ALMOST NO SEQUENCE IS WELL DISTRIBUTED

BY

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Let \( \xi = (x_n) \) be a sequence of real numbers satisfying \( 0 \leq x_n < 1 \) \((n = 1, 2, \ldots)\). Then \( \xi \) is said to be well distributed in \([0, 1] \) (in German: "gleichmässig gleichverteilt"; Hlawka [5], Petersen [10]) if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(\alpha, \beta)}(x_n) = \beta - \alpha
\]

holds uniformly in \( h = 0, 1, 2, \ldots \) for every interval \([\alpha, \beta] \subset [0, 1] \), \( \chi_{(\alpha, \beta)} \) being the characteristic function of this interval.

Let \( I_\infty = \prod_{n=1}^{\infty} I \) be the infinite dimensional unit cube, i.e. \( I_\infty \) is the set of all sequences \( \xi = (x_n) \) with \( 0 \leq x_n < 1 \) \((n = 1, 2, \ldots)\), and let \( \mu_\infty = \prod_{n=1}^{\infty} \mu_n \) be the completed product measure on \( I_\infty \) where each \( \mu_n \) is Lebesgue measure on \( I \) (cf. [3] §38). Let \( \varphi \) be the mapping of \( I_\infty \) onto \([0, 1] \) defined by

\[
\varphi(\xi) = \varphi((x_n)) = \sum_{n=1}^{\infty} \frac{a_n}{n!}
\]

where \( a_n = [nx_n] \), i.e. \( a_n \) is the unique integer such that \( \frac{a_n}{n} \leq x_n < \frac{a_n + 1}{n} \) \((n = 1, 2, \ldots)\).

For any real number \( x \), let \( \{x\} \) denote the fractional part \( \{x\} = x - [x] \). Referring to a result in [7] Dowidar and Petersen showed in [2] that a given sequence \( \xi \in I_\infty \) is well distributed in \([0, 1] \) if and only if the sequence \( \eta = \{n! \varphi(\xi)\} \) is well distributed in \([0, 1] \). Furthermore they showed that the sequence \( \{n! \cdot x\} \) is well distributed in \([0, 1] \) for \( \mu \)-almost no \( x \). Hence, in a certain sense, almost no sequence \( \xi \) is well distributed in \([0, 1] \).

Obviously, the two notions of "almost no" as used in this statement on the one hand and in the sense of product measure on \( I_\infty \) on the other hand do not quite coincide. For instance, the set of all sequences \( \xi \in I_\infty \) with \( x_1 = 0 \), having \( \mu_\infty \)-measure zero, is mapped by \( \varphi \) onto \([0, 1] \), its image thus having \( \mu \)-measure one. We shall show, however, that it actually follows from Dowidars and Petersens result that \( \mu_\infty \)-almost no sequence \( \xi \) is well distributed in \([0, 1] \). We shall also give a direct proof of this statement, using essentially an argument employed by Dowidar and Petersen in order to show that the sequence \( \{kn\theta\} \) is not well distributed in \([0, 1] \) for any integer \( k \) and any real number \( \theta \). This argument also
Theorem 1. The mapping \( \varphi \) is a Borel-measurable transformation on \( I_\infty \) onto \([0, 1]\) and

\[
\mu(B) = \mu_\infty(\varphi^{-1}B) \quad \text{for every Borel-set } B \subset [0, 1].
\]

Proof. Let \( \mathcal{B} \) be the \( \sigma \)-algebra of Borel-sets in \([0, 1]\) and let \( \mathcal{B}_\infty \) be the corresponding \( \sigma \)-algebra of measurable sets in \( I_\infty \). It suffices to show that, for every \( \alpha \in [0, 1] \), \( \varphi^{-1}[0, \alpha[ \subset \mathcal{B}_\infty \) and \( \mu_\infty(\varphi^{-1}[0, \alpha])=\alpha \). The corresponding statements will then follow for every finite union of disjoint half-open intervals \((x, \beta[ \subset [0, 1]\) and, by [3] \( \S \) 15 and \( \S \) 13A, for all Borel sets \( B \).

We observe that for every \( \alpha \in [0, 1] \) we have either a unique expansion

\[
\alpha = \sum_{n=1}^{\infty} \frac{a_n}{n!} \quad (0 \leq a_n < n)
\]

or

\[
\alpha = \sum_{n=1}^{k} \frac{a_n}{n!} = \sum_{n=1}^{\infty} \frac{a_n'}{n!} \quad \text{where } a_k > 0, \ a_n' = \begin{cases} a_n & \text{for } 1 \leq n < k \\ a_k - 1 & \text{for } n = k \\ n - 1 & \text{for } k < n < \infty.
\end{cases}
\]

The set \( A \) of \( \alpha \) having two expansions is countable and dense in \([0, 1]\). Let us first assume that \( \alpha \) is of the form as given in (1). Then

\[
\varphi^{-1}[\alpha] = I \times \left[ \frac{a_2}{2}, \frac{a_2 + 1}{2} \right] \times \ldots \times \left[ \frac{a_{k-1}}{k-1}, \frac{a_{k-1} + 1}{k-1} \right] \times \left[ \frac{a_k}{k}, \frac{a_k + 1}{k} \right] \times \ldots \times \left[ 0, \frac{1}{k+1} \right] \times \left[ 0, \frac{1}{k+2} \right] \times \ldots
\]

\[
\cup I \times \left[ \frac{a_2}{2}, \frac{a_2 + 1}{2} \right] \times \ldots \times \left[ \frac{a_{k-1}}{k-1}, \frac{a_{k-1} + 1}{k-1} \right] \times \left[ \frac{a_k}{k}, \frac{a_k + 1}{k} \right] \times \ldots
\]

\[
\cup I \times \left[ \frac{a_3}{3}, \frac{a_3 + 1}{3} \right] \times \ldots \times \left[ 0, \frac{a_k}{k} \right] \times I \times \ldots
\]

\[
\cup \ldots
\]

\[
\cup I \times \left[ \frac{a_2}{2}, \frac{a_2 + 1}{2} \right] \times \left[ \frac{a_3}{3}, \frac{a_3 + 1}{3} \right] \times \ldots \times \left[ 0, \frac{a_k}{k} \right] \times I \times \ldots
\]

\[
\cup I \times \left[ \frac{a_2}{2}, \frac{a_2 + 1}{2} \right] \times \left[ \frac{a_3}{3}, \frac{a_3 + 1}{3} \right] \times \ldots \times \left[ \frac{a_k-1}{k}, \frac{a_k}{k} \right] \times \ldots
\]

\[
\times \left[ \frac{k}{k+1}, 1 \right] \times \left[ \frac{k+1}{k+2}, 1 \right] \times \ldots
\]
(we define \([0, 0]\) to be the empty set). Therefore, \(\varphi^{-1}[0, \alpha] \in \mathcal{B}_\infty\)
and
\[
\mu_\infty(\varphi^{-1}[0, \alpha]) = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} = 0 = \alpha.
\]

For an \(\alpha = \sum_{n=1}^{\infty} \frac{a_n}{n!} \notin \mathbb{A}\) we put \(\alpha_k = \sum_{n=1}^{k} \frac{a_n}{n!}\) and obtain
\[
\varphi^{-1}[0, \alpha] = \bigcup_{k=1}^{\infty} \varphi^{-1}[0, \alpha_k] \in \mathcal{B}_\infty
\]
and
\[
\mu_\infty(\varphi^{-1}[0, \alpha]) = \lim_{k \to \infty} \mu_\infty(\varphi^{-1}[0, \alpha_k]) = \lim_{k \to \infty} \alpha_k = \alpha.
\]

**Corollary 1.1.** \(\mu_\infty\)-almost no sequence \(\xi \in I_\infty\) is well distributed in \([0, 1]\).

**Proof.** Let \(E \subset I_\infty\) be the set of all \(\xi\) that are well distributed in \([0, 1]\). As Dowidar and Petersen [2] have shown we have \(\mu(\varphi E) = 0\). Let \(B \subset \varphi E\) be a Borel set of Lebesgue measure 0 (cf. [3] §13B and §15). Then we have \(E \subset \varphi^{-1} \varphi E \subset \varphi^{-1} B\) and, by theorem 1, \(\mu_\infty(\varphi^{-1} B) = 0\) which implies \(\mu_\infty(E) = 0\).

Now let \(X\) be any compact Hausdorff space satisfying the second axiom of countability and let \(\mu\) be a normed Borel measure on \(X\). Let \(X_\infty\) be the compact topological product space of countably many copies of \(X\), i.e. \(X_\infty = \prod_{n=1}^{\infty} X_n\) with \(X_n = X\) \((n = 1, 2, \ldots)\), and let \(\mu_\infty\) be the completion of the product measure on \(X_\infty\) corresponding to \(\mu\). A sequence \(\xi = (x_n) \in X_\infty\) is said to be \(\mu\)-uniformly distributed in \(X\) if, for every Borel set \(E \subset X\) whose boundary has \(\mu\)-measure zero and for \(h = 0\), we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N+h} \chi_E(x_n) = \mu(E)
\]
(\(\chi_E\) again denoting the characteristic function of \(E\)); \(\xi\) is said to be \(\mu\)-well distributed in \(X\) if, for every such set \(E\), (2) holds uniformly in \(h = 0, 1, 2, \ldots\) Equivalently we may require

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N+h} j(x_n) = \int_X j(x) d\mu(x)
\]
for every continuous complex-valued function \(j\) on \(X\) and for \(h = 0\) resp. uniformly in \(h = 0, 1, 2 \ldots\) (cf. [5], [6]).

Let \(T\) be the mapping of \(X_\infty\) onto \(X_\infty\) defined by \(T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)\). It is well known that \(T\) is measure preserving and ergodic with respect to \(\mu_\infty\) (cf. [4]). A sequence \(\xi \in X_\infty\) is called completely \(\mu\)-uniformly distributed (Korobov [8]) if the sequence \((T^n \xi)\) is \(\mu_\infty\)-uniformly
distributed in $X_\infty$, which implies, in particular, that $\xi$ is $\mu$-uniformly distributed in $X$.

**Theorem 2.** Suppose that $\mu$ is not a point measure. If the sequence $\xi \in X_\infty$ is completely $\mu$-uniformly distributed in $X$, then $\xi$ is not $\mu$-well distributed in $X$.

**Proof.** Since $\mu$ is not concentrated in one point we can find an open set $E \subset X$ such that $0 < \mu(E) < 1$. Without loss of generality we may assume that the boundary of $E$ has $\mu$-measure zero. (Let, for instance, $x_1$ and $x_2$ be two different points of the support of $\mu$ and let $f$ be a Urysohn function such that $f(x_1) = 0$, $f(x_2) = 1$. Then we may put $E = \{x : f(x) > \varepsilon\}$ for a suitable choice of $\varepsilon$, $0 < \varepsilon < 1$.) Let $N$ be given and let $F_\infty = \prod_{n=1}^{\infty} F_n$ where $F_n = E$ for $1 \leq n \leq N$ and $F_n = X$ for $n > N$. Then $F_\infty$ is open in $X_\infty$ and its boundary has $\mu_\infty$-measure zero. Furthermore, we have $0 < \mu_\infty(F_\infty) < 1$. Since the sequence $(T^n \xi)$ is by assumption $\mu_\infty$-uniformly distributed in $X_\infty$, there exists a positive integer $h_N$ such that $T^{h_N} \xi \in F_\infty$. Hence, for every choice of $N$, we have

$$\frac{1}{N} \sum_{n=1}^{N+h_N} \chi_E(x_n) - \mu(E) = 1 - \mu(E) > 0.$$ 

Thus, the sequence $\xi$ cannot be well distributed.

**Corollary 2.1.** Suppose that $\mu$ is not a point measure. Then $\mu_\infty$-almost no sequence $\xi \in X_\infty$ is $\mu$-well distributed in $X$.

**Proof.** By the individual ergodic theorem, $\mu_\infty$-almost all sequences $\xi \in X_\infty$ are completely $\mu$-uniformly distributed in $X$ (cf. [5] § 6, [1] 3) The assertion then follows from theorem 2.

The two statements "the sequence $\xi$ is $\mu$-well distributed in $X$" and "the sequence $(T^n \xi)$ is $\mu_\infty$-well distributed in $X_\infty$" should well be distinguished:

**Corollary 2.2.** Suppose that $\mu$ is not a point measure. Then there is no sequence $\xi \in X_\infty$ such that $(T^n \xi)$ is $\mu_\infty$-well distributed in $X_\infty$.

**Proof.** Such a sequence $\xi$ would, in particular, have to be completely $\mu$-uniformly distributed in $X$ on the one hand, and $\mu$-well distributed in $X$ on the other hand, a contradiction.

The last corollary is also a consequence of a result of Oxtoby ([9] theorem 5.5) which, extended to not necessarily $1-1$ transformations and applied to the shift transformation $T$ in $X_\infty$, essentially asserts that the sequence $(T^n \xi)$ is $\mu_\infty$-well distributed in $X_\infty$ iff $\mu_\infty$ is the only $T$-invariant normed measure on $X_\infty$ (this remark is due to J. Cigler who also, for special sequences in $I_\infty$, has used a reasoning similar to theorem 2.
in a talk at the Mathematical Center in Amsterdam in February 1964). Corollary 2.2 also contains the statement of Dowidar and Petersen that the sequence \((k^{n})\) is not well distributed in \([0, 1]\) for any real number \(0\) and any integer \(k > 1\). In order to see this one has to identify \([0, 1]\) (via \(k\)-adic expansion) with the infinite product space of the discrete space containing \(k\) elements, each carrying measure \(1/k\); and to observe that in \([0, 1]\) multiplication (mod 1) by \(k\) amounts to applying the shift transformation \(T\) in this product space (cf. [4]).

REFERENCES