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The successive minima of coumpound convex bodies

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Dear Professor Rogers,

First of all, I wish to thank you for the reprints you sent to me in the month of October. They are of great use for me and I am very glad to have received them. I noticed that your last paper was concerned with compound convex bodies and that it dealt with the same problem treated by me in a paper published in the course of this year. I should like to make in this letter some remark on this question. At the same time I send you a copy of my paper.

At the end of your paper you discuss some inequalities for the successive minima of the compound of a given number of convex bodies, which can be derived from your main result. I did not consider such inequalities in my paper, but touched the question in my speech at the colloquium at Oberwolfach on number theory in this year. It turns out that my method leads to inequalities of the type desired (see formula (1) below), as I shall prove.

My proof was based upon the following lemma (see p.285; I now take $\rho=0$).

__Lemma__. Let p and n be integers with $n \geq 2$, $1 \leq p \leq n$, and let $N=\binom{n}{p}$. Let there be given p systems of n points (vectors)

$$\left\{\begin{array}{l} A^{(\pi,1)}=(a_1^{(\pi,1)}, a_2^{(\pi,1)},\ldots, a_n^{(\pi,1)}) \\ ------------------ \\ ------------------ \\ A^{(\pi,n)}=(a_1^{(\pi,n)}, a_2^{(\pi,n)},\ldots, a_n^{(\pi,n)}) \end{array}\right. \qquad (\pi=1,2,\ldots,p)$$

in R_n and suppose that, for $\pi=1,2,\ldots,p$, the points $A^{(\pi,1)},\ldots,A^{(\pi,n)}$ are independent. Further, let $(\nu_{1,i},\nu_{2,i},\ldots,\nu_{p,i})$ $(i=1,2,\ldots,N)$ be the N ascending sets of p positive integers $\leq n$, arranged in some order.

Then there exist N sets of p positive integers $\mu_{1,i}, \mu_{2,i},\ldots,\mu_{p,i}$ $(i=1,2,\ldots,N)$ satisfying the following requirements:

1°. for all i one has

$$\mu_{1,i} \leq \nu_{1,i}, \mu_{2,i} \leq \nu_{2,i},\ldots, \mu_{p,i} \leq \nu_{p,i}$$

2° the N compound points

$$\Xi^{(i)} = \left[A^{(1,\mu_{1,i})}, A^{(2,\mu_{2,i})}, \ldots, A^{(p,\mu_{p,i})}\right]$$

in R_N are independent.

Now let there be given p centrally symmetric, closed, convex bodies in R_n. Choose an arbitrary arrangement of these bodies, and in this arrangement denote them by $K^{(1)}, K^{(2)}, \ldots, K^{(p)}$. Let $\lambda_1^{(\pi)}, \lambda_2^{(\pi)}, \ldots, \lambda_n^{(\pi)}$ be the successive minima of $K^{(\pi)}$ ($\pi=1,2,\ldots,p$). Further, let the arrangement of the sets $(\nu_{1,i}, \nu_{2,i}, \ldots, \nu_{p,i})$ appearing in the enunciation of the lemma, be chosen in such a way that the N products

$$M_i = \lambda^{(1)}_{\nu_{1,i}} \lambda^{(2)}_{\nu_{2,i}} \ldots \lambda^{(p)}_{\nu_{p,i}} \qquad (i=1,2,\ldots,N)$$

form a non-decreasing sequence. Finally, let $F(\Xi)$ denote the distance function of the compound convex body

$$\left[K^{(1)}, K^{(2)}, \ldots, K^{(p)}\right] = \mathcal{K}, \text{ say.}$$

In the following no special properties of this function will be needed, such as the expression for it in terms of the distance functions $f_\pi(x)$ of the bodies $K^{(\pi)}$ derived by Mahler.

Now apply the lemma. For $\pi = 1,2,\ldots,p$ let $A^{(\pi,1)}, A^{(\pi,2)}, \ldots, A^{(\pi,n)}$ be n independent lattice points in R_n with $f_\pi(A^{(\pi,\nu)}) = \lambda_\nu^{(\pi)}$ ($\nu=1,2,\ldots,n$). Then these exist N sets of positive integers $(\mu_{1,i}, \mu_{2,i}, \ldots, \mu_{p,i})$ such that 1° and 2° hold. The points $\Xi^{(i)}$ given by 2° are lattice points. Next, $\{\lambda_\nu^{(\pi)}\}^{-1} A^{(\pi,\nu)} \in K^{(\pi)}$, for all π and ν, and so

$$\left[\{\lambda^{(1)}_{\mu_{1,i}}\}^{-1} A^{(1,\mu_{1,i})}, \{\lambda^{(2)}_{\mu_{2,i}}\}^{-1} A^{(2,\mu_{2,i})}, \ldots, \{\lambda^{(p)}_{\mu_{p,i}}\}^{-1} A^{(p,\mu_{p,i})}\right]$$

$$= \left\{\lambda^{(1)}_{\mu_{1,i}} \lambda^{(2)}_{\mu_{2,i}} \ldots \lambda^{(p)}_{\mu_{p,i}}\right\}^{-1} \Xi^{(i)}$$

is a point of the compound body $\mathcal{K}$. Hence

$$F(\Xi^{(i)}) \leq \lambda^{(1)}_{\mu_{1,i}} \lambda^{(2)}_{\mu_{2,i}} \ldots \lambda^{(p)}_{\mu_{p,i}} .$$

Hence, in virtue of 1^o,

$$F(\Xi^{(1)}) \leq \lambda^{(1)}_{\nu_{1,1}} \lambda^{(2)}_{\nu_{2,1}} \cdots \lambda^{(p)}_{\nu_{p,1}} = M_1 .$$

Since the numbers M_i are non-decreasing and, by 2^o, the points $\Xi^{(i)}$ are independent, it follows that the successive minima of $\mathcal{K}$, say $\mu_1, \mu_2, \ldots, \mu_N$, satisfy the inequalities

$$(1) \qquad \mu_i \leq M_i \quad (i=1,2,\ldots,N).$$

These inequalities are as strong as one could wish. It should be noted that the order of the bodies $K^{(\pi)}$, on which the definition of the M_i depends, is arbitrary.

If the p bodies $K^{(\pi)}$ coincide, then (1) gives precisely the upper bounds for the minima μ_i derived by Mahler in that case. Of course, there is no positive constant c such that one generally has

$$c\, M_i \leq \mu_i \quad \text{for } i=1,2,\ldots,N.$$

For the value of the expression

$$\mu_1 \mu_2 \cdots \mu_N V(\mathcal{K}) . \left\{\prod_{\pi=1}^{p} \lambda_1^{(\pi)} \lambda_2^{(\pi)} \cdots \lambda_n^{(\pi)} V(K^{(\pi)})\right\}^{-P/p} ,$$

where $P = \binom{n-1}{p-1}$, lies between two fixed bounds, and, by the results of Mahler, there is no finite upper bound for the quotient

$$Q = V(\mathcal{K}) \left\{\prod_{\pi=1}^{p} V(K^{(\pi)})\right\}^{-P/p} .$$

With kind regards,

Sincerely yours,

L.

(Dr C.G. Lekkerkerker)