ALMOST NO SEQUENCE IS WELL DISTRIBUTED

BY

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Let $\xi=(x_n)$ be a sequence of real numbers satisfying $0 \leqq x_n < 1$ $(n=1, 2, \ldots)$. Then ξ is said to be *well distributed in* $[0, 1[$ (in German: "*gleichmässig gleichverteilt*"; HLAWKA [5], PETERSEN [10]) if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=h+1}^{h+N} \chi_{[\alpha,\beta[}(x_n) = \beta-\alpha$$

holds uniformly in $h=0, 1, 2, \ldots$ for every interval $[\alpha, \beta[\subset [0, 1[$, $\chi_{[\alpha,\beta[}$ being the characteristic function of this unterval.

Let $I_\infty = \prod_{n=1}^{\infty} I$ be the infinite dimensional unit cube, i.e. I_∞ is the set of all sequences $\xi=(x_n)$ with $0 \leqq x_n < 1$ $(n=1, 2, \ldots)$, and let $\mu_\infty = \prod_{n=1}^{\infty} \mu_n$ be the completed product measure on I_∞ where each μ_n is Lebesgue measure on I (cf. [3] § 38). Let φ be the mapping of I_∞ onto $[0, 1]$ defined by

$$\varphi(\xi) = \varphi((x_n)) = \sum_{n=1}^{\infty} \frac{a_n}{n!}$$

where $a_n=[nx_n]$, i.e. a_n is the unique integer such that $\frac{a_n}{n} \leqq x_n < \frac{a_n+1}{n}$ $(n=1, 2, \ldots)$.

For any real number x, let $\{x\}$ denote the fractional part $\{x\}=x-[x]$. Referring to a result in [7] DOWIDAR and PETERSEN showed in [2] that a given sequence $\xi \in I_\infty$ is well distributed in $[0, 1[$ if and only if the sequence $\eta=(\{n!\,\varphi(\xi)\})$ is well distributed in $[0, 1[$. Furthermore they showed that the sequence $(\{n!\,\alpha\})$ is well distributed in $[0, 1[$ for μ-almost no α. Hence, in a certain sense, almost no sequence ξ is well distributed in $[0, 1[$.

Obviously, the two notions of "*almost no*" as used in this statement on the one hand and in the sense of product measure on I_∞ on the other hand do not quite coincide. For instance, the set of all sequences $\xi \in I_\infty$ with $x_1=0$, having μ_∞-measure zero, is mapped by φ onto $[0, 1]$, its image thus having μ-measure one. We shall show, however, that it actually follows from Dowidars and Petersens result that μ_∞-almost no sequence ξ is well distributed in $[0, 1[$. We shall also give a direct proof of this statement, using essentially an argument employed by DOWIDAR and PETERSEN in order to show that the sequence $(\{k^n\theta\})$ is not well distributed in $[0, 1[$ for any integer k and any real number θ. This argument also carries through in the case of sequences in any compact Hausdorff space, so that the theorem and its proof will be given in this more general setting.

Theorem 1. *The mapping φ is a Borel-measurable transformation on I_∞ onto $[0, 1]$ and*

$$\mu(B) = \mu_\infty(\varphi^{-1}B) \qquad \textit{for every Borel-set } B \subset [0, 1].$$

Proof. Let $\mathfrak{B}$ be the σ-algebra of Borel-sets in $[0, 1]$ and let $\mathfrak{B}_\infty$ be the corresponding σ-algebra of measurable sets in I_∞. It suffices to show that, for every $\alpha \in]0, 1]$, $\varphi^{-1}[0, \alpha[\in \mathfrak{B}_\infty$ and $\mu_\infty(\varphi^{-1}[0, \alpha[) = \alpha$. The corresponding statements will then follow for every finite union of disjoint half-open intervals $[\alpha, \beta[\subset [0, 1]$ and, by [3] § 15 and § 13A, for all Borel sets B.

We observe that for every $\alpha \in]0, 1]$ we have either a unique expansion

$$\alpha = \sum_{n=1}^{\infty} \frac{a_n}{n!} \qquad (0 \leq a_n < n)$$

or

$$(1) \qquad \alpha = \sum_{n=1}^{k} \frac{a_n}{n!} = \sum_{n=1}^{\infty} \frac{a_n'}{n!} \text{ where } a_k > 0,\ a_n' = \begin{cases} a_n & \text{for } 1 \leq n < k \\ a_k - 1 & \text{for } \quad n = k \\ n-1 & \text{for } k < n < \infty. \end{cases}$$

The set A of α having two expansions is countable and dense in $[0, 1]$. Let us first assume that α is of the form as given in (1). Then

$$\begin{aligned} \varphi^{-1}\{\alpha\} = I \times \left[\frac{a_2}{2}, \frac{a_2+1}{2}\right[\times \dots \times \left[\frac{a_{k-1}}{k-1}, \frac{a_{k-1}+1}{k-1}\right[\times \left[\frac{a_k}{k}, \frac{a_k+1}{k}\right[\times \\ \times \left[0, \frac{1}{k+1}\right[\times \left[0, \frac{1}{k+2}\right[\times \dots \\ \cup\, I \times \left[\frac{a_2}{2}, \frac{a_2+1}{2}\right[\times \dots \times \left[\frac{a_{k-1}}{k-1}, \frac{a_{k-1}+1}{k-1}\right[\times \left[\frac{a_k-1}{k}, \frac{a_k}{k}\right[\times \\ \times \left[\frac{k}{k+1}, 1\right[\times \left[\frac{k+1}{k+2}, 1\right[\times \dots \end{aligned}$$

and

$$\begin{aligned} \varphi^{-1}[0, \alpha[= I \times \left[0, \frac{a_2}{2}\right[\times I \times I \times \dots \\ \cup\, I \times \left[\frac{a_2}{2}, \frac{a_2+1}{2}\right[\times \left[0, \frac{a_3}{3}\right[\times I \times I \times \dots \\ \cup \dots \\ \cup\, I \times \left[\frac{a_2}{2}, \frac{a_2+1}{2}\right[\times \left[\frac{a_3}{3}, \frac{a_3+1}{3}\right[\times \dots \times \left[0, \frac{a_k}{k}\right[\times I \times I \times \dots \\ \setminus\, I \times \left[\frac{a_2}{2}, \frac{a_2+1}{2}\right[\times \left[\frac{a_3}{3}, \frac{a_3+1}{3}\right[\times \dots \times \left[\frac{a_k-1}{k}, \frac{a_k}{k}\right[\times \\ \times \left[\frac{k}{k+1}, 1\right[\times \left[\frac{k+1}{k+2}, 1\right[\times \dots \end{aligned}$$

(we define $[0, 0[$ to be the empty set). Therefore, $\varphi^{-1}[0, \alpha[\in \mathfrak{B}_\infty$ and

$$\mu_\infty(\varphi^{-1}[0, \alpha[) = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} - 0 = \alpha.$$

For an $\alpha = \sum_{n=1}^{\infty} \frac{a_n}{n!} \notin A$ we put $\alpha_k = \sum_{n=1}^{k} \frac{a_n}{n!}$ and obtain

$$\varphi^{-1}[0, \alpha[= \bigcup_{k=1}^{\infty} \varphi^{-1}[0, \alpha_k[\in \mathfrak{B}_\infty$$

and

$$\mu_\infty(\varphi^{-1}[0, \alpha[) = \lim_{k\to\infty} \mu_\infty(\varphi^{-1}[0, \alpha_k[) = \lim_{k\to\infty} \alpha_k = \alpha.$$

Corollary 1.1. *μ_∞-almost no sequence $\xi \in I_\infty$ is well distributed in* $[0, 1[$.

Proof. Let $E \subset I_\infty$ be the set of all ξ that are well distributed in $[0, 1[$. As Dowidar and Petersen [2] have shown we have $\mu(\varphi E) = 0$. Let $B \supset \varphi E$ be a Borel set of Lebesgue measure 0 (cf. [3] § 13B and § 15). Then we have $E \subset \varphi^{-1}\varphi E \subset \varphi^{-1}B$ and, by theorem 1, $\mu_\infty(\varphi^{-1}B) = 0$ which implies $\mu_\infty(E) = 0$.

Now let X be any compact Hausdorff space satisfying the second axiom of countability and let μ be a normed Borel measure on X. Let X_∞ be the compact topological product space of countably many copies of X, i.e. $X_\infty = \prod_{n=1}^{\infty} X_n$ with $X_n = X$ $(n = 1, 2, \ldots)$, and let μ_∞ be the completion of the product measure on X_∞ corresponding to μ. A sequence $\xi = (x_n) \in X_\infty$ is said to be *μ-uniformly distributed in X* if, for every Borel set $E \subset X$ whose boundary has μ-measure zero and for $h = 0$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1+h}^{N+h} \chi_E(x_n) = \mu(E) \tag{2}$$

(χ_E again denoting the characteristic function of E); ξ is said to be *μ-well distributed in X* if, for every such set E, (2) holds uniformly in $h = 0, 1, 2, \ldots$. Equivalently we may require

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1+h}^{N+h} f(x_n) = \int_X f(x)\, d\mu(x)$$

for every continuous complex-valued function f on X and for $h = 0$ resp. uniformly in $h = 0, 1, 2 \ldots$ (cf. [5], [6]).

Let T be the mapping of X_∞ onto X_∞ defined by $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. It is well known that T is measure preserving and ergodic with respect to μ_∞ (cf. [4]). A sequence $\xi \in X_\infty$ is called *completely μ-uniformly distributed* (Korobov [8]) if the sequence $(T^n\xi)$ is μ_∞-uniformly distributed in X_∞, which implies, in particular, that ξ is μ-uniformly distributed in X.

Theorem 2. *Suppose that μ is not a point measure. If the sequence $\xi \in X_\infty$ is completely μ-uniformly distributed in X, then ξ is not μ-well distributed in X.*

Proof. Since μ is not concentrated in one point we can find an open set $E \subset X$ such that $0 < \mu(E) < 1$. Without loss of generality we may assume that the boundary of E has μ-measure zero. (Let, for instance, x_1 and x_2 be two different points of the support of μ and let f be a Urysohn function such that $f(x_1) = 0$, $f(x_2) = 1$. Then we may put $E = \{x : f(x) > \varepsilon\}$ for a suitable choice of ε, $0 < \varepsilon < 1$). Let N be given and let $F_\infty = \prod_{n=1}^{\infty} F_n$ where $F_n = E$ for $1 \leq n \leq N$ and $F_n = X$ for $n > N$. Then F_∞ is open in X_∞ and its boundary has μ_∞-measure zero. Furthermore, we have $0 < \mu_\infty(F_\infty) < 1$. Since the sequence $(T^n\xi)$ is by assumption μ_∞-uniformly distributed in X_∞, there exists a positive integer h_N such that $T^{h_N}\xi \in F_\infty$. Hence, for every choice of N, we have

$$\frac{1}{N}\sum_{n=1+h_N}^{N+h_N} \chi_E(x_n) - \mu(E) = 1 - \mu(E) > 0.$$

Thus, the sequence ξ cannot be well distributed.

Corollary 2.1. *Suppose that μ is not a point measure. Then μ_∞-almost no sequence $\xi \in X_\infty$ is μ-well distributed in X.*

Proof. By the individual ergodic theorem, μ_∞-almost all sequences $\xi \in X_\infty$ are completely μ-uniformly distributed in X (cf. [5] § 6, [1] 3) The assertion then follows from theorem 2.

The two statements "*the sequence ξ is μ-well distributed in X*" and "*the sequence $(T^n\xi)$ is μ_∞-well distributed in X_∞*" should well be distinguished:

Corollary 2.2. *Suppose that μ is not a point measure. Then there is no sequence $\xi \in X_\infty$ such that $(T^n\xi)$ is μ_∞-well distributed in X_∞.*

Proof. Such a sequence ξ would, in particular, have to be completely μ-uniformly distributed in X on the one hand, and μ-well distributed in X on the other hand, a contradiction.

The last corollary is also a consequence of a result of Oxtoby ([9] theorem 5.5) which, extended to not necessarily $1-1$ transformations and applied to the shift transformation T in X_∞, essentially asserts that the sequence $(T^n\xi)$ is μ_∞-well distributed in X_∞ iff μ_∞ is the only T-invariant normed measure on X_∞ (this remark is due to J. Cigler who also, for special sequences in I_∞, has used a reasoning similar to theorem 2 in a talk at the Mathematical Center in Amsterdam in February 1964). Corollary 2.2 also contains the statement of DOWIDAR and PETERSEN that the sequence $(\{k^n \theta\})$ is not well distributed in $[0, 1[$ for any real number θ and any integer $k > 1$. In order to see this one has to identify $[0, 1]$ (via k-adic expansion) with the infinite product space of the discrete space containing k elements, each carrying measure $1/k$, and to observe that in $[0, 1[$ multiplication (mod 1) by k amounts to applying the shift transformation T in this product space (cf. [4]).

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