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ON THE STRENGTHENING OF ALEXANDER'S THEOREM

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The notion of compactness is important in topology, and in this lecture we will consider families of compact subsets. Every topological space $(X, \mathcal{O})$ contains a family of subsets, such that a subset is compact if and only if it is a member of the family. This family we will call the family of compact subsets. We will try to strengthen the topology $\mathcal{O}$, in such a way, that the family of compact subsets does not change.

ALEXANDER's theorem implies that the collection of compact subsets in the topological space $(X, \mathcal{O})$ is exactly equal to the collection of subsets of X, whose members are compact relative to some given subbase $\mathcal{B}$ of the topology, and so we will try to strengthen a subbase of a topology too, in such a way, that the family of compact subsets does not change.

It has turned out to be convenient to consider closed subbases instead of open subbases, and hence we will use systems with the finite intersection property instead of covers. To avoid confusion we will define compactness complementary.

Definition. A subset A of a set X is *compact* relative to a collection of subsets $\mathcal{S}$, if and only if for every subcollection $\mathcal{S}'$ of $\mathcal{S}$ with the finite intersection property in the set A, it is true that $(\bigcap \mathcal{S}') \cap A \neq \emptyset$.

Definition. The *compactness operator* ϱ is an operator, defined on the power-set of the power-set of a set X, and it assigns to every collection of subsets $\mathcal{S}$ of X, the collection $\varrho\,\mathcal{S}$ of compact sets relative to $\mathcal{S}$.

The collection $\varrho\mathcal{S}$ is again a family of subsets of X, and hence ϱ assigns to this collection another collection of subsets of X. This family of sets will be called the *square compact* family, relative to $\mathcal{S}$. It will be denoted by $\varrho^2\mathcal{S}$, and from this definition we can derive our main theorem.

Theorem. *For every family of subsets $\mathcal{S}$ of a set X, the family of compact sets relative to $\mathcal{S}$ is exactly equal to the family of compact sets relative to $(\mathcal{S} \cup \varrho^2\mathcal{S})$.*

It is obvious that this theorem can be restated for the topology generated by $\mathcal{S} \cup \varrho^2\mathcal{S}$. We still have two other propositions concerning our question.

Proposition. *If X is a set, and $\mathcal{S}$ and $\mathcal{T}$ are collections of subsets of X, such that* $\varrho\,\mathcal{S} = \varrho\mathcal{T}$, then $\varrho(\mathcal{S} \cup \mathcal{T}) = \varrho\,\mathcal{S}$.

Proposition. *Let X be a set, and let $\mathcal{S}$ be a collection of subsets. Let $\varrho\mathcal{S}$ be the family of compact sets, relative to $\mathcal{S}$, and let $\mathcal{T}$ be the collection of sets, that has a com-*

pact intersection with every compact set, then there exists a maximal family $\tilde{\mathscr{S}}$, *such that* $\varrho\tilde{\mathscr{S}} = \varrho\mathscr{S}$, *if and only if* $\varrho\mathscr{S} = \varrho\mathscr{T}$. *In that case* $\tilde{\mathscr{S}} = \mathscr{T}$.

From this last proposition it follows, that for many topological spaces there is no strongest topology with the same system of compact sets as the original topological space, and so there is no best solution to our problem.

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