stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZW 22/74 MARCH

J.B. van RONGEN

ON THE LARGEST PRIME DIVISOR OF AN INTEGER

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS (MOS) subject classification scheme (1970): 10H25

On the largest prime divisor of an integer *)

by

J.B. van Rongen **)

ABSTRACT

In this report we consider the number-theoretical sequence $\{\lambda_m\}_{m=1}^{\infty}$, where $\lambda_1 = 1$ and $\lambda_m = \frac{\log m}{\log p(m)}$ $(m \geq 2)$, $p(m)$ being the largest prime divisor of m. For a large class of functions f we derive the average limit $\lim \frac{1}{n} \sum_{m=1}^{n} f(\lambda_m)$.

---

*) This paper is not for review; it is meant for publication in a journal.

**) Present address of the author: Hoge Morsweg 111, Leiden, The Netherlands.

0. INTRODUCTION

For integers $m \geq 2$, let $p(m)$ be the largest prime divisor of m, and let λ_m be defined implicitly by $p(m)^{\lambda_m} = m$. It is convenient to take $\lambda_1 = 1$. Recently J. van de Lune ([2]) proposed the following problem. Let f(x) be a function on $[1,\infty)$. Under what conditions on f(x) does

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(\lambda_m)$$

exist?

It was shown in [2] that for bounded and continuous functions f, this limit exists and equals

$$-\int_1^\infty f(x)\, d\rho(x)$$

where $\rho(x)$ is Dickman's function defined below. In this note we extend this result to a class of continuous functions f which includes all polynomials. In particular we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \lambda_m = e^\gamma$$

where γ is Euler's constant.

1. SOME AUXILIARY LEMMAS

<u>Lemma 1</u>. *Dickman's function* $\rho(x)$ *is the continuous function defined by the difference-differential equation*

$$\rho'(x) = -\frac{1}{x}\rho(x-1), \qquad (1 < x),$$

with

$$\rho(x) = 1, \qquad (0 \leq x \leq 1).$$

$\rho(x)$ has the following properties:

(a) $0 < \rho(\alpha) \leq \{\Gamma(\alpha+1)\}^{-1}$, $(\alpha \geq 0)$.

(b) *$\rho(\alpha)$ is non-increasing.*

(c) $\int_0^\infty \rho(\alpha)d\alpha = e^\gamma$ *where γ is Euler's constant.*

(d) *For all positive integers* M,k ,

$$\alpha^k \rho^{(k)}(\alpha) = O_{M,k}(\alpha^{-M}), \qquad (\alpha \to \infty) ,$$

where $\rho^{(k)}$ is the k*-th derivative of* ρ .

<u>Proof</u>. A proof of lemma 1(a) and (b) can be found in [3, p.27-28]. For (c), see [2]. To prove (d), we first show by induction that rational functions

$$R_{k,j}(\alpha), \qquad k = 1,2,\ldots;1 \leq j \leq k ,$$

exist, such that

$$\alpha^k \rho^{(k)}(\alpha) = \sum_{j=1}^{k} R_{k,j}(\alpha)\rho(\alpha-j), \qquad (\alpha > k) \tag{1.1}$$

and such that $R_{k,j}(\alpha)$ has no poles for $\alpha > k$.

By the definition of ρ, clearly $R_{1,1}(\alpha) = -1$. Suppose we have shown (1.1) for $k \leq n$. From

$$\alpha^{n+1}\rho^{(n+1)}(\alpha) = \alpha \frac{d}{d\alpha}\{\alpha^n \rho^{(n)}(\alpha)\} - n\alpha^n\rho^{(n)}(\alpha)$$

we obtain

$$\alpha^{n+1}\rho^{(n+1)}(\alpha) = \alpha \sum_{j=1}^{n} \left[\{\frac{d}{d\alpha} R_{n,j}(\alpha)\}\rho(\alpha-j) - \frac{R_{n,j}(\alpha)}{\alpha-j}\rho(a-j-1)\right] +$$

$$- \sum_{j=1}^{n} n R_{n,j}(\alpha)\rho(\alpha-j) .$$

Hence,

$$\alpha^{n+1}\rho^{(n+1)}(\alpha) = \{\alpha R'_{n,1}(\alpha) - n R_{n,1}(\alpha)\}\rho(\alpha-1) + \sum_{j=2}^{n} \{\alpha R'_{n,j}(\alpha) - n R_{n,j}(\alpha) - \frac{\alpha}{\alpha-j+1} R_{n,j-1}(\alpha)\}\rho(\alpha-j) + - \frac{\alpha}{\alpha-n} R_{n,n}(\alpha)\ \rho(\alpha-n-1) .$$

Defining $R_{n+1,j}(\alpha)$ in an obvious way, this completes the proof of (1.1). It is also clear that the $R_{n+1,j}$'s do not have poles for $\alpha > n+1$. Noting that $\{\Gamma(\alpha+1)\}^{-1} = O_M(\alpha^{-M})$ for all positive M, and using lemma 1(a) in the RHS of (1.1), the proof is finished. □

We define for $y \geq 2$ and $\alpha > 1$,

$$\Psi(n,y) = \text{card}\{m \in \mathbb{Z} \mid 2 \leq m \leq n;\ p(m) \leq y\} ,$$

$$G(n,\alpha) = \text{card}\{m \in \mathbb{Z} \mid 2 \leq m \leq n;\ p(m) \leq m^{1/\alpha}\}.$$

The following two lemmas will give useful estimations for $\Psi(n,n^{1/\alpha})$ and $G(n,\alpha)$.

Lemma 2.

(a) *For* $1 < \alpha \leq (\log n)^{1/2}$ *we have uniformly in* α

$$\Psi(n,n^{1/\alpha}) = n\ \rho(\alpha) + O(\frac{n}{\log n}) .$$

(b) *Let* a_v, $v=0,1,\dots$ *be the coefficients in the power series expansion*

$$s(1+s)^{-1}\ \zeta(1+s) = \sum_{v=0}^{\infty} a_v s^v, \qquad |s| < 1 .$$

Here ζ *is the Riemann* ζ*-function. Let* m *be a positive integer, and suppose* $m < \alpha \leq (\log n)^{1/2}$.

$$\Psi(n,n^{1/\alpha}) = n \sum_{v=0}^{m-1} a_v \alpha^v \rho^{(v)}(\alpha)(\log n)^{-v} + O_m(\frac{\alpha^m n}{(\log n)^m})$$

where $\rho^{(v)}$ *is the* v*-th derivative of* ρ.

Lemma 2(a) is a weakened version of a theorem of Ramaswami [4], see also Norton [3, p.47]. Lemma 2(b) was announced by Ramaswami [4, p.109], but he did not publish a proof. It is an immediate consequence of a theorem of De Bruijn [1].

Lemma 3. *For* $1 < \alpha \le (\log n)^{1/2} \cdot (1 - \frac{\log\log n}{\log n})$ *we have uniformly in* α

$$G(n,\alpha) = n\,\rho(\alpha) + O(\frac{n \log\log n}{\log n}) .$$

Proof. Let $2 \le n_1 < n$, then

$$(1.2) \qquad \Psi(n,n_1^{1/\alpha}) - \Psi(n_1,n_1^{1/\alpha}) = \text{card}\{m \in \mathbb{Z} \mid n_1<m\le n;\ p(m)\le n_1^{1/\alpha}\} \le$$

$$\le \text{card}\{m \in \mathbb{Z} \mid n_1<m\le n;\ p(m)<m^{1/\alpha}\} \le$$

$$\le \text{card}\{m \in \mathbb{Z} \mid 2 \le m\le n;\ p(m)\le m^{1/\alpha}\} =$$

$$= G(n,\alpha) .$$

It is obvious that

$$(1.3) \qquad G(n,\alpha) \le \Psi(n,n^{1/\alpha}) .$$

Suppose that $1 < \alpha \le (\log n)^{1/2}(1 - \log\log n/\log n)$. We take $n_1 = n(\log n)^{-1}$, $\beta = \alpha(1 - \log\log n/\log n)^{-1}$. Hence

$$(1.4) \qquad n_1^{1/\alpha} = n^{1/\beta} \quad \text{and} \quad \beta \le (\log n)^{1/2}.$$

According to lemma 2(a), there is an absolute constant K, such that for $1 < \alpha \le (\log n)^{1/2}$

$$(1.5) \qquad |\Psi(n,n^{1/\alpha}) - n\,\rho(\alpha)| \le K \frac{n}{\log n} .$$

Hence, using the trivial estimate $\Psi(n_1,n_1^{1/\alpha}) \le n_1 = n(\log n)^{-1}$ in (1.2), we have

$$(1.6) \qquad G(n,\alpha) \ge \Psi(n,n^{1/\beta})-\Psi(n_1,n_1^{1/\alpha}) \ge n\,\rho(\beta)-\frac{(K+1)n}{\log n} .$$

On the other hand, (1.3) and (1.5) immediately give

$$(1.7) \qquad G(n,\alpha) \le n\,\rho(\alpha)+K\frac{n}{\log n} ;$$

Finally, we estimate $\rho(\alpha) - \rho(\beta)$. From the definition of ρ, lemma 1(a) and (b) we have

$$0 < \rho(\alpha) - \rho(\beta) = \int_\alpha^\beta t^{-1}\rho(t-1)dt \le \frac{\beta-\alpha}{\alpha}\,\rho(\alpha-1) \le \frac{\beta}{\alpha} - 1 =$$

$$= O(\log\log n/\log n) .$$

A combination of (1.6) and (1.7) now proves the lemma. □

2. MAIN RESULT

<u>Theorem</u>. *Let* f(x) *be a continuous and monotonic function of* x *on* $[1,\infty)$, *such that a positive integer* N *exists with*

$$f(x) = O_N(x^N), \qquad (x \to \infty).$$

Then

$$\frac{1}{n}\sum_{m=1}^{n} f(\lambda_m) = -\int_1^\infty f(\alpha)d\rho(\alpha) + O_N(\log\log n/(\log n)^{1/(N+1)}) .$$

<u>Proof</u>. $G(n,\alpha)$ is already defined for $\alpha > 1$. For $0 < \alpha \le 1$ we define $G(n,\alpha) = [n]$. Fix $n > 2$. $G(n,\alpha)$ is a left-continuous stepfunction of α, with a finite number of jumps, say at $1=\alpha_1<\alpha_2<\ldots<\alpha_\nu$. Clearly $G(n,\alpha) = 0$ for $\alpha > \alpha_\nu$.

Define the characteristic functions $\chi(\alpha,m)$ for $\alpha > 0$, $m \geq 2$ by

$$(2.1) \qquad \chi(\alpha,m) = \begin{cases} 1 & \text{if } p(m) \leq m^{1/\alpha} , \\ 0 & \text{if } p(m) > m^{1/\alpha} . \end{cases}$$

Furthermore, $\chi(\alpha,1) = 1$ if $0 < \alpha \leq 1$, $\chi(\alpha,1) = 0$ elsewhere. Take $\alpha_0 = 0$, $f(0) = 0$. Then we have:

$$(2.2) \qquad \sum_{m=1}^{n} f(\lambda_m) = \sum_{m=1}^{n} \sum_{k=1}^{\nu} \{f(\alpha_k)-f(\alpha_{k-1})\}\chi(\alpha_k,m) =$$

$$= \sum_{k=1}^{\nu} \{f(\alpha_k)-f(\alpha_{k-1})\} \sum_{m=1}^{n} \chi(\alpha_k,m) =$$

$$= \sum_{k=1}^{\nu} \{f(\alpha_k)-f(\alpha_{k-1})\}G(n,\alpha_k) =$$

$$= \int_1^{\infty} G(n,\alpha)df(\alpha) + f(1)G(n,1) .$$

Here the integral and all following integrals are Riemann-Stieltjes integrals. It is easy to see that $\alpha_\nu \leq \log n/\log 2 < 2 \log n$. Therefore, instead of (2.2) we may write as well

$$(2.3) \qquad \sum_{m=1}^{n} f(\lambda_m) = \int_1^{2 \log n} G(n,\alpha)df(\alpha) + f(1)G(n,1) .$$

Suppose that $f(x) = O_N(x^N)$, $(x \to \infty)$, for some $N \geq 2$. We split the above integral into two parts:

$$(2.4) \qquad \int_1^{2 \log n} = \int_1^{z} + \int_z^{2 \log n} .$$

Here $z = (\log n)^{1/(N+1)}$. For the first integral we have, according to lemma 3 and the monotonicity of f:

$$\int_1^z G(n,\alpha)df(\alpha) = n\int_1^z \{\rho(\alpha) + O\left(\frac{\text{loglog } n}{\log n}\right)\}df(\alpha) =$$

(2.5)
$$= n\int_1^z \rho(\alpha)df(\alpha) + O\left(\frac{n \text{ loglog } n}{\log n}\int_1^z |df(\alpha)|\right) =$$

$$= n\int_1^z \rho(\alpha)df(\alpha) + O_N\left(\frac{n \text{ loglog } n}{(\log n)^{1/(N+1)}}\right) .$$

By the definition of ρ, $f(1)G(n,1) = n\ f(1)\rho(1) + O(1)$. Furthermore, using lemma 1(a) we have

$$f(z)\rho(z) = O_N\left((\log n)^{N/(N+1)} \cdot \{\Gamma((\log n)^{1/(N+1)})\}^{-1}\right) =$$

$$= O_N((\log n)^{-1}) .$$

We also have, using the same estimate

$$\int_z^\infty f(\alpha)d\rho(\alpha) = O_N(-\int_z^\infty \alpha^N\ d\rho(\alpha)) = O_N((\log n)^{-1}) .$$

Hence, by partial integration we have

(2.6)
$$n\int_1^z \rho(\alpha)df(\alpha) = -\ f(1)G(n,1) - n\int_1^z f(\alpha)d\rho(\alpha) + O_N(\frac{n}{\log n}) =$$

$$= -\ f(1)G(n,1) - n\int_1^\infty f(\alpha)d\rho(\alpha) + O_N(\frac{n}{\log n}) .$$

Combining (2.5) and (2.6) we get

$$(2.7)\qquad \int_1^z G(n,\alpha)df(\alpha) = - f(1)G(n,1) - n\int_1^\infty f(\alpha)d\rho(\alpha) + $$

$$+ O_N\left(\frac{n \text{ loglog } n}{(\log n)^{1/(N+1)}}\right) .$$

It remains to show that the second integral in the RHS of (2.4) is small. As $\Psi(n,n^{1/\alpha})$, for fixed n, is a non-increasing function of α, we have for $\alpha \geq z$:

$$G(n,\alpha) \leq \Psi(n,n^{1/\alpha}) \leq \Psi(n,n^{1/z}) .$$

Suppose that $\alpha \geq z > N+2$ (i.e. $n > \exp\{(N+2)^{N+1}\}$). Using lemma 2(b) we have

$$(2.8)\qquad G(n,\alpha) \leq n \sum_{v=0}^{N+1} a_v\{z^v\rho^{(v)}(z)\}(\log n)^{-v} + O_N\left(\frac{n}{(\log n)^{N+1-\frac{1}{N+1}}}\right) .$$

According to lemma 1(d), substituting $M = (N+1)(N+1-v)$

$$z^v\rho^{(v)}(z) = O_M(z^{-M}) = O_N((\log n)^{-N-1+v}) .$$

Hence from (2.8) we can conclude

$$G(n,\alpha) = O_N\left(\frac{n}{(\log n)^{N+1-\frac{1}{N+1}}}\right)$$

for $\alpha \geq (\log n)^{1/N+1}$. Therefore we have the following estimate:

$$(2.9)\qquad \int_z^{2\log n} G(n,\alpha)df(\alpha) = O_N\left(\frac{n}{(\log n)^{N+1-\frac{1}{N+1}}} \cdot \int_z^{2\log n} |df(\alpha)|\right) =$$

$$= O_N\left(n\ (\log n)^{\frac{-N}{N+1}}\right).$$

Combining (2.3), (2.4), (2.7) and (2.9), the proof is complete. □

Corollary 1. The theorem is also valid for functions, which are the difference of two monotonic functions, both of order $O_N(x^N)$ for some N. In particular, it holds for all polynomials.

Corollary 2. $\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \lambda_m = e^{\gamma}$.

This result can be arrived at by taking $f(\alpha) = \alpha$. The integral then equals, by partial integration

$$-\int_1^{\infty} \alpha\ d\rho(\alpha) = -\int_0^{\infty} \alpha\ d\rho(\alpha) = \int_0^{\infty} \rho(\alpha)d\alpha = e^{\gamma}$$

according to lemma 1(c).

REFERENCES

[1] N.G. DE BRUIJN, *On the number of positive integers* ≤ x *and free of prime factors* > y, Indag. Math. 13 (1951), 50-60.

[2] J. VAN DE LUNE, *The truncated average limit and some of its applications in analytic number theory*, Mathematical Centre Report ZW 20/74, Amsterdam, 1974.

[3] K.K. NORTON, *Numbers with small prime factors, and the least k-th power non-residue*, A.M.S.-Memoir No. 106 (1971), Amer. Math. Soc., Providence, R.I..

[4] V. RAMASWAMI, *The number of positive integers* ≤ x *and free of prime divisors* > x^c, *and a problem of S.S. Pillai*, Duke Math. J., 16 (1949), 99-109.