which are not divisible by v.

If $u < v$, we have $W = Z = U$. If however $u > v$, we add on both sides of (1) the set with elements $v, 2v, \ldots, \left[\frac{u}{v}\right] v$. So in both cases we get

$$Q \cup (vU) = (Q + u) \cup U. \tag{2}$$

By symmetry we also have

$$Q \cup (uV) = (Q + v) \cup V. \tag{3}$$

We now deduce a formula for Q^k for non negative integers k. First we mention a few properties of the polynomials $B_h(x)$ of Bernoulli which enable us to calculate the U^k.

From

$$u^k + U^k = (U + 1)^k + 1 = \Sigma_{h=0}^{k} \binom{k}{h} U^h + 1$$

it follows that

$$\Sigma_{h=0}^{k-1} \binom{k}{h} U^h = u^k - 1. \tag{4}$$

On the other hand we have

$$B_{k+1}(x) - B_{k+1}(x - 1) = (k + 1)\,(x - 1)^k,$$

so

$$U^k = \frac{1}{k+1} (B_{k+1}(u) - B_{k+1}(1)),$$

hence, using the formula

$$B_{k+1}(x) = \Sigma_{h=0}^{k+1} \binom{k+1}{h} x^h B_{k+1-h} \tag{5}$$

we get

$$U^k = \frac{1}{k+1} \Sigma_{h=1}^{k+1} \binom{k+1}{h} (u^h - 1) B_{k+1-h} =$$

$$= \frac{1}{k+1} \Sigma_{t=0}^{k} \binom{k+1}{t+1} (u^{t+1} - 1) B_{k-t}. \tag{6}$$

We can interpret our result as follows. From the equation (4) taken for $k = 1, \ldots, K$, which equation is linear in the unknowns $U^0, \ldots, U^{K-1}$ these unknowns can be found and obviously are a linear compositum of the right hand members $u - 1, u^2 - 1, \ldots, u^K - 1$ of the equations (4). These values of the unknowns are given by (6).

These results are used now to determine Q^k. Taking the sum of the k^{th} powers of all elements in both sides of the formula (2) we get, since $Q \cap (vU) = (Q + u) \cap U$ is empty, the relation

$$Q^k + v^k U^k = (Q + u)^k + U^k$$

hence

$$\Sigma_{h=0}^{k-1} \binom{k}{h} u^{k-h} Q^h = (v^k - 1) U^k,$$

so

$$\Sigma_{h=0}^{k-1}\binom{k}{h}\frac{Q^h}{u^h}=\frac{v^k-1}{u^k}U^k. \tag{7}$$

Now if in the equations (4) we replace the unknowns U^h by $\frac{Q^h}{u^h}$ and the right hand sides u^k-1 by $\frac{v^k-1}{u^k}\ U^k$, we obtain the equations (7). Hence by the above remark the values of $\frac{Q^h}{u^h}$ must be found from (6) by the same substitution i.e.

$$\frac{Q^k}{u^k}=\frac{1}{k+1}\Sigma_{t=0}^{k}\binom{k+1}{t+1}\frac{v^{t+1}-1}{u^{t+1}}\ U^{t+1}\ B_{k-t},$$

and substituting in this last result for U^{t+1} its value given by (6) we get

$$Q^k=\frac{1}{k+1}\Sigma_{t=0}^{k}\binom{k+1}{t+1}(v^{t+1}-1)u^{k-t-1}B_{k-t}\frac{1}{t+2}\Sigma_{s=0}^{t+1}\binom{t+2}{s+1}.$$
$$(u^{s+1}-1)\ B_{t+1-s}. \tag{8}$$

To reduce the last member of (8) we first calculate the expression

$$\frac{1}{k+1}\Sigma_{t=0}^{k}\binom{k+1}{t+1}(v^{t+1}-1)u^{k-t-1}B_{k-t}\frac{1}{t+2}\Sigma_{s=0}^{t+1}\binom{t+2}{s+1}B_{t+1-s}. \tag{9}$$

Now we have from (5) with $x=1$

$$\Sigma_{h=0}^{t+2}\binom{t+2}{h}B_{t+2-h}=B_{t+2}(1),$$

so

$$\Sigma_{h=1}^{t+2}\binom{t+2}{h}B_{t+2-h}=B_{t+2}(1)-B_{t+2}(0)=0$$

since $t+1\geq 1$. Thus the expression $\Sigma_{s=0}^{t+1}\binom{t+2}{s+1}\ B_{t+1-s}$ vanishes and so does (9). Hence (8) reduces to

$$Q^k=\frac{1}{k+1}\Sigma_{t=0}^{k}\binom{k+1}{t+1}(v^{t+1}-1)u^{k-t-1}\ B_{k-t}\frac{1}{t+2}.$$
$$.\ \Sigma_{s=0}^{t+1}\binom{t+2}{s+1}u^{s+1}\ B_{t+1-s}=$$

$$= \frac{1}{k+1} \sum_{t=-1}^{k} \sum_{s=0}^{t+1} \binom{k+1}{t+1} \binom{t+2}{s+1} \frac{1}{t+2} (v^{t+1} - 1) u^{k+s-t} B_{k-t} B_{t+1-s},$$

where in the first sum the term with $t = -1$ which vanishes, has been added. Putting $k - t = i$, $t + 1 - s = j$ we get

$$Q^k = \frac{1}{k+1} \sum_{i=0}^{k+1} \sum_{j=0}^{k+1-i} \binom{k+1}{i} \binom{k-i+2}{j} \frac{B_i B_j}{k-i+2} (v^{k-i+1} - 1) u^{k-j+1} =$$

$$\sum_{\substack{i,j \geqq 0 \\ i+j \leqq k+1}} \frac{k! B_i B_j}{i! j! (k+2-i-j)!} v^{k-i+1} u^{k-j+1} - C, \qquad (10)$$

where

$$C = \sum_{\substack{i,j \geqq 0 \\ i+j \leqq k+1}} \frac{k! B_i B_j}{i! j! (k+2-i-j)!} u^{k-j+1} =$$

$$= k! \sum_{j=0}^{k+1} \frac{B_j u^{k-j+1}}{j!} \sum_{i=0}^{k+1-j} \frac{B_i}{i! (k+2-i-j)!} =$$

$$= k! \sum_{j=0}^{k+1} \frac{B_j u^{k-j+1}}{j!} \frac{B_{k+2-j}(1) - B_{k+2-j}}{(k+2-j)!}.$$

Here we used (5) with $x = 1$ and $k + 2 - j$ instead of $k + 1$.

Now for $k + 2 - j > 1$ we have $B_{k+2-j}(1) = B_{k+2-j}$ and for $k + 2 - j = 1$ we have $B_{k+2-j}(1) = B_{k+2-j} + 1$. So we find

$$C = k! \frac{B_{k+1}}{(k+1)!} = \frac{B_{k+1}}{k+1}$$

and then from (10) we get

Theorem. If Q^k denotes the sum of the k^{th} powers of the non negative integers which are not representable by u and v, then we have

$$Q^k = \sum_{\substack{i,j \geqq 0 \\ i+j \leqq k+1}} \frac{k! B_i B_j}{i! j! (k+2-i-j)!} v^{k-i+1} u^{k-j+1} - \frac{B_{k+1}}{k+1}. \qquad (11)$$

This result may symbolically be written in the form

$$Q^k = \frac{u^{k+1} v^{k+1}}{(k+1)(k+2)} \left\{ \left(1 + \frac{B}{u} + \frac{B}{v}\right)^{k+2} - \left(\frac{B}{u} + \frac{B}{v}\right)^{k+2} \right\} - \frac{B_{k+1}}{k+1},$$

where in the ordinary expansion of the $(k + 2)^{th}$ powers instead of B^h has to be taken B_h.

If we take $k = 0$ we find the above formula $Q^0 = \frac{1}{2}(u-1)(v-1)$ for the number of elements of Q.

Appendix. Above we used some results of which easily a proof is given by the following considerations.

Let as before u and v denote two integers > 1 with $(u, v) = 1$. Let $\binom{n}{u,\, v}$ denote the number of different ways in which the integer n can be written in the form $n = ux + vy$ with non negative integers x and y. Then obviously

$$\frac{1}{(1 - z^u)(1 - z^v)} = \Sigma_{n=0}^{\infty} \binom{n}{u,\, v} z^n.$$

Since $(u, v) = 1$ the expression

$$\frac{(1 - z^{uv})(1 - z)}{(1 - z^u)(1 - z^v)}$$

is a polynomial in z of degree $N + 1$ where $N = uv - u - v$. Hence we have

$$\frac{(1 - z^{uv})(1 - z)}{(1 - z^u)(1 - z^v)} = \Sigma_{n=0}^{N+1} \binom{n}{u,\, v} z^n - \Sigma_{n=0}^{N} \binom{n}{u,\, v} z^{n+1} =$$

$$= (1 - z)\, \Sigma_{n=0}^{N} \binom{n}{u,\, v} z^n + \binom{N+1}{u,\, v} z^{N+1}.$$

Obviously the coefficient of z^{N+1} in the expansion is equal to 1, so

$$\frac{1 - z^{uv}}{(1 - z^u)(1 - z^v)} = \Sigma_{n=0}^{N} \binom{n}{u,\, v} z^n + \frac{z^{N+1}}{1-z}. \tag{12}$$

Replacing z by $\frac{1}{z}$ and multiplying by z^N we get

$$\frac{z^{uv}-1}{(z^u-1)(z^v-1)} = \Sigma_{n=0}^{N} \binom{n}{u,\, v} z^{N-n} + \frac{1}{z-1} = \Sigma_{n=0}^{N} \binom{N-n}{u,\, v} z^n + \frac{1}{z-1}. \tag{13}$$

Comparing (12) and (13) we get for $n = 0, 1, \ldots, N$

$$\binom{n}{u,\, v} + \binom{N-n}{u,\, v} = 1.$$

Since for all n we have $\binom{n}{u,\, v} \geq 0$, we get for $n = 0, 1, \ldots, N$ the result $\binom{n}{u,\, v} = 0$ or 1, so all these integers n are either not representable or are representable in exactly one way. Further we get from (12)

$$\Sigma_{n=0}^{\infty}\binom{n}{u,\,v}z^n = \frac{1}{(1-z^u)\,(1-z^v)} =$$
$$= \frac{z^{N+1}}{1-z} + \frac{z^{uv}}{(1-z^u)\,(1-z^v)} + \Sigma_{n=0}^{N}\binom{n}{u,\,v}z^n$$

where for $n \geq N + 1$ the coefficient of z^n in the right hand side is obviously ≥ 1. So every integer $n \geq N + 1$ is representable.

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