

On Three Remarkable Affine Connexions in Almost-Hermitian Spaces

by

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§ 1. Almost-Hermitian, almost-Kählerian, pseudo-Hermitian and pseudo-Kählerian spaces.

Let X_{2n} be a $2n$ -dimensional differentiable manifold of class C^3 admitting an almost complex structure ¹⁾ defined by the tensor field F_i^{*h} of class C^2 :

$$(1.1) \quad F_j^{*h} F_i^{*j} = -A_i^h,$$

where the Latin indices h, i, j, k, l, m run over the range $1, 2, \dots, n, n+1, \dots, 2n$ and A_i^h denotes the unit tensor. ²⁾

Since the eigenvalues of the tensor F_i^{*h} satisfying (1.1) are $+1$ and -1 , the introduction of a tensor F_i^{*h} satisfying (1.1) is equivalent to the introduction of two mutually linearly independent linear fields X_{2n}^n 's complex conjugate to each other and of complex dimensions n .

A. Lichnerowicz [11, 12] showed that it is always possible to give an almost-Hermitian structure g_{ih} to an almost complex manifold X_{2n} :

$$(1.2) \quad F_i^{*l} F_h^{*k} g_{lk} = g_{ih}.$$

If we put

$$(1.3) \quad F_{ih} = F_i^{*k} g_{kh},$$

then it is easily seen that F_{ih} is anti-symmetric in its lower indices.

We call almost-Hermitian space a manifold which admits an almost complex structure F_i^{*h} and an almost-Hermitian structure g_{ih} ; that is, a positive definite metric $ds^2 = g_{ih} d\xi^i d\xi^h$ satisfying (1.1) and (1.2).

1) The notion of an almost complex structure was introduced by C. Ehresmann. See for instance, C. Ehresmann [4, 5]. The number between brackets refer to the Bibliography at the end of the paper.

2) As to the notations, we follow J.A. Schouten [15].

The anti-symmetry of the tensor P_{ih} and equation (1.2) show that the operation $v^h P_i^{\cdot h} v^i$ changes a vector v^h into a vector orthogonal to it and does not change its length.

We now put

$$(1.4) \quad P_{jih} = 3 \partial_{[j} P_{ih]}$$

then P_{jih} is a covariant tensor anti-symmetric in all its indices.

Denoting by $\overset{\circ}{\nabla}_j$ the operation of covariant differentiation with respect to the Christoffel symbols $\left\{ \begin{smallmatrix} h \\ ji \end{smallmatrix} \right\}$ formed with g_{ih} , we can write (1.4) also in the form

$$(1.5) \quad P_{jih} = 3 \overset{\circ}{\nabla}_{[j} P_{ih]}$$

In an almost-Hermitian space, when the tensor P_{jih} vanishes identically, the space is called an almost-Kählerian space.

It is well-known that

Theorem 1.1. In order that an almost complex structure $P_i^{\cdot h}$ of class C^ω in a space of class C^ω be induced by a complex structure of the space, it is necessary and sufficient that the tensor P_{jih} satisfies

$$(1.6) \quad N_{ji}^{\cdot \cdot h} \stackrel{\text{def}}{=} 2 P_{[j}^{\cdot l} (\partial_{l|1} P_{i]}^{\cdot h} - \partial_{i|1} P_{j]}^{\cdot h}) = 0.$$

The theorem essentially equivalent to this was stated by de Rham, Ehresmann, Libermann [10], Eckmann, Frölicher [3], Calabi, Spencer [1], Guggenheimer [6, 7] and Hodge [8]. The tensor $N_{ji}^{\cdot \cdot h}$ was introduced by Nijenhuis [13] and the equation of the form (1.6) was given by Eckmann and Frölicher [3].

Equation (1.6) can be written also in the form

$$(1.7) \quad N_{ji}^{\cdot \cdot h} = 2 P_{[j}^{\cdot l} (\overset{\circ}{\nabla}_{|1|} P_{i]}^{\cdot h} - \overset{\circ}{\nabla}_{i|} P_{j]}^{\cdot h})$$

or in covariant form

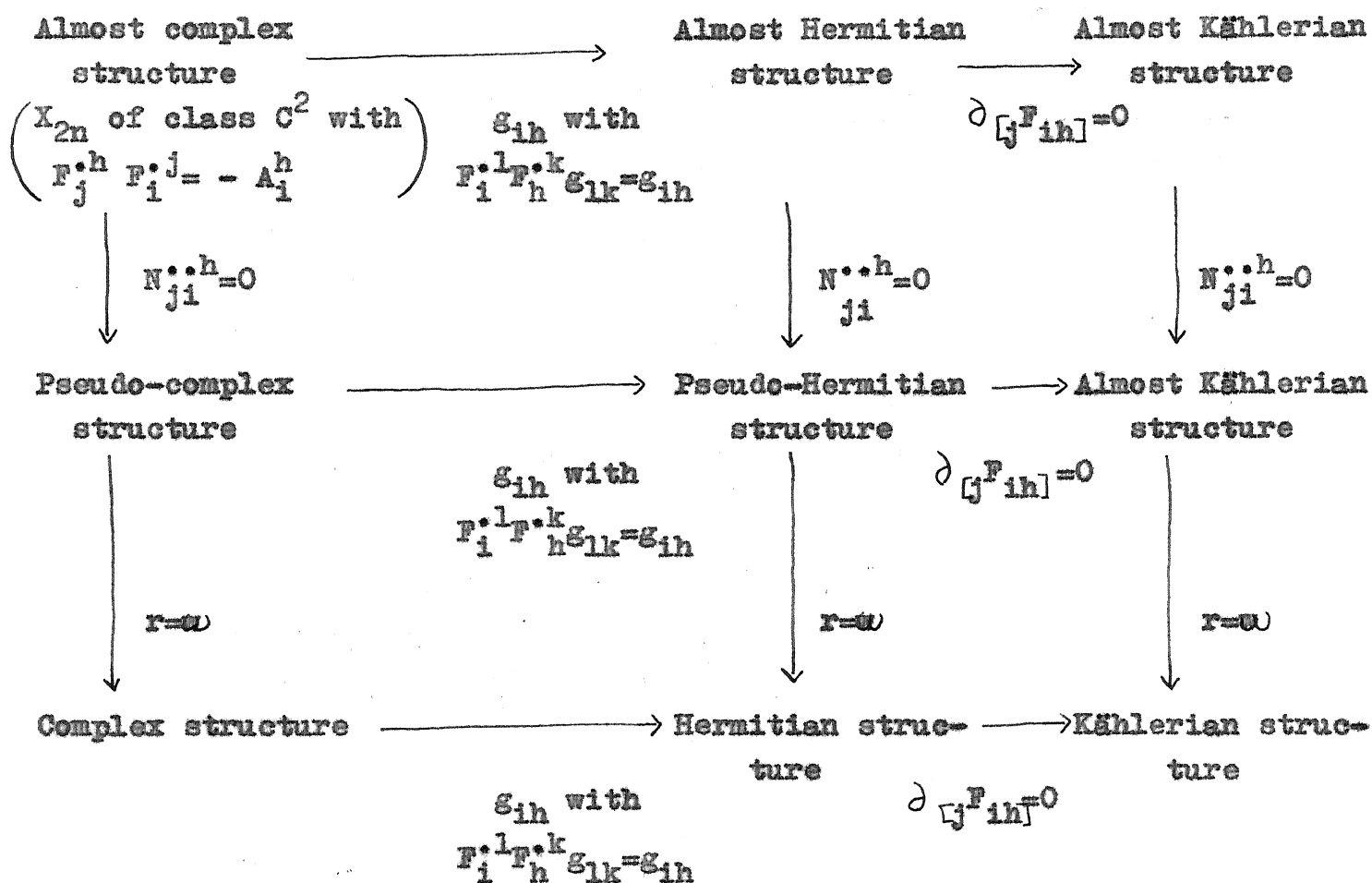
$$(1.8) \quad N_{jih} \stackrel{\text{def}}{=} N_{ji}^{\cdot \cdot k} g_{kh} = 2 P_{[j}^{\cdot l} (\overset{\circ}{\nabla}_{|1|} P_{i]}^{\cdot l} h - \overset{\circ}{\nabla}_{i|} P_{j]}^{\cdot l} h) .$$

It seems quite reasonable to expect that Theorem 1.1 may be true for a tensor $P_i^{\cdot h}$ of class C^{r-1} in a space of class C^r where $r \neq \omega$, but we do not yet have the proof of this fact.

3) See also J.A. Schouten 14 .

In an almost-Hermitian (almost-Kählerian) space of class C^r , when the Nijenhuis tensor $N_{ji}^{..h}$ vanishes identically, the space is called a pseudo-Hermitian (pseudo-Kählerian) space. Following Theorem 1.1, a pseudo-Hermitian (pseudo-Kählerian) space of class C is Hermitian (Kählerian).

The relations between these spaces may be seen in the following diagram:



In an almost-Hermitian space, it is easily seen from (1.5) and (1.8) that if $\overset{\circ}{\nabla}_j F_{ih}$ vanishes, then the tensors F_{jih} and N_{jih} vanish also.

Conversely, since we have from (1.5) and (1.8)

$$(1.9) \quad N_{jih} = 2 (F_j^{..l} \overset{\circ}{\nabla}_h F_{il} - F_{[j}^{..l} F_{l]h}),$$

we can easily see that if two tensors F_{jih} and N_{jih} vanish, then $\overset{\circ}{\nabla}_j F_{ih}$ vanishes also. Thus we have [17].

Theorem 1.2. A necessary and sufficient condition that an almost-Hermitian space be pseudo-Kählerian is that $\overset{\circ}{\nabla}_j F_{ih} = 0$.

§ 2. Affine connexions leaving invariant the tensor F_{ih} .

Let us now introduce an affine connexion $\overset{h}{j}i$ in our almost-Hermitian space and put

$$(2.1) \quad \overset{h}{j}i = \left\{ \overset{h}{j}i \right\} + T_{ji}^{\cdot\cdot h},$$

then the torsion tensor is given by

$$(2.2) \quad S_{ji}^{\cdot\cdot h} = \left[\overset{h}{j}i \right] = T_{[ji]}^{\cdot\cdot h}.$$

Denoting by ∇_j the operation of covariant differentiation with respect to $\overset{h}{j}i$, in order to have $\nabla_j \varepsilon_{ih} = 0$, it is necessary and sufficient that we have

$$(2.3) \quad T_{j(ih)} = 0,$$

which is equivalent to

$$(2.4) \quad T_{jih} = S_{jih} + S_{hij} + S_{hji},$$

where

$$(2.5) \quad T_{jih} \stackrel{\text{def}}{=} T_{ji}^{\cdot\cdot k} \varepsilon_{kh}, \quad S_{jih} \stackrel{\text{def}}{=} S_{ji}^{\cdot\cdot k} \varepsilon_{kh}.$$

On the other hand, in order to have $\nabla_j F_{ih} = 0$, it is necessary and sufficient that we have

$$(2.6) \quad \overset{\circ}{\nabla}_j F_{ih} - T_{ji}^{\cdot\cdot 1} F_{1h} - T_{jh}^{\cdot\cdot 1} F_{i1} = 0$$

or what amounts to the same

$$(2.7) \quad \overset{\circ}{\nabla}_j F_{ih} + T_{jil} F_h^{\cdot 1} - T_{jhl} F_i^{\cdot 1} = 0.$$

We cannot solve equation (2.7) without putting further conditions on T_{jih} . Putting some further conditions on T_{jih} or rather on

$$(2.8) \quad T_{jih} \stackrel{\text{def}}{=} T_{jil} F_h^{\cdot 1},$$

we shall try to solve equation (2.7) with respect to T_{jih} . From now on, we assume only $\nabla_j F_{ih} = 0$ and not $\nabla_j \varepsilon_{ih} = 0$.

§ 3. The first connexion.

We first assume that

$$(3.1) \quad \overset{\cdot}{T}_{j(ih)} = 0,$$

then we have from (2.7)

$$(3.2) \quad \overset{\cdot}{T}_{jih} = -\frac{1}{2} \overset{\circ}{\nabla}_j F_{ih}$$

or

$$(3.3) \quad \overset{\cdot}{T}_{jih} = \frac{1}{2} (\overset{\circ}{\nabla}_j F_{il}) F_h^{*l}$$

and consequently

$$(3.4) \quad \overset{\cdot}{T}_{ji}^{*h} = -\frac{1}{2} (\overset{\circ}{\nabla}_j F_{il}) F^{lh}.$$

From (3.4), we get

$$(3.5) \quad \overset{\cdot}{S}_{ji}^{*h} = \overset{\cdot}{T}_{ji}^{*h} = -\frac{1}{2} (\overset{\circ}{\nabla}_j F_{il}) F^{lh}.$$

Since $F_{il} F_h^{*l} = g_{ih}$ and consequently

$$(\overset{\circ}{\nabla}_j F_{il}) F_h^{*l} + F_i^{*l} (\overset{\circ}{\nabla}_j F_{hl}) = 0,$$

equation (3.3) shows that $\overset{\cdot}{T}_{j(ih)} = 0$ and consequently the connexion is metric and we have

$$(3.6) \quad \overset{\circ}{\nabla}_j g_{ih} = 0, \quad \overset{\circ}{\nabla}_j F_{ih} = 0.$$

From (1.9) and (3.3), we find

$$(3.7) \quad \overset{\cdot}{T}_{jih} = \frac{1}{2} N_{hij} - \frac{1}{2} F_{[h}^{*l} F_{i]gl}.$$

Thus we have, for an almost-Kählerian space,

$$(3.8) \quad \overset{\cdot}{T}_{jih} = \frac{1}{2} N_{hij},$$

for a pseudo-Hermitian space,

$$(3.9) \quad \overset{\cdot}{T}_{jih} = -\frac{1}{2} F_{[h}^{*l} F_{i]jl},$$

and for a pseudo-Kählerian space,

$$(3.10) \quad \overset{\cdot}{T}_{jih} = 0.$$

In the last case the connexion Γ_{ji}^h becomes Riemannian.
In a Hermitian space tensors g_{ih} and $F_i^{h\bar{j}}$ have the components

$$(3.11) \quad g_{ih} = \begin{pmatrix} 0 & g_x \\ g_x & 0 \end{pmatrix}, \quad F_i^{h\bar{j}} = \begin{pmatrix} +i & x & 0 \\ 0 & -i & x \end{pmatrix}$$

respectively and consequently the tensors F_{ih} and F^{ih} defined by $F_i^{h\bar{j}}$ have the components

$$(3.12) \quad F_{ih} = \begin{pmatrix} 0 & -ig_x \\ +ig_x & 0 \end{pmatrix}, \quad F^{ih} = \begin{pmatrix} 0 & +ig_x \\ -ig_x & 0 \end{pmatrix}$$

respectively with respect to a complex coordinate system $(z^x, z^{\bar{x}}, z^x)$, where the Greek indices $x, y, z, \dots$ run over the range $1, 2, \dots, n$ and $\bar{x}, \bar{y}, \bar{z}, \dots$ the range $n+1, n+2, \dots, 2n$.

Thus for the Christoffel symbols Γ_{ji}^h we have 4)

$$(3.13) \quad \Gamma_{ij}^x = g_{ij}^x (g^x), \quad \Gamma_{ij}^{\bar{x}} = g_{ij}^{\bar{x}} (g^{\bar{x}}), \quad \Gamma_{ij}^x = 0, \\ \Gamma_{ij}^{\bar{x}} = g_{ij}^{\bar{x}} (g^{\bar{x}})$$

and four similar expressions for $\Gamma_{ij}^x, \Gamma_{ij}^{\bar{x}}, \Gamma_{ij}^x$ and $\Gamma_{ij}^{\bar{x}}$. We shall denote in the following by the sign "conj." the fact that we have similar formulas as complex conjugates of the formulas already written.

For the tensor F_{jih} and F_j^{ih} we have respectively

$$(3.14) \quad F_{jih} = 0, \quad F_j^{ih} = \text{conj.} \\ F_{jih} = +2i g_{jih}, \quad F_j^{ih} = \text{conj.} \\ F_{jih} = +2i g_{jih}, \quad F_j^{ih} = \text{conj.}$$

4) See for example J.A. Schouten 15, P. 396.

and

$$\begin{aligned}
 (3.15) \quad & F^{*x} = 0, & \text{conj.} \\
 & F^{*x} = -2 \, i \, x & \text{conj.} \\
 & F^{*x} = 0, & \text{conj.} \\
 & F^{*x} = 0. & \text{conj.}
 \end{aligned}$$

Thus for the tensor (3.4), we have

$$(3.16) \quad T^{*x} = 0, \quad T^{*x} = -x, \quad T^{*x} = 0, \quad T^{*x} = 0 \quad \text{conj.}$$

and consequently, for the components of the affine connexion

$$(3.17) \quad h_{ji} = h_{ji} + T_{ji}^{*h},$$

we have

$$(3.18) \quad x = x, \quad x = x \quad \text{conj.}$$

the other 's being zero.

This is the affine connexion introduced by A. Lichnerowicz 12 in an almost-Hermitian space. In a Hermitian space, this connexion seems to be a natural one because this connexion is obtained from Riemannian connection $h_{ji} = h_{ji}$ requiring $j F_i^{*h} = 0$, that is, requiring $jx = jx = 0$, the other 's keeping the same values.

In a classification by Schouten (15 p. 396, formula (3.6)), this corresponds to the case

$$(3.19) \quad S^{*x} = 0, \quad S^{*x} = -\frac{1}{2} g^x, \quad S^{*x} = 0, \quad \text{conj.}$$

§ 4. The second connexion.

We next assume that

$$(4.1) \quad T(j \, i \, h) = 0,$$

then we have from (2.7)

$$(4.2) \quad T_{jih} = -\frac{1}{2} (j F_{ih} + i F_{jh} - h F_{ij})$$

or

$$(4.3) \quad T_{jih} = + \frac{1}{2} (\quad_j F_{il} + \quad_i F_{jl} - \quad_l F_{ij}) F_h^{.1}$$

and consequently

$$(4.4) \quad T_{ji}^{..h} = - \frac{1}{2} (\quad_j F_{il} + \quad_i F_{jl} - \quad_l F_{ij}) F^{lh}.$$

From (4.4), we get

$$(4.5) \quad S_{ji}^{..h} = T_{ji}^{..h} = - \frac{1}{2} (\quad_l F_{ji}) F^{lh}.$$

From (4.3), we find

$$(4.6) \quad T_{jih} = F_h^{.1} \quad_i F_{jl} - \frac{1}{2} F_h^{.1} F_{ijl}.$$

Thus combining (1.9) and (4.6), we find

$$(4.7) \quad T_{jih} = \frac{1}{2} N_{hji} - \frac{1}{2} F_j^{.1} F_{ihl}.$$

Thus we have, for an almost-Kählerian space,

$$(4.8) \quad T_{jih} = \frac{1}{2} N_{hji}.$$

and for a pseudo-Hermitian space

$$(4.9) \quad T_{jih} = - \frac{1}{2} F_j^{.1} F_{ihl}.$$

In the case of pseudo-Hermitian space, the tensor T_{jih} satisfies (2.3) and consequently the connexion is metric and we have

$$(4.10) \quad \quad_j \varepsilon_{ih} = 0, \quad \quad_j F_{ih} = 0.$$

Finally we have, for a pseudo-Kählerian space,

$$(4.11) \quad T_{jih} = 0$$

and the connexion becomes Riemannian.

In a Hermitian space, in addition to formulas (3.11)-(3.15), we have following expressions for $\quad_j F_{ih}$:

$$(4.12) \quad F_x = 0, \quad F_x = 0, \quad F_x = 0,$$

$$F_x = -2i \quad \text{conj.}$$

Thus, for the tensor T_{ji} , we have

$$\begin{aligned}
 (4.13) \quad T^{..x} &= g^{..x} & \text{conj.} \\
 T^{..x} &= -g^{..x} & \text{conj.} \\
 T^{..x} &= 0, & \text{conj.} \\
 T^{..x} &= -g^{..x} & \text{conj.}
 \end{aligned}$$

and consequently for the connexion

$$(4.14) \quad h_{ji} = h_{ji} + T_{ji}^{..h}$$

we have

$$(4.15) \quad x = g^{..x}, \quad x = 0, \quad x = 0, \quad x = 0, \quad \text{conj.}$$

This is the affine connexion introduced by J.A. Schouten and D. van Dantzig 16 and used recently by S.S. Chern 2 and P. Libermann 10 5).

In a classification by Schouten, (15, p. 396, formula (3.6)), this corresponds to the case

$$(4.16) \quad s^{..x} = g^{..x}, \quad s^{..x} = 0, \quad s^{..x} = 0 \quad \text{conj.}$$

§ 5. The third connexion.

We finally assume that

$$(5.1) \quad T_{(ji)h} = 0,$$

then we have from (2.7)

$$(5.2) \quad T_{jih} = -\frac{1}{2} (j F_{ih} - i F_{jh} + h F_{ij})$$

or

$$(5.3) \quad T_{jih} = +\frac{1}{2} (j F_{il} - i F_{jl} - l F_{ij}) F_h^{..l}$$

and consequently

$$(5.4) \quad T_{ji}^{..h} = -\frac{1}{2} (j F_{il} - i F_{jl} + l F_{ij}) F^{..lh}.$$

5) See also G. Legrand 9.

From (5.4) we get

$$(5.5) \quad S_{ji}^{..h} = T_{ji}^{..h} = T_{ji}^{..h},$$

From (5.3) we find

$$(5.6) \quad N_{jih} = 2(\quad_j F_{i1}) F_h^{.1} + F_j^{.1} \quad_1 F_{ih} + F_i^{.1} \quad_1 F_{hj}.$$

Thus combining (5.3) and (5.6), we obtain

$$(5.7) \quad T_{jih} = \frac{1}{2} N_{jih} + \frac{1}{2} (F_j^{.1} \quad_1 F_{ih} + F_i^{.1} \quad_1 F_{hj} + F_h^{.1} \quad_1 F_{ji}).$$

On the other hand, we have from (1.5)

$$(5.8) \quad F_j^{.m} F_i^{.1} F_h^{.k} F_{mlk} = -3 F_j^{.1} \quad_1 F_{ih}.$$

Substituting this into (5.7), we find

$$(5.9) \quad T_{jih} = \frac{1}{2} N_{jih} + \frac{1}{2} F_j^{.m} F_i^{.1} F_h^{.k} F_{mlk}.$$

Thus we have, for an almost-Kählerian space,

$$(5.10) \quad T_{jih} = \frac{1}{2} N_{jih}$$

and, for a pseudo-Hermitian space,

$$(5.11) \quad T_{jih} = \frac{1}{2} F_j^{.m} F_i^{.1} F_h^{.k} F_{mlk}.$$

In the case of pseudo-Hermitian space, the tensor T_{jih} is anti-symmetric in all its indices and consequently satisfies (2.3) and the connexion is metric.

Thus we have

$$(5.12) \quad \quad_j S_{ih} = 0, \quad \quad_j F_{ih} = 0.$$

Finally we have, for a pseudo-Kählerian space,

$$(5.13) \quad T_{jih} = 0,$$

and the connexion becomes Riemannian.

In a Hermitian space, we have, for the tensor $T_{ji}^{..h}$,

$$\begin{aligned}
 (5.14) \quad T^{*+}X &= -\varepsilon X && \varepsilon && \text{conj.} \\
 T^{*+}X &= -\varepsilon X && && \text{conj.} \\
 T^{*+}X &= 0, && && \text{conj.} \\
 T^{*+}X &= +\varepsilon X && && \text{conj.}
 \end{aligned}$$

and consequently, for the components of the affine connection

$$(5.15) \quad h_{ji} = h_{ji} + T_{ji}^{*+}h,$$

we have

$$(5.16) \quad X = \varepsilon \varepsilon, \quad X = 0, \quad X = 0, \quad X = 2\varepsilon \varepsilon \quad \text{conj.}$$

In a classification by J.A. Schouten (15 p. 396, formula (3.6)), this corresponds to the case

$$\begin{aligned}
 (5.17) \quad S^{*+}X &= -\varepsilon \varepsilon, \quad S^{*+}X = -\varepsilon \varepsilon, \\
 S^{*+}X &= 0, \quad \text{conj.}
 \end{aligned}$$

Bibliography

- 1 E. Calabi and D.C. Spencer: Completely integrable almost complex manifolds. Bull.Amer.Math.Soc., 57 (1951), 254-255.
- 2 S.S. Chern: Characteristic classes of Hermitian manifolds. Annals of Math., 47 (1946), 85-121.
- 3 B. Eckmann and A. Frölicher: Sur l'intégrabilité de structures presque complexes. C.R.Acad.Sci. Paris, 232 (1951), 2284-2286.
- 4 C. Ehresmann: Sur la théorie des espaces fibrés. Coll. de Iop. Alg. C.N.R.S. Paris (1947), 3-15.
- 5 ——— : Sur les variétés presque complexes. Proc. Int.Congr. Math., (1952), 412-419.
- 6 H. Guggenheimer: Über KÄhlersche und symplektische Differentialalgebra. Tôhoku Math. Journ. 4 (1952), 157-171.
- 7 ——— : Geometria pseudo-kähleriana. Rend.Acad.Naz.dei Lincei, 14 (1953), 220-222.
- 8 W.V.D. Hodge: Structure problems for complex manifolds. Rend.Math. e Appl.Ser. V, 11 (1952), 101-110.
- 9 G. Legrand: Connexions affines sur une variété presque hermitique. C.R.Acad.Sci. Paris, 237 (1953), 1626-1627.
- 10 P. Libermann: Sur le problème d'équivalence de certaines structures infinitésimales. Thèse (1953).
- 11 A. Lichnerowicz: Généralisations de la géométrie kählérienne globale. Coll.de Géom.Diff. Louvain (1951), 99-122.
- 12 ——— : Un théorème sur les espaces homogènes complexes. Archiv der Math., 5 (1954), 207-215.
- 13 A. Nijenhuis: X_{n-1} -forming sets of eigenvectors. Indagationes Mathematicae 8 (1951), 200-212.
- 14 J.A. Schouten: Sur les tenseurs de aux directions principales $n-1$ -normales. Coll. de Géom. Diff. Louvain (1951), 67-70.

- 15 J.A. Schouten: Ricci Calculus (1954).
- 16 J.A. Schouten and D. van Dantzig: Über unitäre Geometrie.
Math. Ann., 103 (1930), 319-346.
- 17 K. Yano and I. Mogi: Sur les variétés pseudo-kähleriennes
à courbure holomorphique constante, C.R. Acad. Sci.
Paris, 237 (1953), 962-964.